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An abstract point of view on iterative approximation schemes of fixed points for multivalued operators

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Dedicated to the memory of Professor Viorel Radu

Abstract

In this paper we will present an abstract point of view on iterative approximation schemes of fixed points for multivalued operators. More precisely, we suppose that the algorithms are convergent and we will study the impact of this hypothesis in the theory of operatorial inclusiosns: data dependence, stability and Gronwall type lemmas. Some open problems are also presented.

Keywords: multivalued operator, fixed point, strict fixed point, iterative scheme, multivalued Picard operator, multivalued weakly Picard operator.
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1. Introduction

If X is a nonempty set, then we denote

$$\mathcal{P}(X) := \{Y|\ Y \text{ is a subset of } X\},\ P(X) := \{Y \in \mathcal{P}(X)|\ Y \text{ is non-empty}\}.$$

Let (X,d) be a metric space and $T : X \to P(X)$ be a multivalued operator. Throughout this paper the symbol $F_T := \{x \in X|\ x \in T(x)\}$ denotes the fixed point set of T, while $(SF)_T := \{x \in X|\ \{x\} = T(x)\}$ is the strict fixed point set of T.

The aim of this paper is to present an abstract point of view on iterative approximation schemes of (strict) fixed points for multivalued operators and to study the impact of this hypothesis in the theory of operatorial inclusiosns. Data dependence, different kind of stabilities and Gronwall type lemmas are considered. Some open problems are also presented.

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2. Multivalued operator theory: some basic concepts

Let (X, d) be a metric space. We introduce the following notations:

$$P_b(X) := \{Y \in P(X)|\ Y \text{ is bounded }\},\ P_{cl}(X) := \{Y \in P(X)|\ Y \text{ is closed}\},$$

$$P_{cp}(X) := \{Y \in P(X)|\ Y \text{ is compact}\},\ P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

The following (generalized) functional are used throughout the paper.

The gap functional

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ D_d(A, B) := \inf\{d(a, b)|\ a \in A,\ b \in B\}.$$

The diameter generalized functional

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \delta_d(A, B) := \sup\{d(a, b)|\ a \in A,\ b \in B\}.$$

In particular, $\delta(A) := \delta(A, A)$.

The excess generalized functional

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho_d(A, B) := \sup\{D_d(a, B)|\ a \in A\}.$$

The Pompeiu-Hausdorff generalized functional

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) := \max\{\rho_d(A, B), \rho_d(B, A)\}.$$

If no confusion is possible, we will avoid the subscript d from the above notations.

In the main part of this paper, the following results are needed (see [38] p. 76 and [24] p. 12).

Lemma 2.1. *Let (X, d) be a metric space, $A, B \in P(X)$ and $q > 1$. Then, for any $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq qH_d(A, B).$$

Lemma 2.2. *Let (X, d) be a metric space, $A, B \in P(X)$ and $\eta > 0$ such that:*

(1) for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq \eta$;

(2) for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leq \eta$.

Then $H(A, B) \leq \eta$.

If $T : X \to P(X)$ is a multivalued operator, then by

$$Graph(T) := \{(x, y) \in X \times X\ :\ y \in T(x)\}$$

we denote the graphic of the multivalued operator T and by

$$I(T) := \{Y \subset X | T(Y) \subset Y\},$$

the set of all invariant subsets of T. A selection for T is an operator $t : X \to X$ with the property $t(x) \in T(x)$ for each $x \in X$.

We also denote by $T^0 := 1_X,\ T^1 := T, \ldots, T^{n+1} = T \circ T^n,\ n \in \mathbb{N}$ the iterate operators of T. In the same framework, the operator $\hat{T} : P(X) \to P(X)$, defined by

$$\hat{T}(Y) := \bigcup_{x \in Y} T(x),\ \text{for } Y \in P(X)$$

is called the fractal operator generated by T.

