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On the probabilistic stability of the monomial functional equation

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This paper is dedicated to the memory of Professor Viorel Radu

Communicated by Professor D. Miheţ

Abstract

Using the fixed point method, we establish a generalized Ulam - Hyers stability result for the monomial functional equation in the setting of complete random p-normed spaces. As a particular case, we obtain a new stability theorem for monomial functional equations in β-normed spaces.

Keywords: Random p-normed space; Hyers - Ulam - Rassias stability; monomial functional equation.
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1. Introduction

The problem of Ulam - Hyers stability for functional equations concerns deriving conditions under which, given an approximate solution of a functional equation, one may find an exact solution that is near it in some sense. The problem was first stated by Ulam [22] in 1940 for the case of group homomorphisms, and solved by Hyers [10] in the setting of Banach spaces. Hyers's result has since seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution ([2], [21], [5]) and in terms of the methods used for the proofs. Radu [20] noted that the fixed point alternative can be used successfully in the study of Ulam - Hyers stability, to obtain results regarding the existence and uniqueness of the exact solution as a fixed point of a suitably chosen contractive operator on a complete generalized metric space. The fixed point method was subsequently used to obtain stability results for other functional equations in various settings.

The notion of fuzzy stability for functional equations was introduced in the papers [16, 17]. The fixed point method was first used to study the probabilistic stability of functional equations in [12, 13, 14].

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Recently, in [15] and [23], the problem of stability was considered in the more general setting of random p-normed spaces. Following the same approach, we prove a stability result for the monomial functional equation, for mappings taking values in a complete random p-normed space. As random p-normed spaces generalize random normed spaces and β-normed spaces, this allows for a unitary framework in which to discuss several stability results.

Definition 1.1. Let X and Y be linear spaces. A mapping $f : X \to Y$ is called a monomial function of degree N if it is a solution of the monomial functional equation

$$\Delta_y^N f(x) - N! f(y) = 0, \ \forall x, y \in X. \tag{1.1}$$

Here, Δ denotes the difference operator, given by $\Delta_y f(x) = f(x + y) - f(x)$, for all $x, y \in X$, and its iterates are defined inductively by $\Delta_y^1 = \Delta_y$ and $\Delta_y^{n+1} = \Delta_y^1 \circ \Delta_y^n$, for all $n \geq 1$. It can easily be shown that

$$\Delta_y^N f(x) = \sum_{i=0}^{N} (-1)^{N-i} \binom{N}{N-i} f(x + iy).$$

Other well-known functional equations, such as the additive, quadratic or cubic ones, are particular cases of equation (1.1), obtained by setting $N = 1$, 2 or 3 respectively.

The (generalized) Ulam - Hyers stability for the monomial functional equation was previously studied in [1], [7], [8] and [3]. We also mention the recent papers [18] and [19].

We will assume that the reader is familiar with the notations and terminology specific to the theory of random normed spaces. We only recall the definition of a random p-normed space, as given in [9].

Definition 1.2. ([9]) Let $p \in (0, 1]$. A random p-normed space is a triple (X, μ, T) where X is a real vector space, T is a continuous t-norm, and μ is a mapping from X into D_+ so that the following conditions hold:

(P1) $\mu_x(t) = 1$ for all $t > 0$ iff $x = 0$;

(P2) $\mu_{\alpha x}(t) = \mu_x \left(\frac{t}{|\alpha|^p}\right)$, for all $x \in X$, $\alpha \neq 0$ and $t > 0$;

(P3) $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$, for all $x, y \in X$, $t, s \geq 0$.

If (X, μ, T) is a random p-normed space with T - a continuous t-norm such that $T \geq T_L$, then

$$\mathcal{V} = \{V(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1)\}, \ V(\varepsilon, \lambda) = \{x \in X : \mu_x(\varepsilon) > 1 - \lambda\}$$

is a complete system of neighborhoods of the null vector for a linear topology on X generated by the p-norm μ ([9]).

