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Fixed points for non-self operators in gauge spaces

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Dedicated to the memory of Professor Viorel Radu

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Abstract

The purpose of this article is to present some local fixed point results for generalized contractions on (ordered) complete gauge space. As a consequence, a continuation theorem is also given. Our theorems generalize and extend some recent results in the literature.

Keywords: gauge space, generalized contraction, fixed point, ordered gauge space, continuation theorem.
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1. Introduction

Throughout this paper $\mathbb{E}$ will denote a nonempty set E endowed with a separating gauge structure $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$, where Λ is a directed set (see [5] for definitions). Let $\mathbb{N} := \{0, 1, 2, \cdots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We also denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+ := [0, +\infty)$.

A sequence (x_n) of elements in $\mathbb{E}$ is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an N with $d_\alpha(x_n, x_{n+p}) \le \varepsilon$ for all $n \ge N$ and $p \in \mathbb{N}^*$. The sequence (x_n) is called convergent if there exists an $x_0 \in \mathbb{E}$ such that for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an $N \in \mathbb{N}^*$ with $d_\alpha(x_0, x_n) \le \varepsilon$, for all $n \ge N$.

A gauge space $\mathbb{E}$ is called complete if any Cauchy sequence is convergent. A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements. See also J. Dugundji [5] for other definitions and details.

If $f : E \to E$ is an operator, then $x \in E$ is called fixed point for f if and only if $x = f(x)$. The set $F_f := \{x \in E | x = f(x)\}$ denotes the fixed point set of f.

On the other hand, Ran and Reurings [19] proved the following Banach-Caccioppoli type principle in ordered metric spaces.

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Theorem 1.1 (Ran and Reurings [19]) *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete. Let $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing) operator. Suppose that the following two assertions hold:*

1) there exists $a \in]0, 1[$ such that $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$ with $x \geq y$

2) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$.

Then f has an unique fixed point $x^* \in X$, i. e. $f(x^*) = x^*$, and for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f starting from x converges to $x^* \in X$.

Since then, several authors considered the problem of existence (and uniqueness) of a fixed point for contraction-type operators on partially ordered sets.

In 2005, J.J. Nieto and R. Rodríguez-López in [12] proved a modified variant of Theorem 1.1, by removing the continuity of f. The case of decreasing operators is treated in J.J. Nieto and R. Rodríguez-López [14]. It is also worth mentioning that A. Petruşel, I.A. Rus in [16] and J.J. Nieto, R.L. Pouso, R. Rodríguez-López, improved part of the above mentioned results working in the setting of abstract L-spaces in the sense of Fréchet. D. O'Regan and A. Petruşel in [15] extended these theoretical results to the case of nonlinear contractions and gave some interesting applications to integral equations. Moreover, since then a lot of different generalizations and extensions of these results are proved in the literature (see [1], [2], [9], [10], [11], [22], etc.).

The aim of this paper is to present some local fixed point theorems for generalized contractions on ordered complete gauge space. As a consequence, a continuation theorem is also given. Our theorems generalize some of the above mentioned theorems, as well as, some other ones in the recent literature.

2. Preliminaries

Let X be a nonempty set and $f : X \to X$ be an operator. Then,

$$f^0 := 1_X,\ f^1 := f, \ldots, f^{n+1} = f \circ f^n,\ n \in \mathbb{N}$$

denote the iterate operators of f. Let X be a nonempty set and let $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [6]; see also [21]) if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence, $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$.

In this setting, if $U \subset X \times X$, then an operator $f : X \to X$ is called orbitally U-continuous (see [13]) if: $[x \in X$ and $f^{n(i)}(x) \to a \in X$, as $i \to +\infty$ and $(f^{n(i)}(x), a) \in U$ for any $i \in \mathbb{N}]$ imply $[f^{n(i)+1}(x) \to f(a)$, as $i \to +\infty]$. In particular, if $U = X \times X$, then f is called orbitally continuous.

Let $(X, \leq)$ be a partially ordered set, i.e., X is a nonempty set and $\leq$ is a reflexive, transitive and anti-symmetric relation on X. Denote

$$X_{\leq} := \{(x, y) \in X \times X | x \leq y \text{ or } y \leq x\}.$$

In the same setting, consider $f : X \to X$. Then,

$$(LF)_f := \{x \in X | x \leq f(x)\}$$

is the lower fixed point set of f, while

$$(UF)_f := \{x \in X | x \geq f(x)\}$$

is the upper fixed point set of f.