If (X, d) is a metric space, then a multivalued operator $T : X \to P(X)$ is called upper semicontinuous (briefly u.s.c.) on X if and only if $T^+(V) := \{x \in X|\ T(x) \subset V\}$ is open, for each open set $V \subset X$ and it is said to be lower semicontinuous (briefly l.s.c.) on X if and only if $T^-(W) := \{x \in X|\ T(x) \cap W \neq \emptyset\}$ is open, for each open set $W \subset X$. If T is u.s.c. and l.s.c. on X then it is called continuous on X.

Lemma 2.3. *(see e.g. [1], [2], [13], [19]) If (X, d) is a metric space and $T : X \to P_{cp}(X)$ is a multivalued operator, then the following conclusions hold:*

(a) if T is upper semicontinuous, then $T(Y) \in P_{cp}(X)$, for every $Y \in P_{cp}(X)$;

(b) the continuity of T implies the continuity of $\hat{T} : P_{cp}(X) \to P_{cp}(X)$;

(c) If T is a multivalued α-contraction (i.e., $\alpha \in [0,1[$ and $H_d(T(x),T(y)) \le \alpha d(x,y)$, for each $x,y \in X$) (see [19], [8]), then the operator $\hat{T} : (P_{cp}(X), H_d) \to (P_{cp}(X), H_d)$ is a (singlevalued) α-contraction.

For the theory of multivalued operators see [1], [2], [13], [15], [18], [24], [42], [49], [52], etc.

3. Multivalued weakly Picard operators and fixed points

Let X be a nonempty set. Denote $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [11]) if the following conditions are satisfied:

(i) If $x_n = x,\ \forall\ n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition an element of $c(X)$ is convergent sequence and $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$. From now on, we will denote an L-space by $(X, \to)$.

Example 3.1. (L-structures on Banach spaces) Let X be a Banach space. We denote by $\to$ the strong convergence in X and by $\rightharpoonup$ the weak convergence in X. Then $(X, \to)$, $(X, \rightharpoonup)$ are L-spaces.

Remark 3.2. Notice that an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov' sense (i.e., $d(x,y) \in \mathbb{R}^m_+$), in Luxemburg-Jung' sense (i.e., $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$)), cone metric spaces (i.e., $d(x,y) \in K$ where K a cone in a Banach space), 2-metric spaces, probabilistic metric spaces, syntopogenous spaces, etc. are other examples of L-spaces. Notice also that, in general, the L-space convergence is not a topological one, in the sense that, in general, there is no topology which generates this convergence. In spite of this, we can define notions as "closed set", "continuity of an operator" in the terms of sequences, as in a metric space. For more details see Fréchet [11], Blumenthal [4] and I.A. Rus [35].

We recall now the concept of multivalued weakly Picard operator.

Definition 3.3. ([41]; see also [24], [42]) Let $(X, \to)$ be an L-space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$;

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$;

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit $x^*(x,y)$ is a fixed point of T.

The sequence $(x_n)_{n\in\mathbb{N}} \subset X$ satisfying (i) and (ii) from the above definition is called a sequence of successive approximations of T starting from $(x,y) \in Graph(T)$.

In the singlevalued case, we have the following concept.

Definition 3.4. (see [35]; see also [42]) Let $(X, \to)$ be an L-space. Then, we say that $t : X \to X$ is a Picard operator if and only if:

(i) $F_t = \{x^*\}$

(ii) $(t^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Definition 3.5. ([26], [27]) Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a MWP operator. Then we define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x,y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x,y) that converges to $z\ \}$.

