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Coupled coincidence point theorems for nonlinear contractions under (F, g)-invariant set in cone metric spaces

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Abstract

We extend the recent results of coupled coincidence point theorems of Shatanawi et. al. (2012) by weakening the concept of mixed g-monotone property. We also give an example of a nonlinear contraction mapping, which is not applied to the existence of coupled coincidence point by the results of Shatanawi et. al. but can be applied to our results. The main results extend and unify the results of Shatanawi et. al. and many results of the coupled fixed point theorems of Sintunavarat et. al. (2012).

Keywords: Coincidence point, Cone metric space, C-distance, Fixed point, (F,g)-invariant set. 2010 MSC: Primary 47H10; Secondary 54H25, 55M20.

1. Introduction

Since Banach's fixed point theorem in 1922, because of its simplicity and usefulness, it has become a very important tool in solving the existence problems in many branches of non-linear analysis. Ran and Reurings [12] extended the Banach contraction principle to metric spaces endowed with a partial ordering and they gave application of their results to matrix equations. In [11] Nieto and López extended the result of Ran and Reurings [12] for non-decreasing mappings and applied their results to get a unique solution for a first order differential equation.

The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in

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the Banach spaces. The concept of cone metric space was introduced in the work of Huang and Zhang [5] where they also established the Banach contraction mapping principle in this space. Then, several authors have studied fixed point problems in cone metric spaces. For some of the work on cone metric spaces, one may refer to ([1, 3, 5, 6, 17]).

Bhaskar and Lakshmikantham [2] introduced the notion of a coupled fixed point of a mapping F from $X \times X$ into X. They established some coupled fixed point results and applied their results to the study of existence and uniqueness of solution for a periodic boundary value problem. Lakshmikantham and Ćirić [9] introduced the concept of coupled coincidence points and proved coupled coincidence and coupled common fixed point results for mappings F from $X \times X$ into X and g from X into X satisfying nonlinear contraction in ordered metric space. For more study on coupled fixed point theory see ([1, 4, 8, 9, 10, 13, 14, 16]).

Recently Cho et. al. [3] introduced a new concept of c-distance in cone metric spaces which is a cone version of w-distance of Kada et. al. In [16] Sintunavarat et. al. established coupled fixed point theorems for weak contraction mappings by using the concept of F-invariant set and c-distance in partially ordered cone metric spaces. Further, In [15] Shatanawi et. al. established coupled coincidence point theorems for nonlinear contractions by using the concept of mixed g-monotone property and c-distance in partially ordered cone metric spaces. In this paper we introduce the concept of an (F,g)-invariant set and extend the results of Shatanawi et. al. [16] as we establish the existence of coupled coincidence point for mappings $F : X \times X \to X$ and $g : X \to X$ satisfying nonlinear contraction under c-distance in cone metric spaces having an (F,g)-invariant subset.

Throughout this paper (X, \sqsubseteq) denotes a partially ordered set with partial order \sqsubseteq .

Definition 1.1. [2] A mapping $F: X \times X \to X$ is said to have mixed monotone property if for any $x, y \in X$

$$\begin{array}{cccc} x_1, x_2 \in X, \ x_1 \ \sqsubseteq \ x_2 \implies F(x_1, y) \ \sqsubseteq \ F(x_2, y), \\ y_1, y_2 \in X, \ y_1 \ \sqsubseteq \ y_2 \implies F(x, y_1) \ \sqsupseteq \ F(x, y_2). \end{array}$$

Definition 1.2. [9] A mapping $F : X \times X \to X$ is said to have mixed g-monotone property if for any $x, y \in X$

Definition 1.3. [2] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mappings $F : X \times X \to X$ if F(x, y) = x and F(y, x) = y.

Definition 1.4. [9] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $g: X \to X$ if F(x, y) = gx and F(y, x) = gy.

Definition 1.5. [9] Let $F: X \times X \to X$ and $g: X \to X$. The mappings F and g are said to commute if gF(x,y) = F(gx,gy) for all $x, y \in X$.