If $f : X \to X$ and $g : Y \to Y$, then the cartesian product of f and g is denoted by $f \times g$ and it is defined in the following way:

$$f \times g : X \times Y \to X \times Y, (f \times g)(x, y) := (f(x), g(y)).$$

Definition 2.1 Let X be a nonempty set. By definition $(X, \to, \leq)$ is an ordered L-space if and only if:
(i) $(X, \to)$ is an L-space;
(ii) $(X, \leq)$ is a partially ordered set;
(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$, for each $n \in \mathbb{N} \ \Rightarrow x \leq y$.

If $\mathbb{E} := (E, \mathcal{D})$ is a gauge space, then the convergence structure is given by the family of gauges $\mathcal{D} = \{d_\alpha\}_{\alpha\in\Lambda}$. Hence, $(E, \mathcal{D}, \leq)$ is an ordered L-space and it will be called an ordered gauge space, see also [17], [18], [4].

If $\mathbb{E} := (E, \mathcal{D})$ is a gauge structure, then for $r := \{r_\alpha\}_{\alpha\in A} \in (0, \infty)^A$ and $x_0 \in E$, we will denote by $\overline{B}_d(x_0, r)$ the closure of $B_d(x_0, r)$ in $(\mathbb{E}, \mathcal{D})$, where

$$B_d(x_0, r) := \{x \in \mathbb{E} \ : d_\alpha(x_0, x) < r_\alpha, \text{ for all } \alpha \in A\}.$$

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0. For example, $\varphi(t) = at$ (where $a \in [0, 1[$), $\varphi(t) = \frac{t}{1+t}$ and $\varphi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$ are comparison functions.

3. Fixed point results

Our first main result is the following existence, uniqueness and approximation fixed point theorem.

Theorem 3.3 *Let $(E, \mathcal{D}, \leq)$ be an ordered complete gauge space, $x_0 \in E$ and $r := \{r_\alpha\}_{\alpha\in A} \in (0, \infty)^A$. Let $f : \overline{B} := \overline{B}_d(x_0, r) \to E$ be an operator. Suppose that:*
(i) $\overline{B}_\leq \in I(f \times f)$;
(ii) $(x, y) \in \overline{B}_\leq$ and $(y, z) \in \overline{B}_\leq$ imply $(x, z) \in \overline{B}_\leq$;
(iii) $(x_0, f(x_0)) \in \overline{B}_\leq$;
(iv) f is orbitally continuous;
(v) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $\alpha \in \Lambda$ we have

$$d_\alpha(f(x), f(y)) \leq \varphi(d_\alpha(x, y)), \text{ for each } (x, y) \in \overline{B}_\leq.$$

Then, f has at least one fixed point $x^* \in \overline{B}$ and, for each $x \in \overline{B}_\leq$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges to x^*. Moreover, the fixed point is unique in $\overline{B}_\leq$.

Proof. Let $x_0 \in E$ be such that $(x_0, f(x_0)) \in \overline{B}_\leq$. Suppose first that $x_0 \neq f(x_0)$. From (i) we obtain

$$(f(x_0), f^2(x_0)), (f^2(x_0), f^3(x_0)), \cdots, (f^n(x_0), f^{n+1}(x_0)), \cdots \in \overline{B}_\leq.$$

From (v), by induction, we get, for each $\alpha \in \Lambda$, that

$$d_\alpha(f^n(x_0), f^{n+1}(x_0)) \leq \varphi^n(d_\alpha(x_0, f(x_0)), \text{ for each } n \in \mathbb{N}.$$

Since $\varphi^n(d_\alpha(x_0, f(x_0)) \to 0$ as $n \to +\infty$, for an arbitrary $\varepsilon > 0$ we can choose $N \in \mathbb{N}^*$ such that $d_\alpha(f^n(x_0), f^{n+1}(x_0)) < \varepsilon - \varphi(\varepsilon)$, for each $n \geq N$. Since $(f^n(x_0), f^{n+1}(x_0)) \in \overline{B}_\leq$ for all $n \in \mathbb{N}$, we have that:

$$d_\alpha(f^n(x_0), f^{n+2}(x_0)) \leq d_\alpha(f^n(x_0), f^{n+1}(x_0)) + d_\alpha(f^{n+1}(x_0), f^{n+2}(x_0))$$
$$< \varepsilon - \varphi(\varepsilon) + \varphi(d_\alpha(f^n(x_0), f^{n+1}(x_0)) \leq \varepsilon, \text{ for all } n \geq N.$$