Definition 1.3. Let (X, μ, T) be a random p-normed space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $m, n \geq N$.

(iii) A random p-normed space (X, μ, T) is said to be complete iff every Cauchy sequence in X is convergent to a point in X.

2. Main results

In the following, N is a fixed positive integer.

Definition 2.1. Let X be a linear space, (Y, μ, T_M) be a random p-normed space and Φ be a mapping from X^2 to D_+. A mapping $f : X \to Y$ is said to be probabilistic Φ-approximately monomial of degree N if

$$\mu_{\Delta_y^N f(x) - N! f(y)}(t) \geq \Phi_{x,y}(t), \ \forall x, y \in X, t > 0. \tag{2.1}$$

We will prove that, under suitable conditions on the function Φ, every probabilistic Φ-approximately monomial mapping can be approximated, in a probabilistic sense, by a monomial mapping of the same degree. In doing so, we will need the following lemmas:

Lemma 2.2. *([11]) Let (X, d) be a complete generalized metric space and $A : X \to X$ be a strict contraction with the Lipschitz constant $L \in (0, 1)$, such that $d(x_0, A(x_0)) < +\infty$ for some $x_0 \in X$. Then A has a unique fixed point in the set $Y := \{y \in X, d(x_0, y) < \infty\}$ and the sequence $(A^n(x))_{n\in N}$ converges to the fixed point x^* for every $x \in Y$. Moreover, $d(x_0, A(x_0)) \le \delta$ implies $d(x^*, x_0) \le \frac{\delta}{1-L}$.*

Lemma 2.3. *([6])Let n, $\lambda \ge 2$ be integers,*

$$A = \begin{pmatrix} \alpha_0^{(0)} & \cdots & \alpha_0^{(\lambda n)} \\ \vdots & \ddots & \vdots \\ \alpha_{(\lambda-1)n}^{(0)} & \cdots & \alpha_{(\lambda-1)n}^{(\lambda n)} \end{pmatrix}$$

where for $i = 0, ..., (\lambda - 1)n$ and $k = -i, ..., \lambda n - i$

$$\alpha_i^{(i+k)} = \begin{cases} (-1)^k \binom{n}{n-k}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Let a_i denote the i^{th} row in A, $i = 0, ..., (\lambda - 1)n$, and $b = (\beta^{(0)} \cdots \beta^{(\lambda n)})$, where

$$\beta^{(k)} = \begin{cases} (-1)^{\frac{k}{\lambda}} \binom{n}{n-\frac{k}{\lambda}}, & \text{if } \lambda \mid k, \\ 0, & \text{if } \lambda \nmid k, \end{cases}$$

for $k = 0, ..., \lambda n$.

Then there exist positive integers $K_0, ..., K_{(\lambda-1)n}$ so that

$$K_0 + \cdots K_{(\lambda-1)n} = \lambda^n$$

and

$$K_0 a_0 + \cdots + K_{(\lambda-1)n} a_{(\lambda-1)n} = b.$$

Remark 2.4. In the case of $\lambda = 2$, $K_i = \binom{n}{n-i}$, for all $i = \overline{0, N}$ (see [6]).

Next, given linear spaces X and Y and a mapping $f : X \to Y$, using the notations of the previous lemma for $\lambda = 2$, one can write

$$\Delta_x^N f(ix) = (-1)^N \sum_{k=0}^{2N} \alpha_i^{(k)} f(kx),\ \forall i = \overline{0, N},$$

and

$$\Delta_{2x}^N f(0) = (-1)^N \sum_{k=0}^{2N} \beta^{(k)} f(kx).$$

By Lemma 2.3, $\sum\limits_{i=0}^{N} K_i \alpha_i^{(k)} = \beta^{(k)}$ for all $k = \overline{0, 2N}$, with $K_i = \binom{N}{N-i}$. Therefore we have shown that

$$\sum_{i=0}^{N} \binom{N}{N-i} \Delta_x^N f(ix) = \Delta_{2x}^N f(0),\ \forall x \in X. \tag{2.2}$$