Definition 3.6. ([26], [27]) Let (X,d) be a generalized metric space and $T : X \to P(X)$ a MWP operator. Then T is said to be a ψ-multivalued weakly Picard operator (briefly ψ-MWP operator) if and only $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and satisfies $\psi(0) = 0$ and there exists a selection t^∞ of T^∞ such that

$$d(x, t^\infty(x,y)) \leq \psi(d(x,y)), \text{ for all } (x,y) \in Graph(T). \tag{3.1}$$

In particular, if ψ has a linear representation, i.e., there exists $c > 0$ such that $\psi(t) = ct$ for all $t \in \mathbb{R}_+$, then T is called a c-multivalued weakly Picard operator (briefly c-MWP operator).

Example 3.7. ([41]) Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued α-contraction. Then, T is a $\frac{1}{1-\alpha}$-MWP operator.

Example 3.8. ([25]) Let (X,d) be a generalized complete metric space (in the sense that $d(x,y) \in \mathbb{R}^m_+$) and $T : X \to P_{cl}(X)$ be a multivalued A-contraction, i.e., there exists a matrix $A \in \mathcal{M}_{mm}(\mathbb{R})$ which converges to zero such that for each $x,y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u,v) \leq Ad(x,y)$. Then T is a $(I-A)^{-1}$-MWP operator.

Definition 3.9. If (X,d) is a metric spece, then a multivalued operator $T : X \to P(X)$ is said to satisfies Condition (I) (see [47], [48], [22], [43], [32]) if there exists an increasing function $\theta : \mathbb{R}_+\mathbb{R}_+$ such that:

(a) $\theta(0) = 0$ and $\theta(r) > 0$ for every $r > 0$;

(b) $D_d(x, T(x)) \geq \theta(D_d(x, F_T))$ for every $x \in X$.

Now the following problem arises.

Problem 3.10. Compare Condition (3.1) in Definition 3.6 with Condition (I) in Definition 3.9.

For basic notions and results on the theory of multivalued weakly Picard operators see [26], [25], [27], [35], [42]. For related results concerning metric and Banach spaces, operators on metric and Banach spaces and fixed points see [13], [17], [18], [49], [9], [20].

4. Multivalued Picard operators and strict fixed points

Let (X,d) be a metric space. By definition, $T : X \to P(X)$ is called a multivalued Picard operator (briefly MP operator) (see [26], [27]) if and only if:

(i) $(SF)_T = F_T = \{x^*\}$;

(ii) $T^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Example 4.1. Let (X,d) be a complete metric space and $T : X \to P_b(X)$ be a multivalued δ-contraction of Reich type with coefficients α, β, γ (see S. Reich [33]), i.e., there exist $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that

$$\delta(T(x), T(y)) \leq \alpha d(x,y) + \beta\delta(x, T(x)) + \gamma\delta(y, T(y)), \text{ for all } x,y \in X.$$

Additionally suppose that $\alpha + 2\beta < 1$. Then, T is a MP operator.

Example 4.2. Let (X,d) be a complete metric space, $T : X \to P_b(X)$ be a multivalued operator and $\varphi : \mathbb{R}^5_+ \to \mathbb{R}_+$ be a mapping. Suppose:

(i) $r, s \in \mathbb{R}^5_+, r \leq s$ implies that $\varphi(r) \leq \varphi(s)$;

(ii) there exists $p > 1$ such that the mapping $\Phi_p : \mathbb{R}_+ \to \mathbb{R}_+$ given by $t \longmapsto \varphi(t, pt, pt, t, t)$ is a strict comparison function;

(iii) $\delta(T(x), T(y)) \leq \varphi(d(x,y), \delta(x, T(x)), \delta(y, T(y)), \delta(x, T(y)), \delta(y, T(x)))$, for all $x, y \in X$. (see I.A. Rus [38], [37], [44]).

If, additionally, there exists a comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$r_0, r_1 \in \mathbb{R}_+ \text{ with } r_1 \leq \varphi(r_0, r_0 + r_1, 0, r_0, r_1) \text{ implies that } r_1 \leq \psi(r_0),$$

then T is a multivalued Picard operator.

