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Fixed point theorems for partial α-ψ contractive mappings in generalized metric spaces

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Abstract

In this paper, we introduce the concept of partial α-ψ contractive mappings along with generalized metric distance. We also establish the existence of fixed point theorems for such mappings in generalized metric spaces. Our results extend and unify main results of Karapinar [E. Karapinar, Abstr. Appl. Anal., **2014** (2014), 7 pages] and several well-known results in literature. We give some examples to illustrate the usability of our results. Moreover, we prove the fixed point results in generalized metric space endowed with an arbitrary binary relation and the fixed point results in generalized metric space endowed with graph.

Keywords: Fixed point theory, partial α-ψ contractive mappings, generalized metric spaces, binary relation.
2010 MSC: 47H10, 54H25.

1. Introduction

In 2000, Branciari [1] defined the generalized metric space by replacing the triangle inequality in metric space with a more general inequality called quadrilateral inequality. He also established the generalization of the Banach fixed point theorem in the setting of generalized metric space. In 2012, Samet *et al.* [5] introduced the concept of α-admissible mappings and established fixed point theorems for nonlinear mappings satisfying the α-admissibility in complete metric spaces. Recently, Karapinar [2] proved the existence and uniqueness

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of fixed points for α-ψ contractive mappings satisfying the α-admissibility in complete generalized metric spaces.

The aim of this paper is to defined the concept of partial α-ψ contractive mappings. Under some suitable condition, we study and establish fixed point theorems for such mappings in generalized metric spaces. These results extend, unify and generalize main results of Karapinar [2] and various well known results in the existing literature. Also, we give some examples to illustrate the usability of our results. Moreover, we give the fixed point results in generalized metric space endowed with an arbitrary binary relation and the fixed point results in generalized metric space endowed with graph.

2. Preliminaries

In this paper, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}^+$ denotes the set of positive real numbers. In what follows, we recall the notion of generalized metric spaces.

Definition 2.1 ([1])**.** Let X be a nonempty set and $d : X \times X \to [0, \infty)$ satisfies the following conditions for all $x, y \in X$ and all distinct $u, v \in X$, each of them different from x and y:

(GMS1) $d(x, y) = 0 \iff x = y$;
(GMS2) $d(x, y) = d(y, x)$;
(GMS3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *(quadrilateral inequality)*.

Then the mapping d is called *generalized metric*. Here, the pair (X, d) is called *generalized metric space* and abbreviated as GMS.

Example 2.2. Let $X = \{a_1, a_2, a_3, a_4, a_5\}$. Define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(a_i, a_j) = \begin{cases} 0, & i = j, \\ 1, & |i - j| = 1, \\ 5, & i \cdot j = 5, \\ 3, & \text{otherwise.} \end{cases}$$

Then d is a generalized metric on X, but d is not a metric on X, because

$$d(a_2, a_4) = 3 > 2 = d(a_2, a_3) + d(a_3, a_4).$$

Example 2.3. Let $X = \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{0\}$. Define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(x, y) = \begin{cases} 0, & \text{for } x = y, \\ \frac{1}{n}, & \text{for } x \neq y \quad \text{and} \quad x, y \in \{0, \frac{1}{n}\}, \\ 1, & \text{for } x \neq y \quad \text{and} \quad x, y \in X \backslash \{0\}. \end{cases}$$

Then d is a generalized metric on X, but d is not a metric on X, because

$$d\left(\frac{1}{5}, \frac{1}{2}\right) = 1 > 0.7 = d\left(\frac{1}{5}, 0\right) + d\left(0, \frac{1}{2}\right).$$

Definition 2.4 ([1])**.** Let (X, d) be a generalized metric space and $\{x_n\}$ be a sequence of elements of X.

(1) A sequence $\{x_n\}$ is said to be *GMS convergent* to a limit $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ is said to be *GMS Cauchy* if and only if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n > m > N(\epsilon)$.

(3) A generalized metric space X is said to be *complete* if every GMS Cauchy sequence in X is GMS convergent.