Now since $(f^n(x_0), f^{n+2}(x_0)) \in \overline{B}_\leq$ (see (iii)) we have for any $n \geq N$ that

$$d_\alpha(f^n(x_0), f^{n+3}(x_0)) \leq d_\alpha(f^n(x_0), f^{n+1}(x_0)) + d_\alpha(f^{n+1}(x_0), f^{n+3}(x_0))$$

$< \varepsilon - \varphi(\varepsilon) + \varphi(d_\alpha(f^n(x_0), f^{n+2}(x_0)) \leq \varepsilon.$
By induction, for each $\alpha \in \Lambda$, we have
$d_\alpha(f^n(x_0), f^{n+k}(x_0)) < \varepsilon,$ for any $k \in \mathbb{N}^*$ and $n \geq N$.
Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\overline{B}, \mathcal{D})$. From the completeness of the gauge space $(E, \mathcal{D})$ we have $(f^n(x_0))_{n\in\mathbb{N}} \to x^* \in \overline{B}$ as $n \to +\infty$.

Let $x \in E$ be such that $(x, x_0) \in \overline{B}_\leq$ then $(f^n(x), f^n(x_0)) \in \overline{B}_\leq$ and thus, for each $\alpha \in \Lambda$, we have $d_\alpha(f^n(x), f^n(x_0)) \leq \varphi^n(d_\alpha(x, x_0))$, for each $n \in \mathbb{N}$. Letting $n \to +\infty$ we obtain that $(f^n(x))_{n\in\mathbb{N}} \to x^*$. By the orbital continuity of f we get that $x^* \in F_f$. Thus $x^* = f(x^*)$.

If $f(x_0) = x_0$, then x_0 plays the role of x^*. □

Remark 3.4 *Equivalent representation of condition (iii) are:*
(iv)' There exists $x_0 \in E$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$
(iv)" $(LF)_f \bigcup (UF)_f \neq \emptyset$.

Remark 3.5 *Condition (i) is implied by each of the following assertions:*
(ii)' $f : (\overline{B}, \leq) \to (E, \leq)$ is increasing
(ii)" $f : (\overline{B}, \leq) \to (E, \leq)$ is decreasing.

In a similar way to Theorem 3.3, we can prove the following result, which is useful for applications (see [17]).

Theorem 3.6 *Let $(E, \mathcal{D}, \leq)$ be an ordered complete gauge space, $x_0 \in E$ and $r := \{r_\alpha\}_{\alpha\in A} \in (0, \infty)^A$. Let $f : \overline{B} := \overline{B}_d(x_0, r) \to E$ be an operator. We suppose that:*
(i) $f : (\overline{B}, \leq) \to (E, \leq)$ is increasing;
(ii) $x_0 \leq f(x_0)$;
$(iii)_A$ f is orbitally continuous
or
$(iii)_B$ if an increasing sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in $\overline{B}$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
(iv) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_\alpha(f(x), f(y)) \leq \varphi(d_\alpha(x, y)), \text{ for each } (x, y) \in \overline{B} \text{ with } x \leq y \text{ and for all } \alpha \in \Lambda;$$

(v) $d_\alpha(x_0, f(x_0)) < r - \varphi(r)$, for each $\alpha \in \Lambda$;
Then f has at least one fixed point in $\overline{B}$. Moreover, the fixed point is unique in the set $\overline{B}_\leq$ of comparable elements from $\overline{B}$.