Theorem 2.5. *Let X be a real linear space, (Y,μ,T_M) be a complete random p-normed space, and $\Phi : X^2 \to D_+$ be a mapping such that, for some $\alpha \in (0, 2^{Np})$, the following relations hold:*

$$\min_{i=\overline{0,N}}\{\Phi_{2ix,2x}(\alpha t), \Phi_{0,4x}(\alpha t)\} \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t), \Phi_{0,2x}(t)\},\ \forall x \in X, t > 0, \tag{2.3}$$

and

$$\lim_{n\to\infty} \Phi_{2^n x,2^n y}(2^{nNp}t) = 1,\ \forall x, y \in X, t > 0. \tag{2.4}$$

If $f : X \to Y$ is a probabilistic Φ-approximately monomial mapping of degree N with $f(0) = 0$, then there exists a unique monomial mapping of degree N, $M : X \to Y$, so that

$$\mu_{f(x)-M(x)}(t) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}\left(\frac{(N!)^p(2^{Np}-\alpha)}{1+\sum\limits_{i=0}^{N}\binom{N}{N-i}^p}t\right), \Phi_{0,2x}\left(\frac{(N!)^p(2^{Np}-\alpha)}{1+\sum\limits_{i=0}^{N}\binom{N}{N-i}^p}t\right)\},\quad \forall x \in X, t > 0. \tag{2.5}$$

In addition,

$$M(x) = \lim_{n\to\infty}\frac{f(2^n x)}{2^{nN}},\ \forall x \in X. \tag{2.6}$$

Proof. We will follow an idea of Gilányi (see [8]) to obtain an estimate of $\mu_{f(x)-\frac{f(2^n x)}{2^{nN}}}(t)$. For $i = \overline{0,N}$, substitute (x, y) with (ix, x) in (2.1) to get

$$\mu_{\Delta_x^N f(ix)-N!f(x)}(t) \geq \Phi_{ix,x}(t),\ \forall x \in X, t > 0, \tag{2.7}$$

which implies

$$\mu_{\binom{N}{N-i}\Delta_x^N f(ix)-\binom{N}{N-i}N!f(x)}\left(\binom{N}{N-i}^p t\right) \geq \Phi_{ix,x}(t),\ \forall x \in X, t > 0.$$

By using (P3), we obtain

$$\mu_{\sum\limits_{i=0}^{N}\binom{N}{N-i}\Delta_x^N f(ix)-2^N N!f(x)}\left(\sum_{i=0}^{N}\binom{N}{N-i}^p t\right) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t)\}, \forall x \in X, t > 0,$$

or equivalently, via (2.2),

$$\mu_{\Delta_{2x}^N f(0)-2^N N!f(x)}\left(\sum_{i=0}^{N}\binom{N}{N-i}^p t\right) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t)\},\ \forall x \in X, t > 0.$$

Also, by setting $i = 0$ and replacing x with $2x$ in (2.7), we get

$$\mu_{\Delta_{2x}^N f(0)-N!f(2x)}(t) \geq \Phi_{0,2x}(t),\ \forall x \in X, t > 0.$$

Consequently,

$$\mu_{N!2^N f(x)-N!f(2x)}\left(\left(1+\sum_{i=0}^{N}\binom{N}{N-i}^p\right)t\right) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t), \Phi_{0,2x}(t)\},\quad \forall x \in X, t > 0,$$

or

$$\mu_{f(x)-\frac{f(2x)}{2^N}}\left(\frac{1+\sum\limits_{i=0}^{N}\binom{N}{N-i}^p}{2^{Np}(N!)^p}t\right) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t),\Phi_{0,2x}(t)\},\ \forall x\in X, t>0. \tag{2.8}$$