Proposition 2.5 ([3]). *Assume that $\{x_n\}$ is a GMS Cauchy sequence in a generalized metric space (X, d) with $\lim_{n\to\infty} d(x_n, u) = 0$ for some $u \in X$. Then $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ for all $z \in X$. In particular, if $z \neq u$, then the sequence $\{x_n\}$ does not converge to z.*

Definition 2.6. Let (X, d) be a generalized metric space. A mapping $T : X \to X$ is *continuous* at a point $x \in X$ if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$, we get $Tx_n \to Tx$ as $n \to \infty$.

Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is nondecreasing;

(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k \tag{2.1}$$

for all $k \geq k_0$ and any $t \in \mathbb{R}^+$.

Lemma 2.7 ([4]). *If $\psi \in \Psi$, then the following assertions hold:*

(i) *$\{\psi^n(t)\}_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;*
(ii) *$\psi(t) \leq t$ for any $t \in \mathbb{R}^+$;*
(iii) *ψ is continuous at 0;*
(iv) *the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.*

Definition 2.8 ([2]). Let (X, d) be a generalized metric space. We say that $T : X \to X$ is an *α-ψ contractive mapping* if there exist $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \tag{2.2}$$

for all $x, y \in X$.

Definition 2.9 ([5]). Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that $T : X \to X$ is an *α-admissible mapping* if the following condition holds:

$$x, y \in X \quad \text{with} \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Example 2.10. Let $X = [0, \infty)$. Define mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} 2, & x \geq y, \\ 0, & x < y. \end{cases}$$

Then T is an α-admissible mapping.

Example 2.11. Let $X = \mathbb{R}$. Define mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \ln|x|, & x \neq 0, \\ 7, & x = 0 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} e^{x-y}, & 0 < y \leq x, \\ 0, & \text{otherwise}. \end{cases}$$

Then T is an α-admissible mapping.

3. Main results

First we give the notion of partial α-ψ contractive mapping in the setting of generalized metric spaces as follows:

Definition 3.1. Let (X, d) be a generalized metric space. We say that $T : X \to X$ is a *partial α-ψ contractive mapping* if there exist $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that the following condition hold:

$$x, y \in X \quad \text{with} \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y)). \tag{3.1}$$

Now, we establish the following fixed point theorems for partial α-ψ contractive mappings in generalized metric spaces.

Theorem 3.2. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a partial α-ψ contractive mapping. Then T has a fixed point provided that the following conditions hold:*

(i) T is an α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;

(iii) T is a continuous mapping.

Proof. From condition (ii) in hypotheses, there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. We construct a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If $x_{\tilde{n}} = x_{\tilde{n}+1}$ for some $\tilde{n} \in \mathbb{N} \cup \{0\}$, then $x_{\tilde{n}} = u$ is a fixed point of T. So we will assume that

$$x_n \neq x_{n+1} \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Since T is an α-admissible mapping, we have

$$\begin{aligned} \alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \quad &\Longrightarrow \quad \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1 \\ &\Longrightarrow \quad \alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1 \\ &\Longrightarrow \quad \alpha(x_3, x_4) = \alpha(Tx_2, Tx_3) \geq 1 \\ &\quad\vdots \end{aligned} \tag{3.3}$$

Continuing above process, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Similarly, we have

$$\alpha(x_n, x_{n+2}) \geq 1 \qquad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

Consider (3.1) and (3.4), we obtain that

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \psi(d(x_n, x_{n-1})) \qquad \text{for all } n \in \mathbb{N}. \tag{3.6}$$

Repeating the process (3.6), we get

$$d(x_{n+1}, x_n) \leq \psi^n(d(x_1, x_0)) \qquad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

From Lemma 2.7, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Consider (3.1) and (3.5), we have

$$d(x_{n+2}, x_n) = d(Tx_{n+1}, Tx_{n-1}) \leq \psi(d(x_{n+1}, x_{n-1})) \qquad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Continuing the process (3.9), we get

$$d(x_{n+2}, x_n) \le \psi^n(d(x_2, x_0)) \qquad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