In [5], cone metric space was introduced in the following manner:

Let $(E, \|.\|)$ be a real Banach space and θ denote the zero element in E. Assume that P is a subset of E. Then P is called a cone if and only if:

- 1. P is non empty, closed and $P \neq \{\theta\}$,
- 2. If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$.
- 3. $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subseteq E$ and $x, y \in E$, the partial ordering \preceq on E with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of \prec stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in intP$. It can be easily shown that $\lambda .intP \subseteq intP$ for all $\lambda > 0$. and $intP + intP \subseteq intP$. A cone P is called normal if there is a number K > 0 such that for all $x, y \in E, \theta \preceq x \preceq y$ implies $||x|| \leq K ||y||$. The least positive number K satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space, P is a cone in E with $intP \neq \phi$. and \leq is partial ordering with respect to P.

Definition 1.6. [5] Let X be a non empty set and E be a real Banach space equipped with the partial ordering \leq with respect to the cone P. Suppose that the mapping $d: X \times X \to E$ satisfies the following condition:

- (i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta \Leftrightarrow x = y$
- (ii) d(x,y) = d(y,x) for all $x, y \in X$
- (iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.7. [5] Let (X, d) be a cone metric space $\{x_n\}$ be a sequence in X and $x \in X$.

- 1. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all n > N then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.
- 2. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all n, m > N then $\{x_n\}$ is called a Cauchy sequence in X.
- 3. A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 1.8. [5] Let (X, d) be a cone metric space, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then,

- 1. the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $||d(x_n, x)|| \to 0$),
- 2. the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $||d(x_n, x_m)|| \to 0$).
- 3. the sequence $\{x_n\}$ (respectively, $\{y_n\}$) converges to x (respectively, y) then $d(x_n, y_n) \to d(x, y)$.

Lemma 1.9. [17] Every cone metric space (X, d) is a topological space. For $c \gg 0$, $c \in E$, $x \in X$ let $B(x,c) = \{y \in X : d(y,x) \ll c\}$ and $\beta = \{B(x,c) : x \in X, c \gg 0\}$. Then $\tau = \{U \subseteq X : \forall x \in U \exists B_x \in \beta with x \in B_x \subseteq U\}$ is a topology on X.

Definition 1.10. [17] Let (X, d) be a cone metric space. A map $T : (X, d) \to (X, d)$ is called sequentially continuous if $x_n \in X, x_n \to x$ implies $Tx_n \to Tx$.

Lemma 1.11. [17] Let (X,d) be a cone metric space, and $T : (X,d) \to (X,d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.

Let (X, d) be a cone metric space and $X^2 = X \times X$. Define a function $\rho : X^2 \times X^2 \to E$ by $\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)$ for all (x_1, y_1) and $(x_2, y_2) \in X^2$. Then (X^2, ρ) is a cone metric space [8].

Lemma 1.12. [8] Let $z_n = (x_n, y_n) \in X^2$ be a sequence and $z = (x, y) \in X^2$. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

Next we give the notation of c-distance on a cone metric space which is generalization of w-distance of Kada et. al. [7] with some properties.

Definition 1.13. [3] Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a *c*-distance on X if the following conditions hold:

- (q1) $\theta \leq q(x, y)$ for all $x, y \in X$,
- (q2) $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x, y, z \in X$,
- (q3) For each $x \in X$ and $n \in \mathbb{N}$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
- (q4) For all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 1.14. The c-distance q is a w-distance on X if we let (X, d) be a metric space, $E = \mathbb{R}$, $P = [0, \infty)$, and q3 is replaced by the following condition: for any $x \in X, q(x, .) : X \to \mathbb{R}$ is lower semicontinuous. Moreover, q3 holds whenever q(x, .) is lower semi-continuous. Thus, if (X, d) is a metric space, $E = \mathbb{R}$, and $P = [0, \infty)$, then every w-distance is a c-distance. But the converse is not true in the general case. Therefore, the c-distance is a generalization of the w-distance.

Example 1.15. [16] Let $E = \mathbb{R}$ and $P = \{x \in E : x \ge 0\}$. Let $X = [0, \infty)$ and define a mapping $d: X \times X \to E$ by d(x, y) = |x - y| for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q: X \times X \to E$ by q(x, y) = y for all $x, y \in X$. Then q is a c-distance on X.