Now, let $G(x,t) := \min\limits_{i=\overline{0,N}}\{\Phi_{ix,x}(t),\Phi_{0,2x}(t)\}$. Note that, by (2.3), G has the property $G(2x,\alpha t)\geq G(x,t)$, for all $x\in X$ and all $t>0$. We denote by E the space of all mappings $g: X\to Y$ with $g(0)=0$, and define the mapping $d_G: E\times E\to[0,\infty]$ as

$$d_G(g,h)=\inf\{a\in\mathbb{R}:\mu_{g(x)-h(x)}(at)\geq G(x,t),\ \forall x\in X, t>0\}.$$

Following the same reasoning as in [13], it can be shown that (E,d_G) is a complete generalized metric space.

We claim that $J: E\to E$, $Jg(x)=\frac{g(2x)}{2^N}$, is a strict contraction, with the Lipschitz constant $\frac{\alpha}{2^{Np}}$. Indeed, let $g,h\in E$ be so that $d_G(g,h)<\varepsilon$. This implies

$$\mu_{g(x)-h(x)}(\varepsilon t)\geq G(x,t),\ \forall x\in X, t>0.$$

Then

$$\mu_{Jg(x)-Jh(x)}\left(\frac{\alpha}{2^{Np}}\varepsilon t\right)=\mu_{g(2x)-h(2x)}(\alpha\varepsilon t)\geq G(2x,\alpha t)\geq G(x,t), \forall x\in X, t>0,$$

so $d_G(Jg,Jh)\leq\frac{\alpha}{2^{Np}}\varepsilon$. Therefore $d_G(Jg,Jh)\leq\frac{\alpha}{2^{Np}}d_G(g,h)$, and our claim is proved. Moreover, from (2.8),

$$d_G(f,Jf)\leq\frac{1+\sum\limits_{i=0}^{N}\binom{N}{N-i}^p}{2^{Np}(N!)^p}.$$

By Lemma 2.2, J has a fixed point $M: X\to Y$ with the following properties:

(i) $d_G(J^nf,M)\to 0$ when $n\to\infty$, so $\lim\limits_{n\to\infty}\frac{f(2^nx)}{2^{nN}}=M(x)$, for all $x\in X$.

(ii) $d_G(f,M)\leq\frac{1}{1-\frac{\alpha}{2^N p}}d_G(f,Jf)$, so the estimation (2.5) holds.

(iii) M is the unique fixed point of J in the set $\{g\in E: d_G(f,g)<\infty\}$.

Finally, we must show that M is a monomial mapping of degree N. Substituting x and y by 2^nx and 2^ny in (2.1), we obtain

$$\mu_{\Delta^N_{2^ny}f(2^nx)-N!f(2^ny)}(t)\geq\Phi_{2^nx,2^ny}(t)$$

or

$$\mu_{\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{N}{N-i}f(2^n(x+iy))-N!f(2^ny)}(t)\geq\Phi_{2^nx,2^ny}(t)$$

for all $x\in X$ and all $t>0$, so

$$\mu_{\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{N}{N-i}\frac{f(2^n(x+iy))}{2^{nN}}-N!\frac{f(2^ny)}{2^{nN}}}(t)\geq\Phi_{2^nx,2^ny}(2^{nNp}t),\ \forall x\in X, t>0. \tag{2.9}$$

Now,

$$\begin{aligned}\mu_{\Delta^N_yM(x)-N!M(y)}(t)&=\mu_{\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{N}{N-i}M(x+iy)-N!M(y)}(t)\\&\geq\min\{\mu_{\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{N}{N-i}(M(x+iy)-\frac{f(2^n(x+iy))}{2^{nN}})-N!(M(y)-\frac{f(2^ny)}{2^{nN}})}\left(\frac{t}{2}\right),\\&\qquad\mu_{\sum\limits_{i=0}^{N}(-1)^{N-i}\binom{N}{N-i}\frac{f(2^n(x+iy))}{2^{nN}}-N!\frac{f(2^ny)}{2^{nN}}}\left(\frac{t}{2}\right)\},\ \forall x\in X, t>0.\end{aligned}$$