From Lemma 2.7, we get

$$\lim_{n\to\infty} d(x_{n+2}, x_n) = 0. \tag{3.11}$$

Let $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Without loss of generality, we will assume that $m > n$. Since $\alpha(x_m, x_{m-1}) \ge 1$, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, x_n) \\ &= d(Tx_m, x_m) \\ &= d(Tx_m, Tx_{m-1}) \\ &\le \psi(d(x_m, x_{m-1})) \\ &\le \psi^{m-n}(d(x_{n+1}, x_n)) \end{aligned} \tag{3.12}$$

for all $m, n \in \mathbb{N}$. By Lemma 2.7 (ii) and (3.12), we have

$$d(x_{n+1}, x_n) \le \psi^{m-n}(d(x_{n+1}, x_n)) < d(x_{n+1}, x_n), \tag{3.13}$$

which is a contradiction. Thus, $x_n \neq x_m$ for all $m, n \in \mathbb{N}$ with $m \neq n$.

Next, we will prove that $\{x_n\}$ is a GMS Cauchy sequence. Fix $\epsilon > 0$ and $n(\epsilon)$ such that

$$\sum_{n \ge n(\epsilon)} [\psi^n(d(x_1, x_0)) + \psi^n(d(x_2, x_0))] < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n(\epsilon)$. Assume that $m = n + k$, where $k > 2$. We will consider the only two cases as follows:

Case(I): For k is odd. Let $k = 2l + 1$, where $l \in \mathbb{N}$. By using the quadrilateral inequality together with (3.7), we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+2l+1}) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+2l}, x_{x+2l+1}) \\ &\le \sum_{k=n}^{n+k-1} \psi^k(d(x_1, x_0)) \\ &\le \sum_{n \ge n(\epsilon)} [\psi^n(d(x_1, x_0)) + \psi^n(d(x_2, x_0))] < \epsilon. \end{aligned} \tag{3.14}$$

Case(II): For k is even. Let $k = 2l$, where $l \in \mathbb{N}$. By using the quadrilateral inequality together with (3.10), we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+2l}) \\ &\le d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+2l-1}, x_{n+2l}) \\ &\le \sum_{k=n}^{n+k} \psi^k(d(x_2, x_0)) \\ &\le \sum_{n \ge n(\epsilon)} [\psi^n(d(x_1, x_0)) + \psi^n(d(x_2, x_0))] < \epsilon. \end{aligned} \tag{3.15}$$

It follows that $\{x_n\}$ is a GMS Cauchy sequence in (X, d). By the completeness of (X, d), there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{3.16}$$

Also, from T is continuous and (3.16), we have

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0. \tag{3.17}$$

This implies that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tu$. By Proposition 2.5, we obtain that $Tu = u$, that is, T has a fixed point. □

Definition 3.3. Let (X, d) be a generalized metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. We say that X *satisfies condition* $(\star)$ if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Theorem 3.4. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a partial α-ψ contractive mapping. Therefore, T has a fixed point provided that the following conditions hold:*

(i) T is an α-admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;

(iii) X satisfies condition $(\star)$.

Proof. Following the proof of Theorem 3.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$ converges for some $u \in X$. From (3.4) and condition (iii), we get $\alpha(x_n, \mu) \geq 1$ for all $n \in \mathbb{N}$. By using (3.1), we get

$$\begin{aligned} d(x_{n+1}, Tu) &= d(Tx_n, Tu) \\ &\leq \psi(d(x_n, u)) \qquad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{3.18}$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\lim_{k\to\infty} d(x_{n+1}, Tu) = 0. \tag{3.19}$$

By Proposition 2.5, we conclude that u is a fixed point of T. □

Example 3.5. Let $X = A \cup B$ where $A = (-\infty, 0)$, $B = \{1, 2, 3, 4\}$. Define the generalized metric d on X follows:

$$\begin{aligned} &d(1, 2) = d(2, 1) = d(3, 4) = d(4, 3) = 3, \\ &d(1, 3) = d(3, 1) = d(2, 4) = d(4, 2) = 8, \\ &d(1, 4) = d(4, 1) = d(2, 3) = d(3, 2) = 4, \\ &d(x, y) = |x - y| \quad \text{otherwise}. \end{aligned}$$