Example 1.16. [16] Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q: X \times X \to P$ by q(x, y) = d(x, y) for all $x, y \in X$. Then, q is c-distance.

Example 1.17. [16] Let $E = C^1_{\mathbb{R}}[0,1]$ with $||x||_1 = ||x||_{\infty} + ||x'||_{\infty}$ and $P = \{x \in E : x(t) \ge 0, t \in [0,1]\}$. Let $X = [0, +\infty)$ (with usual order) and $d(x, y)(t) = |x - y|\varphi(t)$ where $\varphi : [0,1] \to \mathbb{R}$ is given by $\varphi(t) = e^t$ for all $t \in [0,1]$. Then (X,d) is an ordered cone metric space (see [3] Example 2.9). This cone is not normal. Define a mapping $q : X \times X \to P$ by $q(x, y) = (x + y)\varphi$ for all $x, y \in X$. Then q is a c-distance.

Example 1.18. [16] Let (X, d) be a cone metric space and P a normal cone. Define a mapping $q: X \times X \to P$ by q(x, y) = d(u, y) for all $x, y \in X$, where u is a fixed point in X. Then q is a c-distance.

Lemma 1.19. [3] Let (X, d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $y, z \in X$. Suppose that u_n is a sequence in P converging to θ . Then the following hold:

- 1. If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then y = z.
- 2. If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then y_n converges to z.
- 3. If $q(x_n, x_m) \preceq u_n$ for m > n, then $\{x_n\}$ is a Cauchy sequence in X.
- 4. If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

Remark 1.20. [3]

- 1. q(x,y) = q(y,x) may not be true for all $x, y \in X$.
- 2. $q(x,y) = \theta$ is not necessarily equivalent to x = y for all $x, y \in X$.

Samet et. al. in [14] introduced an F-invariant set.

Definition 1.21. [14] Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a non empty subset of X^4 . We say that M is an F-invariant subset of X^4 if and only if for all $x, y, z, w \in X$ we have

- (a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and
- (b) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M.$

We observe that the set $M = X^4$ is trivially *F*-invariant.

2. Main Results

We begin with the introduction of an (F, g)-invariant set which is a generalization of an F-invariant set introduced by Samet et. al. in [14].

Definition 2.1. Let (X, d) be a metric space and $F : X \times X \to X$, $g : X \to X$ be given mappings. Let M be a non empty subset of X^4 . We say that M is an (F, g)-invariant subset of X^4 if and only if for all $x, y, z, w \in X$ we have

- (a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$ and
- (b) $(gx, gy, gz, gw) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M.$

We observe that

- 1. The set $M = X^4$ is trivially (F, g)-invariant.
- 2. Every F-invariant set is (F, I_X) -invariant. Here I_X denotes identity map on X.

Following example shows that we may have (F, g)-invariant set which is not F-invariant.

Example 2.2. Let $X = \mathbb{R}$ and $F : X \times X \to X$ be defined by $F(x, y) = 1 - x^2$. Let $g : X \to X$ be given by gx = 1 + x. Then $M = \{(x, y, z, w) \in X^4 : y = z = 0\}$ is not *F*-invariant as $(1, 0, 0, 1) \in M$ but (F(1, 0), F(0, 1), F(0, 1), F(1, 0)) = (0, 1, 1, 0) does not belong to M. It is easy to see that M is (F, g)-invariant.

Example 2.3. Let (X, d) be a cone metric space endowed with a partial order \sqsubseteq . Let $F : X \times X \to X$ and $g : X \to X$ be any two mappings such that F satisfies mixed g-monotone property. Define a subset M of X^4 by $M = \{(a, b, c, d) : c \sqsubseteq a, b \sqsubseteq d\}$. Then M is (F, g)-invariant.