Both expressions on the right hand side of the inequality above tend to 1 as n tends to infinity, the latter due to (2.4) and (2.9). Thus, we have shown that $\Delta^N_yM(x)-N!M(y)=0$, which concludes the proof. □

Similarly, one can obtain the following result for $\alpha > 2^{Np}$.

Theorem 2.6. *Let X be a real linear space, (Y, μ, T_M) be a complete random p-normed space, and $\Phi : X^2 \to D_+$ be a mapping such that, for some $\alpha > 2^{Np}$,*

$$\min_{i=\overline{0,N}}\{\Phi_{ix,x}(t), \Phi_{0,2x}(t)\} \geq \min_{i=\overline{0,N}}\{\Phi_{2ix,2x}(\alpha t), \Phi_{0,4x}(\alpha t)\},\ \forall x \in X, t > 0, \tag{2.10}$$

and

$$\lim_{n\to\infty} \Phi_{2^{-n}x,2^{-n}y}\left(\frac{t}{2^{nNp}}\right) = 1,\ \forall x, y \in X, t > 0. \tag{2.11}$$

If $f : X \to Y$ is a probabilistic Φ-approximately monomial mapping of degree N with $f(0) = 0$, then there exists a unique monomial mapping of degree N, $M : X \to Y$, so that

$$\mu_{f(x)-M(x)}(t) \geq \min_{i=\overline{0,N}}\{\Phi_{\frac{ix}{2},\frac{x}{2}}\left(\frac{(N!)^p(\alpha - 2^{Np})}{\alpha(1 + \sum_{i=0}^{N}\binom{N}{N-i}^p)}t\right), \Phi_{0,x}\left(\frac{(N!)^p(\alpha - 2^{Np})}{\alpha(1 + \sum_{i=0}^{N}\binom{N}{N-i}^p)}t\right)\},$$
$$\forall x \in X, t > 0. \tag{2.12}$$

Moreover, $M(x) = \lim_{n\to\infty} 2^{nN} f\left(\frac{x}{2^n}\right)$, for all $x \in X$.

Proof. Relation (2.8) implies

$$\mu_{2^N f(\frac{x}{2})-f(x)}\left(\frac{1 + \sum_{i=0}^{N}\binom{N}{N-i}^p}{(N!)^p}t\right) \geq \min_{i=\overline{0,N}}\{\Phi_{\frac{ix}{2},\frac{x}{2}}(t), \Phi_{0,x}(t)\},\ \forall x \in X, t > 0.$$

Set $G(x, t) := \min_{i=\overline{0,N}}\{\Phi_{\frac{ix}{2},\frac{x}{2}}(t), \Phi_{0,x}(t)\}$ and note that, by (2.10), it has the property $G\left(\frac{x}{2}, \frac{t}{\alpha}\right) \geq G(x, t)$. We define

$$d_G(g, h) = \inf\{a \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(at) \geq G(x, t),\ \forall x \in X, t > 0\}$$

on the space $E = \{g : X \to Y : g(0) = 0\}$ and note that (E, d_G) is a complete generalized metric space.

