Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 6 (2013), 35–40
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces

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Dedicated to the memory of Professor Viorel Radu

Abstract

We prove some common coupled fixed point theorems for contractive mappings in fuzzy metric spaces under geometrically convergent t-norms.

Keywords: Fuzzy metric space, g-convergent t-norm, coupled common fixed point.
2010 MSC: Primary 54E70; Secondary 54H25.

1. Introduction

Many common coupled fixed point theorems for contractions in fuzzy metric spaces and probabilistic metric spaces under either a t-norm of Hadžić-type or the t-norm $T_P = Prod$ can be found in the recent literature, see, e.g., [10], [6], [11], [2], [3], [1], [7], [11]. The aim of this paper is to obtain similar results in a larger class of fuzzy metric spaces, namely in fuzzy metric spaces endowed with geometrically convergent t-norms.

We assume that the reader is familiar with the basic concepts and terminology of the theory of fuzzy metric spaces. We only recall that a t-norm T is said to be *of Hadžić-type* (denoted $T \in \mathcal{H}$) if the family $\{T^n(t)\}_{n=1}^{\infty}$ defined by

$$T^1(t) = t, T^{n+1}(t) = T(t, T^n(t)) \quad (n = 1, 2, ..., t \in [0, 1])$$

is equicontinuous at $t = 1$, and that a t-norm T is called *geometrically convergent (or g-convergent)* ([5]) if, for all $q \in (0, 1)$,

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1 - q^i) = 1.$$

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Received 2012-6-11

It is worth noting (see e.g. [5]) that if for a t-norm there exists $q_0 \in (0,1)$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1-q_0^i)=1,$$

then

$$\lim_{n\to\infty} T_{i=n}^{\infty}(1-q^i)=1$$

for every $q \in (0,1)$.

The well-known t-norms $T_M = Min$, $T_P = Prod$, T_L (Łukasiewicz t-norm) are g-convergent. Also, every member of the *Domby family* $(T_\lambda^D)_{\lambda\in(0,\infty)}$, *Aczel-Alsina family* $(T_\lambda^{AA})_{\lambda\in(0,\infty)}$ and *Sugeno-Weber family* $(T_\lambda^{SW})_{\lambda\in(-1,\infty)}$ is g-convergent ([5]). A large class of g-convergent t-norms, in terms of the generators of strict t-norms is described in [5] (also see [4], Ch. 1.8).

In the following we consider M-complete fuzzy metric spaces in the sense of Kramosil and Michalek ([8]), satisfying the condition (FM-6): $\lim\limits_{t\to\infty} M(x,y,t)=1$ for all $x, y \in X$.

2. Main Results

We start by recalling two definitions from [9].

Definition 2.1. Let X be a nonempty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to commute if $gF(x,y) = F(gx,gy)$ for all $x, y \in X$.

Definition 2.2. An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

The mappings F and g have a common fixed point if there exists $x \in X$ such that $x = gx = F(x,x)$.

Our main theorem states as follows.

Theorem 2.3. *Let (X,M,T) be a complete fuzzy metric space, satisfying (FM-6), with T a g-convergent t-norm. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that, for some $k \in (0,1)$,*

$$M(F(x,y),F(u,v),kt) \geq Min\{M(gx,gu,t),M(gy,gv,t)\} \tag{2.1}$$

for all $x, y, u, v \in X, t > 0$.

Suppose that $F(X \times X) \subset g(X)$, and that g is continuous and commutes with F. If there exist $a > 0$ and $x_0, y_0 \in X$ such that

$$\sup_{t>0} t^a(1-M(gx_0,F(x_0,y_0),t)) < \infty$$

and

$$\sup_{t>0} t^a(1-M(gy_0,F(y_0,x_0),t)) < \infty,$$

then F and g have a unique common fixed point in X.

We note that if (x_0,y_0) is a coupled coincidence point of F and g, then the conditions $\sup\limits_{t>0} t^a(1-M(gx_0,F(x_0,y_0),t)) < \infty$ and $\sup\limits_{t>0} t^a(1-M(gy_0,F(y_0,x_0),t)) < \infty$ are satisfied.

Proof. Let x_0, y_0 be as in the statement of the theorem. Since $F(X \times X) \subset g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$. Continuing in this way one can construct two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X with the properties

$$gx_{n+1} = F(x_n,y_n),\ gy_{n+1} = F(y_n,x_n),\ \forall n \in \mathbb{N}.$$

We divide the proof into 5 steps.

Step 1. We show that $\{gx_n\}_{n\in\mathbb{N}}$ and $\{gy_n\}_{n\in\mathbb{N}}$ are Cauchy sequences.

Indeed, let $\alpha > 0$ be such that

$$t^a(1 - M(gy_0, F(y_0, x_0), t)) \le \alpha$$

and

$$t^a(1 - M(gx_0, F(x_0, y_0), t)) \le \alpha$$

for all $t > 0$. Then $M\left(gx_0, gx_1, \frac{1}{t^n}\right) \ge 1 - \alpha(t^a)^n$ and $M\left(gy_0, gy_1, \frac{1}{t^n}\right) \ge 1 - \alpha(t^a)^n$ for every $t > 0$ and $n \in \mathbb{N}$.