Example 4.3. ([38] p. 67) Let (X,d) be a complete metric space and let $T : X \to P_{b,cl}(X)$ be a multivalued α-contraction with $(SF)_T \neq \emptyset$. Then, T is a multivalued Picard operator.

Let X be a topological space. By definition, $T : X \to P_{cl}(X)$ is called a topological contraction (Tarafdar-Yuan [51], see also [52]) if:

a) T is u.s.c.

b) $Y \in P_{cl}(X)$ with $T(Y) = Y \;\Rightarrow\; Y = \{x^*\}$.

Example 4.4. Let (X,d) be a compact metric space and $T : X \to P_{cl}(X)$ be a l.s.c. topological contraction. Then T is a multivalued Picard operator.

Example 4.5. Let (X,d) be a complete metric space and $t_1, ..., t_m : X \to X$ be Picard operators, such that $F_{t_i} = \{x^*\}$ for each $i \in \{1,2,...,m\}$. Consider the multivalued operator $T : X \to P_{cp}(X)$ defined by

$$T(x) = \{t_1(x), t_2(x), \cdots t_m(x)\}.$$

Then, T is a multivalued Picard operator.

For basic notions and results on the theory of multivalued Picard operators see [26], [27], [28] and [25].

5. Admissible perturbation of a multivalued operator

Let X be a nonempty set, $T : X \to P(X)$ be a multivalued operator and $G : X \times X \to X$ be and operator. We suppose:

(A_1) $G(x,x) = x$, for all $x \in X$;

(A_2) $x, y \in X$ and $G(x,y) = x$ imply $y = x$.

We define now the operator $T_G : X \to P(X)$ by

$$T_G(x) := G(x, T(x)) := \{G(x,u) | u \in T(x)\}.$$

Lemma 5.1. *$F_{T_G} = F_T$ and $(SF)_{T_G} = (SF)_T$.*

Proof. (a) We shall prove that $F_{T_G} = F_T$. Indeed, if $x \in F_T$, then $x = G(x,x) \in G(x,T(x))$. Thus $x \in F_{T_G}$. For the reverse inclusion, let $x \in F_{T_G}$. Since $x \in G(x,T(x))$, there exists $u \in T(x)$ such that $G(x,u) = x$. Hence we get that $x \in F_T$.
(b) We shall prove now that $(SF)_{T_G} = (SF)_T$. Indeed, if $x \in (SF)_T$, then $G(x,T(x)) = G(x,\{x\}) = \{x\}$. Thus $x \in (SF)_{T_G}$. For the reverse inclusion, if $x \in (SF)_{T_G}$, then $\{x\} = G(x,T(x))$. Thus, for all $u \in T(x)$ we have that $G(x,u) = x$. This implies that $u = x$ and so $T(x) = \{x\}$. □

Definition 5.2. If X is a nonempty set and the operator $G : X \times X \to X$ satisfies (A_1) and (A_2), then the multivalued operator T_G is called the admissible perturbation of T corresponding to G.

Example 5.3. Let $(V, +, \mathbb{R})$ be a vector space, $X \subset X$ be a convex set, $\lambda \in]0,1[$, $T : X \to P(X)$ be a multivalued operator and $G : X \times X \to X$ be defined by $G(x,y) := (1-\lambda)x + \lambda y$. Then T_G is an admissible perturbation of T corresponding to G.

Example 5.4. Let $(V, +, \mathbb{R})$ be a vector space, $X \subset X$ be a convex set, $\chi : X \times X \to]0,1[$, $T : X \to P(X)$ be a multivalued operator and $G : X \times X \to X$ be defined by $G(x,y) := (1-\chi(x,y))x + \chi(x,y)y$. Then T_G is an admissible perturbation of T corresponding to G.