As in the proof of Theorem 2.5, we can show that $J : E \to E$, $Jg(x) = 2^N g\left(\frac{x}{2}\right)$, is a strict contraction, with the Lipschitz constant $\frac{2^{Np}}{\alpha}$, and its only fixed point $M : X \to Y$ so that $d_G(f, M) < \infty$ is the unique monomial mapping with the required properties. □

Remark 2.7. Note that, by (2.8),

$$\min_{i=\overline{0,N}}\{\Phi_{\frac{ix}{2},\frac{x}{2}}\left(\frac{(N!)^p(\alpha - 2^{Np})}{\alpha(1 + \sum_{i=0}^{N}\binom{N}{N-i}^p)}t\right), \Phi_{0,x}\left(\frac{(N!)^p(\alpha - 2^{Np})}{\alpha(1 + \sum_{i=0}^{N}\binom{N}{N-i}^p)}t\right)\} \geq$$
$$\geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}\left(\frac{(N!)^p(\alpha - 2^{Np})}{1 + \sum_{i=0}^{N}\binom{N}{N-i}^p}t\right), \Phi_{0,2x}\left(\frac{(N!)^p(\alpha - 2^{Np})}{1 + \sum_{i=0}^{N}\binom{N}{N-i}^p}t\right)\},$$

so that the estimation (2.12) is comparable to that in Theorem 2.5.

Remark 2.8. Instead of the hypothesis (2.3) + (2.4), one can consider a condition with a simpler formulation, namely

$$\Phi_{2x,2y}(\alpha t) \geq \Phi_{x,y}(t),\ \forall x, y \in X, t > 0. \tag{2.13}$$

However, we note that the two are not equivalent. It is immediate that (2.13) implies (2.3) and (2.4). The following example shows that the converse does not hold:

Example 2.9. Let $(X, \|\cdot\|)$ be a normed space. The mapping $\Phi : X \times X \to D_+$ defined by

$$\Phi_{x,y}(t) = \begin{cases} 1, & \text{if there exists } a \in \mathbb{R} \text{ so that } y = ax, \\ \frac{t}{t+\|x-y\|^{Np+1}}, & \text{otherwise,} \end{cases}$$

satisfies the conditions (2.3) and (2.4), but, for all linearly independent $x, y \in X$, $\Phi_{2x,2y}(\alpha t) < \Phi_{x,y}(t)$.

Similarly, the condition

$$\Phi_{2x,2y}(\alpha t) \leq \Phi_{x,y}(t),\ \forall x, y \in X, t > 0$$

can be considered instead of the hypothesis (2.10) + (2.11) in Theorem 2.6.

3. Applications

As consequences of Theorem 2.5, we will obtain generalized Ulam - Hyers stability results for the case of random normed spaces and β-normed spaces and compare them with those already existing in the literature. Results regarding the case $\alpha > 2^{Np}$ can be derived in an identical manner from Theorem 2.6.

In the setting of random normed spaces, our theorem reads as follows:

Theorem 3.1. *(compare with [4, Theorem 4.1]) Let X be a real linear space and (Y, μ, T_M) be a complete random normed space. Suppose that the mapping $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\mu_{\Delta_y^N f(x)-N!f(y)}(t) \geq \Phi_{x,y}(t),\ \forall x, y \in X, t > 0,$$

where $\Phi : X^2 \to D_+$ is a given function. If there exists $\alpha \in (0, 2^N)$ such that

$$\min_{i=\overline{0,N}}\{\Phi_{2ix,2x}(\alpha t), \Phi_{0,4x}(\alpha t)\} \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}(t), \Phi_{0,2x}(t)\},\ \forall x \in X, t > 0$$

and

$$\lim_{n\to\infty} \Phi_{2^n x,2^n y}(2^{nN}t) = 1,\ \forall x \in X, t > 0,$$

then there exists a unique monomial mapping of degree N, $M : X \to Y$, which satisfies the inequality

$$\mu_{f(x)-M(x)}(t) \geq \min_{i=\overline{0,N}}\{\Phi_{ix,x}\left(\frac{N!(2^N-\alpha)}{2^N+1}t\right), \Phi_{0,2x}\left(\frac{N!(2^N-\alpha)}{2^N+1}t\right)\},$$
$$\forall x \in X, t > 0.$$

Proof. Set $p = 1$ in Theorem 2.5. □

In view of Remark 2.8, our two hypotheses on Φ could have been replaced with $\Phi_{2x,2y}(\alpha t) \geq \Phi_{x,y}(t)$, which is the condition that appears in [4].