Proof. Since $f : (\overline{B}, \leq) \to (E, \leq)$ is increasing and $x_0 \leq f(x_0)$ we immediately have $x_0 \leq f(x_0) \leq f^2(x_0) \leq \cdots f^n(x_0) \leq \cdots$. Notice that, by (v), we have that $f(x_0) \in \overline{B}$. Thus, by (v) and (iv) we get that, for each $\alpha \in \Lambda$, we have $d_\alpha(f^2(x_0), f(x_0)) \leq \varphi(d_\alpha(f(x_0), x_0)) < \varphi(r)$. Hence, for each $\alpha \in \Lambda$, we obtain $d_\alpha(f^2(x_0), x_0) \leq d_\alpha(f^2(x_0), f(x_0)) + d_\alpha(f(x_0), x_0) < \varphi(r) + r - \varphi(r) = r$, proving that $f^2(x_0) \in \overline{B}$. By induction, we get that $f^n(x_0) \in \overline{B}$, for each $n \in \{1, 2, \cdots\}$.

Now using (iv), we obtain $d_\alpha(f^n(x_0), f^{n+1}(x_0)) \leq \varphi^n(d_\alpha(x_0, f(x_0))$, for each $n \in \mathbb{N}$. By a similar approach as in the proof of Theorem 3.3 we obtain:
$d_\alpha(f^n(x_0), f^{n+k}(x_0)) < \varepsilon,$ for any $k \in \mathbb{N}^*$ and $n \geq N$.
Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{E}$. From the completeness of the gauge space we have that $(f^n(x_0))_{n\in\mathbb{N}} \to x^* \in \overline{B}$ as $n \to +\infty$.

By the orbital continuity of the operator f we get that $x^* \in F_f$. If $(iii)_B$ takes place, then, since $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, given any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^*$ such that for each $n \geq N_\epsilon$ we have $d_\alpha(f^n(x_0), x^*) < \epsilon$. On the other hand, for each $n \geq N_\epsilon$, since $f^n(x_0) \leq x^*$, we have, for each $\alpha \in \Lambda$, that:
$d_\alpha(x^*, f(x^*)) \leq d_\alpha(x^*, f^{n+1}(x_0)) + d_\alpha(f(f^n(x_0)), f(x^*)) \leq d_\alpha(x^*, f^{n+1}(x_0)) + \varphi(d_\alpha(f^n(x_0), x^*)) < 2\epsilon.$
Thus $x^* \in F_f$.

The uniqueness of the fixed point follows by contradiction. Suppose there exists $y^* \in F_f$, with $x^* \neq y^*$ and $(x^*, y^*) \in \overline{B}_\leq$. Then we have $0 < d_\alpha(y^*, x^*) = d_\alpha(f^n(y^*), f^n(x^*)) \leq \varphi^n(d_\alpha(y^*, x^*)) \to 0$ as $n \to +\infty$, which is a contradiction. Hence $x^* = y^*$. □

Remark 3.7 *A kind of dual result also holds, with the following modified assumptions:*
(ii)' $x_0 \geq f(x_0)$*;*
$(iii)'_B$ *if a decreasing sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* x *in* $\overline{B}$*, then* $x_n \geq x$ *for all* $n \in \mathbb{N}$*.*

Remark 3.8 *It is worth to mention that could be of interest to extend the above technique for other metrical fixed point theorems, see [3], [8], etc. It is also an open problem to present a fixed point theory (in the sense of [20]) for contractions and generalized contractions in ordered complete gauge spaces.*

As a consequence of the above results a continuation result can be given now. For a nice survey on this topic see Frigon [7].

Theorem 3.9 *Let* $(\mathbb{E}, \mathcal{D})$ *be a complete gauge space, where* $\mathcal{D} := \{d_\alpha\}_{\alpha\in\Lambda}$ *is a gauge structure on* $\mathbb{E}$*. Let* U *be an open subset of* $\mathbb{E}$ *and* $G : \overline{U} \times [0,1] \to \mathbb{E}$ *be an operator. Assume that the following assumptions are satisfied:*

(i) $x \neq G(x,t)$*, for each* $x \in \partial U$ *(the boundary of* U*) and each* $t \in [0,1]$*;*

(ii) there exists $a := \{a_\alpha\}_{\alpha\in A} \in]0,+\infty[^\Lambda$ *such that* $a_\alpha < 1$ *and*

$$d_\alpha(G(x,\cdot), G(y,\cdot)) \leq a_\alpha \cdot d_\alpha(x,y), \text{ for each } x, y \in \overline{U}_\leq \text{ and for each } \alpha \in \Lambda.$$