If is easy to see that d does not satisfy triangle inequality. Indeed,

$$8 = d(1, 3) \geq d(1, 2) + d(2, 3) = 5.$$

Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be defined as

$$Tx = \begin{cases} \dfrac{2x - 1}{7}, & x \in A, \\ 3, & x \in B \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} x^2 y, & x, y \in B, \\ \dfrac{|xy|}{(|x| + |y|)^2}, & \text{otherwise}. \end{cases}$$

Next, we will show that T is a partial α-ψ contractive mapping $\psi(t) = t/2$ for all $t \in [0, \infty)$. For $\alpha(x, y) \geq 1$, we have $x, y \in B$ and thus

$$d(Tx, Ty) = d(3, 3) \leq \psi(d(x, y)).$$

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we have

$$\alpha(x_0, Tx_0) = \alpha(1, T1) = \alpha(1, 3) = 3 \quad \text{and} \quad \alpha(x_0, T^2x_0) = \alpha(1, T^21) = \alpha(1, T3) = \alpha(1, 3) = 3.$$

Also, we have T is an α-admissible mapping. Indeed, assume that $x, y \in X$ with $\alpha(x, y) \geq 1$. It follows that $x, y \in B$. By definition of the mapping T, we have $\alpha(Tx, Ty) = \alpha(3, 3) = 27 \geq 1$.

Finally, we will prove that condition ($\star$) hold. Let $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we get $x_n \in B$ for $n \in \mathbb{N}$. By the closedness of B, we obtain that $x \in B$. Therefore, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Hence, T satisfies all the conditions of Theorem 3.4 which proves the required result. So T has a (unique) fixed point on X, that is, $x = 3$.

4. Fixed point theorems in generalized metric spaces endowed with an arbitrary binary relations

In this section, we present fixed point theorems in generalized metric spaces endowed with an arbitrary binary relations. Before presenting our results, we give some definitions and notions which are useful for our results as follow:

Let (X, d) be a generalized metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}.$$

Clearly,

$$x, y \in X, \quad x\mathcal{S}y \quad \Longleftrightarrow \quad x\mathcal{R}y \quad \text{or} \quad y\mathcal{R}x.$$

It is easy to see that $\mathcal{S}$ is the symmetric relation attached to $\mathcal{R}$.

Definition 4.1. Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. The mapping $T : X \to X$ is called a *comparative mapping* if

$$x, y \in X \quad \text{with} \quad x\mathcal{S}y \quad \Longrightarrow \quad (Tx)\mathcal{S}(Ty). \tag{4.1}$$

Definition 4.2. Let (X, d) be a generalized metric space and $\mathcal{R}$ be a binary relation over X. The mapping $T : X \to X$ is called a *partial ψ contractive mapping with respect to $\mathcal{S}$* if there exists a function $\psi \in \Psi$ such that the following condition hold:

$$x, y \in X \quad \text{with} \quad x\mathcal{S}y \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y)). \tag{4.2}$$

Theorem 4.3. *Let (X, d) be a generalized metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to X$ be a partial ψ contractive mapping with respect to $\mathcal{S}$. Then T has a fixed point provided that the following conditions hold:*

(i) T is a comparative mapping;
(ii) there exists $x_0 \in X$ such that $(x_0)\mathcal{S}(Tx_0)$ and $(x_0)\mathcal{S}(T^2x_0)$;
(iii) T is a continuous mapping.