If $t > 0$ and $\varepsilon \in (0, 1)$ are given, we choose μ in the interval $(k, 1)$ such that $T_{i=n+1}^{\infty}(1 - (\mu^a)^i) > 1 - \varepsilon$ and $\delta = \frac{k}{\mu}$. As $\delta \in (0, 1)$, we can find $n_1 (= n_1(t))$ such that $\sum_{n=n_1}^{\infty} \delta^n < t$.

Condition (2.1) implies that, for all $s > 0$,

$$\begin{aligned} M(gx_1, gx_2, ks) &= M(F(x_0, y_0), F(x_1, y_1), s) \\ &\ge Min\{M(gx_0, gx_1, s), M(gy_0, gy_1, s)\}, \end{aligned}$$

and

$$\begin{aligned} M(gy_1, gy_2, ks) &= M(F(y_0, x_0), F(y_1, x_1), s) \\ &\ge Min\{M(gy_0, gy_1, s), M(gx_0, gx_1, s)\}. \end{aligned}$$

It follows by induction that

$$M(gx_n, gx_{n+1}, k^n s) \ge Min\{M(gx_0, gx_1, s), M(gy_0, gy_1, s)\},$$

$$M(gy_n, gy_{n+1}, k^n s) \ge Min\{M(gy_0, gy_1, s), M(gx_0, gx_1, s)\},$$

for all $n \in \mathbb{N}$. Then for all $n \ge n_1$ and all $m \in \mathbb{N}$ we obtain

$$\begin{aligned} M(gx_n, gx_{n+m}, t) &\ge M\left(gx_n, gx_{n+m}, \sum_{i=n_1}^{\infty} \delta^i\right) \\ &\ge M\left(gx_n, gx_{n+m}, \sum_{i=n}^{n+m-1} \delta^i\right) \\ &\ge T_{i=n}^{n+m-1} M\left(gx_i, gx_{i+1}, \delta^i\right) \\ &\ge T_{i=n}^{n+m-1}\left(Min\left\{M\left(gx_0, gx_1, \frac{1}{\mu^i}\right), M\left(gy_0, gy_1, \frac{1}{\mu^i}\right)\right\}\right) \\ &\ge T_{i=n}^{n+m-1}(1 - \alpha\mu^{ai}). \end{aligned}$$

If we choose $l_0 \in \mathbb{N}$ such that $\alpha\mu^{al_0} \le \mu^a$, then

$$1 - \alpha(\mu^a)^{n+l_0} \ge 1 - (\mu^a)^{n+1}$$

for all n. Thus,

$$M(gx_{n+l_0}, gx_{n+l_0+m}, t) \ge T_{i=n+1}^{\infty}(1 - (\mu^a)^i) > 1 - \varepsilon,$$

for every $n \ge n_1$ and $m \in \mathbb{N}$, hence $\{gx_n\}$ is a Cauchy sequence.

Similarly one can show that $\{gy_n\}$ is a Cauchy sequence.

Step 2. We prove that g and F have a coupled coincidence point.

Since X is complete, there exist $x, y \in X$ such that $\lim_{n\to\infty} gx_n = x$, $\lim_{n\to\infty} gy_n = y$. We show that $F(x, y) = gx$ and $F(y, x) = gy$.

From the continuity of g it follows that $\lim_{n\to\infty} ggx_n = gx$ and $\lim_{n\to\infty} ggy_n = gy$.

As F and g commute,

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n).$$

Consequently, for all $t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} M(ggx_{n+1}, F(x,y), kt) &= M(F(x_n, y_n), F(x,y), kt) \\ &= M(F(gx_n, gy_n), F(x,y), kt) \\ &\geq Min\{M(ggx_n, gx, t), M(ggy_n, gy, t). \end{aligned}$$

Letting $n \to \infty$ yields $M(gx, F(x, y, kt) \geq 1$ for all $t > 0$, hence $gx = F(x, y)$. Similarly one can deduce that $F(y, x) = gy$.

Step 3. We show that $gx = y$ and $gy = x$.

Indeed, letting $n \to \infty$ in the inequality

$$M(gx, gy_{n+1}, kt) \geq Min\{M(gx, gy_n, t), M(gy, gx_n, t)\} \quad (t > 0)$$

(obtained from $M(gx, gy_{n+1}, kt) = M(F(x, y), F(y_n, x_n), t)$), we get

$$M(gx, y, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}$$

and similarly

$$M(gy, x, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}.$$

Thus

$$Min\{M(gx, y, t), M(gy, x, t)\} \geq Min\left\{M\left(gx, y, \frac{t}{k^n}\right), M\left(gy, x, \frac{t}{k^n}\right)\right\}$$

for all $n \in \mathbb{N}$, implying $Min\{M(gx, y, t), M(gy, x, t)\} = 1$ for all $t > 0$. It follows that $M(gx, y, t) = M(gy, x, t) = 1$ for all $t > 0$, whence $gx = y$ and $gy = x$, as claimed.