Theorem 2.4. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let q be a c-distance on X and M be an (F, g)-invariant subset of X^4 . Let

$$q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \leq k(q(gx,gu) + q(gy,gv))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx.gy) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. Choose $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$. Since $F(X \times X) \subseteq g(X)$, one can find $x_1, y_1 \in X$ in a way that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Repeating the same argument one can find $x_2, y_2 \in X$ in a way that $gx_2 = F(x_1, y_1)$ and $F(y_1, x_1) = gy_2$. Continuing this process one can construct sequences $\{x_n\}$ and $\{y_n\}$ in X that satisfy $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$ for all $n \ge 0$. It is asserted that

$$(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M \text{ for all } n \ge 0.$$
 (2.1)

For n = 0, (2.1) follows by the choice of x_0 and y_0 . Let us assume that (2.1) holds good for $n = k, k \ge 0$. Then we have $(gx_{k+1}, gy_{k+1}, gx_k, gy_k) \in M$. (F, g)-invariance of M now implies that

$$(F(x_{k+1}, y_{k+1}), F(y_{k+1}, x_{k+1}), F(x_k, y_k), F(y_k, x_k)) \in M$$

That is, $(gx_{k+2}, gy_{k+2}, gx_{k+1}, gy_{k+1}) \in M$. Thus (2.1) follows for k + 1. Hence, by induction, our assertion follows. Now for all $n \in \mathbb{N}$

$$q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1}) = q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n))$$

$$\leq k(q(gx_{n-1}, gx_n) + q(gy_{n-1}, gy_n))$$

Put $q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$. Then, we have

$$q_n = q(gx_n, gx_{n+1}) + q(gy_n, gy_{n+1})$$
$$\leq k q_{n-1}$$
$$\vdots$$
$$\leq k^n q_0$$

Let $m > n \ge 1$. It follows that

$$q(gx_n, gx_m) \leq q(gx_n, gx_{n+1}) + q(gx_{n+1}, gx_{n+2}) + \dots + q(gx_{m-1}, gx_m) \quad and q(gy_n, gy_m) \leq q(gy_n, gy_{n+1}) + q(gy_{n+1}, gy_{n+2}) + \dots + q(gy_{m-1}, gy_m).$$

Then we have

$$q(gx_n, gx_m) + q(gy_n, gy_m) \leq q_n + q_{n+1} + \dots + q_{m-1}$$

$$\leq k^n q_0 + k^{n+1} q_0 + \dots + k^{m-1} q_0$$

$$\leq \frac{k^n}{1-k} q_0$$
(2.2)

From (2.2) we have

$$q(gx_n, gx_m) \preceq \frac{k^n}{1-k} q_0 \tag{2.3}$$

and also

$$q(gy_n, gy_m) \preceq \frac{k^n}{1-k} q_0 \tag{2.4}$$

Thus, Lemma 1.19(3) shows that gx_n and gy_n are Cauchy sequences in X. Since X is complete, there exists there exists $x^*, y^* \in X$ such that $gx_n \to x^*$ and $gy_n \to y^*$ as $n \to \infty$. By continuity of g we get

$$\lim_{n \to \infty} ggx_n = gx^* \text{ and } \lim_{n \to \infty} ggy_n = gy$$

Commutativity of F and g now implies that

$$ggx_n = g(F(x_{n-1}, y_{n-1})) = F(gx_{n-1}, gy_{n-1}) \text{ for all } n \in \mathbb{N}$$

and $ggy_n = gF(y_{n-1}, x_{n-1}) = F(gy_{n-1}, gx_{n-1}) \text{ for all } n \in \mathbb{N}.$

Since F is continuous, therefore,

$$gx^* = \lim_{n \to \infty} ggx_n$$

=
$$\lim_{n \to \infty} F(gx_{n-1}, gy_{n-1})$$

=
$$F(\lim_{n \to \infty} gx_{n-1}, \lim_{n \to \infty} gy_{n-1})$$

=
$$F(x^*, y^*)$$

and
$$gy^* = \lim_{n \to \infty} ggy_n$$

 $= \lim_{n \to \infty} F(gy_{n-1}, gx_{n-1})$
 $= F(\lim_{n \to \infty} gy_{n-1}, \lim_{n \to \infty} gx_{n-1})$
 $= F(y^*, x^*)$

Thus (x^*, y^*) is a coupled coincidence point of F and g.