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in x\mathcal{S}y, \\ 0, & \text{otherwise.} \end{cases}$$

By condition (ii), we have $\alpha(x_0, Tx_0) = 1$ and $\alpha(x_0, T^2x_0) = 1$. It follows from T is comparative mapping that T is an α-admissible mapping. Since T is a partial ψ contractive mapping with respect to $\mathcal{S}$, we have, for all $x, y \in X$,

$$x\mathcal{S}y \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y))$$

and thus

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y)).$$

Therefore, T is a partial α-ψ contractive mapping. Now all the hypotheses of Theorem 3.2 are satisfied. Therefore, T has a fixed point. □

Definition 4.4. Let (X, d) be a generalized metric space and $\mathcal{R}$ be a binary relation over X. We say that X *satisfies condition* ($\star_{\mathcal{S}}$) if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n\mathcal{S}x$ for all $n \in \mathbb{N}$.

Theorem 4.5. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a partial ψ contractive mapping with respect to $\mathcal{S}$. Then T has a fixed point provided that the following conditions hold:*

(i) T is a comparative mapping;
(ii) there exists $x_0 \in X$ such that $(x_0)\mathcal{S}(Tx_0)$ and $(x_0)\mathcal{S}(T^2x_0)$;
(iii) X satisfies condition ($\star_{\mathcal{S}}$).

Proof. This proof is similar to Theorem 4.3. □

5. Fixed point theorem analysis with graph

In this section, we give the existence of fixed point theorems on a generalized metric space endowed with graph. Before presenting our results, we give some definitions and notions which are useful for our results as follow:

Let (X, d) be a generalized metric space. A set $\{(x, x) : x \in X\}$ is called a diagonal of the Cartesian product $X \times X$ and is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 5.1. Let X be a nonempty set endowed with a graph G. The mapping $T : X \to X$ is called *preserve edge* if the following condition hold:

$$x, y \in X \quad \text{with} \quad (x, y) \in E(G) \quad \Longrightarrow \quad (Tx, Ty) \in E(G). \tag{5.1}$$

Definition 5.2. Let (X, d) be a generalized metric space endowed with a graph G. The mapping $T : X \to X$ is called a *partial ψ contractive mapping with respect to* $E(G)$ if there exists a function $\psi \in \Psi$ such that the following condition hold:

$$x, y \in X \quad \text{with} \quad (x, y) \in E(G) \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y)). \tag{5.2}$$

Theorem 5.3. *Let (X, d) be a generalized metric space endowed with a graph G and $T : X \to X$ be a partial ψ contractive mapping with respect to $E(G)$. Then T has a fixed point provided that the following conditions hold:*

(i) T is preserve edge;
(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $(x_0, T^2x_0) \in E(G)$;
(iii) T is a continuous mapping.

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

From condition (ii), we have $\alpha(x_0, Tx_0) = 1$ and $\alpha(x_0, T^2x_0) = 1$. It follows from T is preserve edge that T is an α-admissible mapping. Since T is a partial ψ contractive mapping with respect to $E(G)$, we have, for all $x, y \in X$,

$$(x, y) \in E(G) \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y))$$

and thus

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi(d(x, y)).$$

Therefore, T is a partial α-ψ contractive mapping. Now all the hypotheses of Theorem 3.2 are satisfied. Therefore, T has a fixed point. □

Definition 5.4. Let (X, d) be a generalized metric space endowed with a graph G. We say that X *satisfies condition* ($\star_{\mathcal{E}}$) if $\{x_n\}$ is sequence in X such that $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Theorem 5.5. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be a partial ψ contractive mapping with respect to $E(G)$. Then T has a fixed point provided that the following conditions hold:*

(i) T is preserve edge;
(ii) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $(x_0, T^2x_0) \in E(G)$;
(iii) X satisfies condition ($\star_{\mathcal{E}}$).

Proof. This proof is similar to Theorem 5.3. □

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References

[1] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen, **57** (2000), 31–37. 1, 2.1, 2.4
[2] E. Karapinar, *Discussion on α-ψ contractions on generalized metric spaces*, Abstr. Appl. Anal., **2014** (2014), 7 pages. 1, 2.8
[3] W. A. Kirk, N. Shahzad, *Generalized metrics and Caristis theorem*, Fixed Point Theory Appl., **2013** (2013), 9 pages. 2.5
[4] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, Cluj-Napoca, (2001). 2.7
[5] B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1, 2.9