Example 5.5. Let X is a nonempty set endowed with the F-convex structure of Gudder and Schroeck (see [14]), where $F : [0,1] \times X \times X \to X$ is an operator satisfying some conditions (see Gudder-Schroeck [14], A. Petruşel [23]). Let Y be an F-convex subset of X, $\lambda \in]0,1[$, $T : Y \to P(Y)$ be a multivalued operator and $G : Y \times Y \to Y$ be defined by $G(x,y) := F(\lambda, x, y)$. Then T_G is an admissible perturbation of T corresponding to G.

Example 5.6. Let (X, d) be a metric space endowed with the W-convex structure of Takahashi (see [50]; see also [39]), where $W : X \times X \times [0, 1] \to X$ is an operator satisfying the condition

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y), \text{ for all } x, y \in X.$$

Additionally, we suppose that

$$\lambda \in]0, 1[, \; W(x, y, \lambda) = x \;\Rightarrow\; y = x.$$

Let $\lambda \in]0, 1[$, Y be a W-convex subset of X, $T : Y \to P(Y)$ be a multivalued operator and $G : Y \times Y \to Y$ be defined by $G(x, y) := W(x, y, \lambda)$. Then T_G is an admissible perturbation of T corresponding to G.

Remark 5.7. For the case of admissible perturbation of a singlevalued operator see I.A. Rus [40].

6. Iterative algorithms in terms of admissible perturbations

Let $(X, \to)$ be an L-space, $T : X \to P(X)$ be a multivalued operator and $G, G_n, G_n^1, G_n^2 : X \times X \to X$ $(n \in \mathbb{N})$ be operators such that the multivalued operators $T_G, T_{G_n}, T_{G_n^1}, T_{G_n^2} : X \to P(X)$ are, respectively, admissible perturbations of T corresponding to G, G_n, G_n^1, G_n^2.

Example 6.1. (GK-algorithm) Consider the following iterative algorithm:

$$x_0 \in X \text{ be arbitrary, } x_{n+1} \in G(x_n, T(x_n)), \text{ for } n \in \mathbb{N}.$$

The above algorithm will be called Krasnoselskii's algorithm corresponding to G.

By definition, the above algorithm is convergent if and only if for each $x \in X$ and each $y \in G(x, T(x))$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x, \; x_1 = y$;
(ii) $x_{n+1} \in G(x_n, T(x_n))$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point $x^*(x, y)$ of T;
(iv) $x^*(x, x) = x$, for all $x \in F_T$.

Hence, in terms of multivalued weakly Picard operators language, we have that if the GK-algorithm is convergent, then T_G is a MWP operator.

Notice that, if the GK-algorithm is convergent and we define

$$t^\infty : Graph(T_G) \to X \text{ by } t^\infty(x, y) := x^*(x, y),$$

then we have:

(a) $t^\infty(Graph(T_G)) = F_T$;
(b) $t^\infty(x, x) = x$, for all $x \in F_T$.

Example 6.2. (GM-algorithm) Consider the following iterative algorithm:

$$x_0 \in X \text{ be arbitrary, } x_{n+1} \in G_n(x_n, T(x_n)), \text{ for } n \in \mathbb{N}.$$

The above algorithm will be called Mann's algorithm corresponding to $G := (G_n)_{n\in\mathbb{N}}$.

By definition, the above algorithm is convergent if and only if for each $x \in X$ and each $y \in G_0(x, T(x))$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x, \; x_1 = y$;
(ii) $x_{n+1} \in G_n(x_n, T(x_n))$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point $x^*(x, y)$ of T;
(iv) $x^*(x, x) = x$, for all $x \in F_T$.

Hence, if we use again the multivalued weakly Picard operators language, then if the GM-algorithm is convergent it follows that T_{G_n} is a MWP operator, for each $n \in \mathbb{N}$.