Corollary 2.5. [15] Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let F satisfy mixed g-monotone property and

$$q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \leq k(q(gx,gu) + q(gy,gv))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. Take M as in Example 2.3.

Corollary 2.6. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous and commuting functions with $F(X \times X) \subseteq g(X)$. Let q be a c-distance on X and M be an (F, g)-invariant subset of X^4 . Let

$$q(F(x,y),F(u,v)) \preceq aq(gx,gu) + bq(gy,gv))$$

$$(2.5)$$

for some $a, b \in [0, 1)$ with a + b < 1 and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx.gy) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. Given $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx. gy) \in M$. So by (2.5) we have

$$q(F(x,y),F(u,v)) \preceq aq(gx,gu) + bq(gy,gv))$$

and

$$q(F(y,x),F(v,u)) \preceq aq(gy,gv) + bq(gx,gu)$$

Thus $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \leq (a+b)(q(gx, gu) + q(gy, gv))$ where a+b < 1. Result follows by Theorem 2.4.

Corollary 2.7. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ is a continuous functions, M is an F-invariant subset of X^4 and

$$q(F(x,y),F(u,v)) \preceq aq(x,u) + bq(y,v))$$

for some $a, b \in [0,1)$ with a + b < 1 and all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*) .

Proof. Take $g = I_X$, the identity map on X in Corollary 2.6.

Corollary 2.8. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X. Suppose $F : X \times X \to X$ is a continuous functions, M is an F-invariant subset of X^4 and

$$q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \leq k(q(x,u) + q(y,v))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*)

Proof. Take $g = I_X$ in Theorem 2.4.

The continuity of F in Theorem 2.4 can be dropped. For this, we refer to the following useful lemma which is a variant of Lemma 1.19(1).

Lemma 2.9. [15] Let (X, d) be a cone metric space and q be a c-distance on X. Let (x_n) be a sequence in X. Suppose that (α_n) and (β_n) are sequences in P converging to θ . If $q(x_n, y) \preceq \alpha_n$ and $q(x_n, z) \preceq \beta_n$, then y = z.

Theorem 2.10. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let $F: X \times X \to X$ and $g: X \to X$ be given functions with $F(X \times X) \subseteq g(X)$ and (g(X), d) is a complete subspace of X. Let q be a c-distance on X and M be an (F, g)-invariant subset of X^4 . Let

$$q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \leq k(q(gx,gu) + q(gy,gv))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx.gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. Consider Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ as in the proof of Theorem 2.4. Since (g(X), d) is complete, there exists $x^*, y^* \in X$ such that $gx_n \to gx^*$ and $gy_n \to gy^*$. By q3,(2.3) and (2.4) we have

$$q(gx_n, gx^*) \preceq \frac{k^n}{1-k} q_0 \quad \text{for all } n \ge 0$$
(2.6)

and

$$q(gy_n, gy^*) \preceq \frac{k^n}{1-k} q_0 \quad \text{for all } n \ge 0$$
(2.7)

Adding (2.6) and (2.7) we get

$$q(gx_n, gx^*) + q(gy_n, gy^*) \preceq \frac{2k^n}{1-k} q_0 \quad \text{for all } n \ge 0$$

Since $gx_n \to gx^*, gy_n \to gy^*$ and $(gx_{n+1}, gy_{n+1}, gx_n, gy_n) \in M$ for all $n \ge 0$, therefore, $(gx^*, gy^*, gx_n, gy_n) \in M$ for all $n \ge 0$. Thus for all $n \in \mathbb{N}$

$$q(gx_n, F(x^*, y^*)) + q(gy_n, F(y^*, x^*)) = q(F(x_{n-1}, y_{n-1}), F(x^*, y^*)) + q(F(y_{n-1}, x_{n-1}), F(y^*, x^*))$$

$$\preceq k[q(gx_{n-1}, gx^*) + q(gy_{n-1}, gy^*)]$$

$$\preceq k \frac{k^{n-1}}{1-k} q_0 + \frac{k^{n-1}}{1-k} q_0$$

$$= \frac{2k^n}{1-k} q_0$$

This implies that

$$q(gx_n, F(x^*, y^*)) \preceq \frac{2k^n}{1-k}q_0$$
 (2.8)

and

$$q(gy_n, F(y^*, x^*)) \preceq \frac{2k^n}{1-k}q_0$$
(2.9)