Step 4. We prove that $x = y$.

Indeed, from

$$\begin{aligned} M(gx_{n+1}, gy_{n+1}, kt) &= M(F(x_n y_n), F(y_n, x_n), kt) \\ &\geq Min\{M(gx_n, gy_n, t), M(gy_n, gx_n, t)\} \quad (t > 0) \end{aligned}$$

it follows that $M(x, y, kt) \geq M(x, y, t)$ for all $t > 0$, and so $x = y$.

Step 5. We show that the fixed point is unique.

Let z, w be common fixed points for F and g. Then from (2.1) we obtain

$$M(F(z, z), F(w, w), kt) \geq Min\{M(gz, gw, t), M(gz, gw, t)\} \quad (t > 0),$$

that is, $M(z, w, kt) \geq M(z, w, t)\ \forall t > 0$, implying $z = w$. □

Our next theorem shows that, if the t-norm T is of Hadžić-type, then the conditions

$$sup_{t>0} t^a (1 - M(gx_0, F(x_0, y_0), t)) < \infty$$

and

$$sup_{t>0} t^a (1 - M(gy_0, F(y_0, x_0), t)) < \infty$$

can be dropped.

Theorem 2.4. *Let (X, M, T) be a complete fuzzy metric space satisfying (FM6), with $T \in \mathcal{H}$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that, for some $k \in (0, 1)$,*

$$M(F(x, y), F(u, v), kt) \geq Min\{M(gx, gu, t), M(gy, gv, t)\}$$

for all $x, y, u, v \in X, t > 0$. Suppose that $F(X \times X) \subset g(X)$ and g is continuous and commutes with F. Then F and g have a unique common fixed point in X.

Proof. We only have to verify Step 1 in Theorem 2.3, that is, to prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Let $t > 0$ and $\varepsilon \in (0, 1)$ be given. Since T is a t-norm of Hadžić-type, then there exists $\mu > 0$ such that $T^k(1 - \mu) > 1 - \varepsilon$ for all $k \in \mathbb{N}$.

By (FM-6), we can find $s > 0$ such that

$$M(gx_0, gx_1, s) > 1 - \mu,\ M(gy_0, gy_1, s) > 1 - \mu.$$

Let $n_0 \in \mathbb{N}$ be such that $t > \sum_{i=n_0}^{\infty} k^i s$.

As in Step 1 in the proof of Theorem 2.3 it can be proved that

$$M(gx_n gx_{n+1}, k^n s) \geq Min\{M(gx_0, gx_1, s), M(gy_0, gy_1, s\} > 1 - \mu,$$

and

$$M(gy_n, gy_{n+1}, k^n s) \geq Min\{M(gy_0, gy_1, s), M(gx_0, gx_1, s)\} > 1 - \mu$$

for all $n \in \mathbb{N}$. Therefore, for all $n \geq n_0$ and all $m \in \mathbb{N}$ the following inequalities hold:

$$\begin{aligned} M(gx_n, gx_{n+m}, t) &\geq M\left(gx_n, gx_{n+m}, \sum_{i=n_1}^{\infty} k^i s\right) \geq M\left(gx_n, gx_{n+m}, \sum_{i=n}^{n+m-1} k^i s\right) \\ &\geq T_{i=n}^{n+m-1} M(gx_i, gx_{i+1}, k^i s) \geq T_{i=n}^{n+m-1}(1 - \mu) > 1 - \varepsilon. \end{aligned}$$

□

We conclude with an example to illustrate Theorem 2.3.

Example 2.5. Let $X = [-2, 2]$ and $M(x, y, t) = \left(\frac{t}{t+1}\right)^{|x-y|}$. It is easy to verify that (X, M, T_P) is a complete fuzzy metric space.

Let $F : X \times X \to X$, $F(x, y) = \frac{x^2}{16} + \frac{y^2}{16} - 2$ and $g : X \to X$, $g(x) = x$. Then $F(X \times X) = [-2, -\frac{3}{2}]$ and (2.1) is verified with $k = \frac{1}{2}$.

Indeed, since $\frac{t/2}{t/2+1} \geq \left(\frac{t}{t+1}\right)^2$ for all $t \geq 0$, then

$$\begin{aligned} M\left(F(x, y), F(u, v), \frac{t}{2}\right) &= \left(\frac{\frac{t}{2}}{\frac{t}{2} + 1}\right)^{\frac{|x^2-u^2+y^2-v^2|}{16}} \\ &\geq \left(\frac{t}{t+1}\right)^{\frac{|x^2-u^2+y^2-v^2|}{8}} \geq \left(\frac{t}{t+1}\right)^{\frac{|x-u|+|y-v|}{2}} \\ &\geq Min\left\{\left(\frac{t}{t+1}\right)^{|x-u|}, \left(\frac{t}{t+1}\right)^{|y-v|}\right\} \\ &= Min\{M(gx, gu, t), M(gy, gv, t)\} \quad (x, y \in X, t > 0). \end{aligned}$$

The point $x = 4(1 - \sqrt{2})$ belongs to X and it is the unique common fixed point of F and g.

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