(iii) there exists a continuous function $\phi : [0,1] \to \mathbb{R}$ *such that*

$$d_\alpha(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)|, \text{ for all } t, s \in [0,1] \text{ and each } x \in \overline{U};$$

(iv) $G : \overline{U} \times [0,1] \to \mathbb{E}$ *is continuous;*

(v) $G(\cdot,t) : \overline{U} \to \mathbb{E}$ *is increasing.*

Then $G(\cdot,0)$ *has a fixed point if and only if* $G(\cdot,1)$ *has a fixed point.*

Proof. Suppose that $z \in F_{G(\cdot,0)}$. From (i) we have that $z \in U$. Consider the set

$$S := \{(t,x) \in [0,1] \times U : x = G(x,t)\}.$$

Since $(0,z) \in S$, we have that $S \neq \emptyset$. We introduce a partial order defined on S by the formula:

$$(t,x) \leq (s,y) \text{ if and only if } t \leq s \text{ and } d_\alpha(x,y) \leq \frac{2}{1-a_\alpha}[\phi(s) - \phi(t)].$$

Let M be a totally ordered subset of S, $t^* := \sup\{t \ : (t,x) \in M\}$ and let $(t_n, x_n)_{n\in\mathbb{N}^*} \subset M$ be a sequence such that $(t_n, x_n) \leq (t_{n+1}, x_{n+1})$ for each $n \in \mathbb{N}^*$ and let $t_n \to t^*$ as $n \to \infty$. Then

$$d_\alpha(x_m, x_n) \leq \frac{2}{1-a_\alpha}[\phi(t_m) - \phi(t_n)], \text{ for each } m, n \in \mathbb{N}^*, \ m > n.$$

Letting $m, n \to +\infty$ we obtain that $d_\alpha(x_m, x_n) \to 0$, proving that $(x_n)_{n\in\mathbb{N}^*}$ is Cauchy. Denote by $x^* \in \mathbb{E}$ its limit. Since $x_n = G(x_n, t_n)$, $n \in \mathbb{N}^*$ and using the fact that G is continuous, we get that $x^* = G(x^*, t^*)$. From (i) we note that $x^* \in U$. Thus $(t^*, x^*) \in S$.

From the fact that M is totally ordered we have that $(t,x) \leq (t^*, x^*)$, for each $(t,x) \in M$. Thus (t^*, x^*) is an upper bound of M. We can apply Zorn's Lemma, so S admits a maximal element $(t_0, x_0) \in S$. Notice here that $G(x_0, t) \leq G(x_0, t_0) = x_0$, for each $t \in [0,1]$. We now prove that $t_0 = 1$.

Suppose that $t_0 < 1$. Let $r = \{r_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ and $t \in]t_0, 1]$ be such that $B_d(x_0, r_\alpha) \subset U$ and $r_\alpha := \frac{2}{1-a_\alpha}[\phi(t) - \phi(t_0)]$ for every $\alpha \in A$. Then for each $\alpha \in A$ we have:

$$\begin{aligned} d_\alpha(x_0, G(x_0,t)) &\leq d_\alpha(x_0, G(x_0,t_0)) + d_\alpha(G(x_0,t_0), G(x_0,t)) \\ &\leq \phi(t) - \phi(t_0) = \frac{r_\alpha(1-a_\alpha)}{2} < (1-a_\alpha) r_\alpha. \end{aligned}$$

Since $\overline{B}_d(x_0, r_\alpha) \subset \overline{U}$, the operator $G(\cdot, t) : \overline{B}_d(x_0, r) \to \mathbb{E}$ satisfies, for all $t \in [0, 1]$, the assumptions of the dual variant of Theorem 3.6 (with $\varphi_\alpha(t) := a_\alpha t$, for each $t \in [0, 1]$). Hence there exists $x \in \overline{B}_d(x_0, r_\alpha)$ such that $x = G(x, t)$. Thus $(t, x) \in S$. Since we have that

$$d_\alpha(x_0, x) \leq r_\alpha = \frac{2}{1 - a_\alpha}[\phi(t) - \phi(t_0)],$$

thus we have that

$$(t_0, x_0) < (t, x),$$

which contradicts the maximality of (t_0, x_0). Thus $t_0 = 1$ and the proof is complete. □

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