By Lemma 2.9, (2.6) and (2.8) we have $F(x^*, y^*) = gx^*$. Similarly, by Lemma 2.9, (2.7) and (2.9) we have $F(y^*, x^*) = gy^*$. Thus (x^*, y^*) is a coupled coincidence point of F and g.

Corollary 2.11. [15] Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let $F : X \times X \to X$ and $g : X \to X$ be given functions with $F(X \times X) \subseteq g(X)$ and (g(X), d) is a complete subspace of X. Let q be a c-distance on X. Let F satisfy mixed g-monotone property and

$$q(F(x,y),F(u,v)) + q(F(y,x),F(v,u)) \leq k(q(gx,gu) + q(gy,gv))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(gx \sqsubseteq gu)$ and $(gy \sqsupseteq gv)$ or $(gx \sqsupseteq gu)$ and $(gy \sqsubseteq gv)$. Suppose X has the following property:

(i) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \sqsubseteq x$ for all n.

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y \sqsubseteq y_n$ for all n.

If there exist $x_0, y_0 \in X$ satisfying $gx_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq gy_0$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. Take M as in Example 2.3.

Corollary 2.12. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let $F: X \times X \to X$ and $g: X \to X$ be given functions with $F(X \times X) \subseteq g(X)$ and (g(X), d) is a complete subspace of X. Let q be a c-distance on X and M be an (F, g)-invariant subset of X^4 . Let

$$q(F(x,y),F(u,v)) \preceq aq(gx,gu) + bq(gy,gv)$$

for some $a, b \in [0, 1)$ with a + b < 1 and all $x, y, u, v \in X$ with $(gx, gy, gu, gv) \in M$ or $(gu, gv, gx, gy) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), gx_0, gy_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$, that is, F and g have a coupled coincidence point (x^*, y^*) .

Proof. It follows from Theorem 2.10 by similar arguments to those given in proof of Corollary 2.6. \Box

Corollary 2.13. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let $F: X \times X \to X$ be a given function. Let q be a c-distance on X and M be an F-invariant subset of X^4 . Let

$$q(F(x,y),F(u,v)) \preceq aq(x,u) + bq(y,v)$$

for some $a, b \in [0, 1)$ with a + b < 1 and all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*) .

Proof. Take $g = I_X$ in Corollary 2.12.

Corollary 2.14. Let (X, \sqsubseteq) be a partially ordered set and suppose that (X, d) is a cone metric space. Let $F: X \times X \to X$ be a given function. Let q be a c-distance on X and M be an F-invariant subset of X^4 . Let

$$q(F(x,y), F(u,v)) + q(F(y,x), F(v,u)) \leq k(q(x,u) + q(y,v))$$

for some $k \in [0,1)$ and all $x, y, u, v \in X$ with $(x, y, u, v) \in M$ or $(u, v, x, y) \in M$. Suppose $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$ and $x_n \to x, y_n \to y$ implies $(x, y, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. If there exist $x_0, y_0 \in X$ satisfying $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$, then there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$, that is, F has a coupled fixed point (x^*, y^*) .

Proof. It follows from Theorem 2.10 by taking $g = I_X$.

Theorem 2.15. In addition to the hypothesis of either Theorem 2.4 or Theorem 2.10 if $(gx, gy, gx, gy) \in M$ or $(gy, gx, gy, gx) \in M$ for all $x, y \in X$ then we have $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$.

Proof.