Recall that a β*-normed space* $(0 < \beta \leq 1)$ is a pair $(Y, \|\cdot\|_\beta)$, where Y is a real linear space and $\|\cdot\|_\beta$ is a real valued function on Y (called a β-norm) satisfying the following conditions:

(i) $\|x\|_\beta \geq 0$ for all $x \in Y$ and $\|x\|_\beta = 0$ if and only if $x = 0$;
(ii) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $x \in Y$ and $\lambda \in \mathbb{R}$;
(iii) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in Y$.

In [3], Cădariu and Radu used the fixed point method to obtain the following generalized Ulam - Hyers stability result for the monomial functional equation in β-normed spaces:

Theorem 3.2. *([3, Theorem 2.1]) Let X be a linear space, Y be a complete β-normed space, and assume we are given a function $\varphi : X \times X \to [0, \infty)$ with the following property:*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^{nN\beta}} = 0, \ \forall x, y \in X. \tag{3.1}$$

Suppose that the mapping $f : X \to Y$ with $f(0) = 0$ verifies the control condition

$$\|\Delta_y^N f(x) - N!f(y)\|_\beta \le \varphi(x, y), \ \forall x, y \in X. \tag{3.2}$$

If there exists a positive constant $L < 1$ such that the mapping

$$x \mapsto \psi(x) = \frac{1}{(N!)^\beta} \left(\varphi(0, x) + \sum_{i=0}^{N} \binom{N}{N-i} \varphi\left(\frac{ix}{2}, \frac{x}{2}\right)\right), \ \forall x \in X,$$

satisfies the inequality

$$\psi(2x) \le 2^{N\beta} L\psi(x), \ \forall x \in X, \tag{3.3}$$

then there exists a unique monomial mapping of degree N, $M : X \to Y$, with the following property:

$$\|f(x) - M(x)\|_\beta \le \frac{L}{1-L}\psi(x), \ \forall x \in X. \tag{3.4}$$

By noting that every β-normed space $(Y, \|\cdot\|_\beta)$ induces a random p-normed space (Y, μ, T_M) with $\beta = p$ and $\mu_x(t) = \frac{t}{t+\|x\|_\beta}$, from Theorem 2.5 we obtain the following new stability result.

Theorem 3.3. *Let X be a real linear space, $(Y, \|\cdot\|_\beta)$ be a complete β-normed space, and $\varphi : X^2 \to [0, \infty)$ be a mapping so that (3.1) holds and, for some $\alpha \in (0, 2^{N\beta})$,*

$$\max_{i=\overline{0,N}} \{\varphi(2ix, 2x), \varphi(0, 4x)\} \le \alpha \max_{i=\overline{0,N}} \{\varphi(ix, x), \varphi(0, 2x)\}, \ \forall x \in X. \tag{3.5}$$

Suppose that $f : X \to Y$ with $f(0) = 0$ verifies the control condition (3.2). Then there exists a unique monomial mapping of degree N, $M : X \to Y$, with the following property:

$$\|f(x) - M(x)\|_\beta \le \frac{1 + \sum\limits_{i=0}^{N} \binom{N}{N-i}^\beta}{(N!)^\beta (2^{N\beta} - \alpha)} \max_{i=\overline{0,N}} \{\varphi(ix, x), \varphi(0, 2x)\}, \ \forall x \in X. \tag{3.6}$$

Proof. Consider the induced random p-normed space (Y, μ, T_M) and apply Theorem 2.5 with $\Phi_{x,y}(t) = \frac{t}{t+\varphi(x,y)}$. □

Remark 3.4. Theorem 3.3 provides an alternative version for the stability result obtained in [18] in the particular case of quasi-p-normed spaces.

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