We have
$$q(gx^*, gx^*) + q(gy^*, gy^*) = q(F(x^*, y^*), F(x^*, y^*) + q(F(y^*, x^*), F(y^*, x^*)))$$

 $\leq k(q(gx^*, gx^*) + q(gy^*, gy^*))$

That is $q(gx^*, gx^*) + q(gy^*, gy^*) \leq k(q(gx^*, gx^*) + q(gy^*, gy^*))$ Since $0 \leq k < 1$, we have $q(gx^*, gx^*) + q(gy^*, gy^*) = \theta$. But $q(gx^*, gx^*) \geq \theta$ and $q(gy^*, gy^*) \geq \theta$, hence $q(gx^*, gx^*) = \theta$ and $q(gy^*, gy^*) = \theta$.

Theorem 2.16. In addition to hypothesis of either Theorem 2.4 or Theorem 2.10, suppose that any two elements x and y of X satisfy $(gx, gy, gy, gx) \in M$ or $(gy, gx, gx, gy) \in M$ and g is one-one. Then there exists a coupled coincidence point of F and g which is of the form (x^*, x^*) for some $x^* \in X$.

Proof. Consider coupled coincidence point (x^*, y^*) of F and g. Then

we have
$$q(gx^*, gy^*) + q(gy^*, gx^*) = q(F(x^*, y^*), F(y^*, x^*) + q(F(y^*, x^*), F(x^*, y^*))$$

 $\leq k(q(gx^*, gy^*) + q(gy^*, gx^*))$

That is $q(gx^*, gy^*) + q(gy^*, gx^*) \preceq k(q(gy^*, gx^*) + q(gx^*, gy^*))$ Since $0 \leq k < 1$, we have $q(gx^*, gy^*) + q(gy^*, gx^*) = \theta$.

But $q(gx^*, gy^*) \ge \theta$ and $q(gy^*, gx^*) \ge \theta$, hence $q(gx^*, gy^*) = \theta$ and $q(gy^*, gx^*) = \theta$. Let $u_n = \theta, x_n = gx^*$ for all $n \ge 0$, then we have $q(x_n, gx^*) \preceq u_n$ for all $n \ge 0$ and $q(x_n, gy^*) \preceq u_n$ for all $n \ge 0$. By Lemma 1.19(1) we have $gx^* = gy^*$.since g is one-one, therefore, $x^* = y^*$. Thus there exists a coupled coincidence point of the form (x^*, x^*) for some $x^* \in X$. This completes the proof.

Corollary 2.17. In addition to hypothesis of either Corollary 2.5 or Corollary 2.11, suppose that any two elements of g(X) are comparable and g is one-one. Then there exists a coupled coincidence point of F and g which is of the form (x^*, x^*) for some $x^* \in X$.

Example 2.18. Let $E = C^1_{\mathbb{R}}[0,1]$ with $||x||_1 = ||x||_{\infty} + ||x'||_{\infty}$ and $P = \{x \in E : x(t) \ge 0, t \in [0,1]\}$. Let $X = [0, +\infty)$ (with usual order), and $d(x, y)(t) = ||x - y|| e^t$. Then (X, d) is an ordered cone metric space (see [3] Example 2.9). Further, let $q : X \times X \to E$ be defined by $q(x, y)(t) = y e^t$. It is easy to check that q is a c-distance on X. Consider now the function defined by

$$F(x,y) = \begin{cases} \frac{1}{7}(x+y) & \text{if } x \ge y\\ 0 & \text{if } x < y \end{cases}$$

and $g(x) = \frac{3}{2}x$ for all x. Then $F(X \times X) \subseteq g(X) = X$ and (g(X), d) = (X, d) is complete. For $y_1 = 2$ and $y_2 = 3$ we have $gy_1 \sqsubseteq gy_2$ but $F(x, y_1) \sqsubseteq F(x, y_2)$ for all x > 3. So F does not satisfy mixed g-monotone property. Hence main result of [15] can not be applied to this example. Also it can be seen easily that $q(F(x, y), F(u, v)) + q(F(y, x), F(v, u)) \preceq \frac{1}{3}(q(gx, gu) + q(gy, gv))$ for all $(x, y, u, v) \in X^4 = M$. It is easy to see that all other conditions of Theorem 2.10 are satisfied for $M = X^4$. Thus, by Theorem 2.10, F and g have a coincidence point. Here F and g have a unique coincidence point at (0, 0).

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