Notice that, if the GM-algorithm is convergent, then we can define

$$t^\infty : Graph(T_{G_0}) \to X \text{ by } t^\infty(x, y) := x^*(x, y).$$

Example 6.3. (GH-algorithm) Let $u \in X$. Consider the following iterative algorithm:

$$x_0 \in X \text{ be arbitrary, } x_{n+1} \in G_n(u, T(x_n)), \text{ for } n \in \mathbb{N}.$$

The above algorithm will be called Halpern's algorithm corresponding to $G := (G_n)_{n\in\mathbb{N}}$.

By definition, the above algorithm is convergent if and only if for each $x \in X$ and each $y \in G_0(u, T(x))$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x,\ x_1 = y$;
(ii) $x_{n+1} \in G_n(u, T(x_n))$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point $x^*(x, y)$ of T;
(iv) $x^*(x, x) = x$, for all $x \in F_T$.

Notice that, if the GH-algorithm is convergent, then we can define

$$t^\infty : Graph(G_0(u, T(\cdot))) \to X \text{ by } t^\infty(x, y) := x^*(x, y).$$

Example 6.4. (G_1G_2I-algorithm) Consider the following iterative algorithm:

$$x_0 \in X \text{ be arbitrary, } x_{n+1} \in G^2_n(x_n, T(G^1_n(x_n, T(x_n)))), \text{ for } n \in \mathbb{N}.$$

The above algorithm will be called Ishikawa's algorithm corresponding to $G^1 := (G^1_n)_{n\in\mathbb{N}}$ and $G^2 := (G^2_n)_{n\in\mathbb{N}}$.

By definition, the above algorithm is convergent if and only if for each $x \in X$ and each $y \in G^2_0(x, T(G^1_0(x, T(x))))$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x,\ x_1 = y$;
(ii) $x_{n+1} \in G^2_n(x_n, T(G^1_n(x_n, T(x_n))))$, for all $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point $x^*(x, y)$ of T;
(iv) $x^*(x, x) = x$, for all $x \in F_T$.

Notice that, if the G^1G^2I-algorithm is convergent, then we can define

$$t^\infty : Graph(G^2_0(\cdot, T(G^1_0(\cdot, T(\cdot))))) \to X \text{ by } t^\infty(x, y) := x^*(x, y).$$

For some particular cases of the above algorithms see [3], [7], [45], [47], [48], [22], [32], [6], etc.

Moreover, the above considerations give rise to the following open question.

Problem 6.1. Study the convergence of above algorithms in terms of the operators T and G.

7. Data dependence

In this section, as an example, we will study the data dependence of the fixed points for case of GK-algorithm.

Let (X, d) be a metric space and $T, S : X \to P(X)$ be two multivalued operators. Consider $G : X \times X \to X$ and let T_G and respectively S_G be the corresponding admissible perturbations. If T and S are "close enough" (i.e., there is $\eta > 0$ such that $H_d(T(x), S(x)) \leq \eta$, for all $x \in X$) and they have fixed points, we are interested to estimate $H_d(F_T, F_S)$.

Definition 7.1. Let (X, d) be a metric space and $G : X \times X \to X$. We say, by definition, that the GK-algorithm satisfies the condition (ψ) with respect to the multivalued operator $T : X \to P(X)$ if and only if the following assumptions are satisfied:

(i) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi(0) = 0$;
(ii) the GK-algorithm is convergent;
(iii) $d(x, t^\infty(x, y)) \leq \psi(d(x, y))$, for all $(x, y) \in Graph(T_G)$.

Our first result is the following theorem.

Theorem 7.2. *Let (X,d) be a metric space and $G : X \times X \to X$ be an operator which satisfies (A_1) and (A_2). Let $T, S : X \to P(X)$ be two multivalued operators. We suppose that:*

(i) the GK-algorithm satisfies the condition (ψ) with respect to the multivalued operators T and S;

(ii) there exist $l_G > 0$ such that $d(G(x,y), G(x,z)) \leq l_G d(y,z)$, for all $x, y, z \in X$;

(iii) there exists $\eta > 0$ such that $H_d(T(x), S(x)) \leq \eta$, for all $x \in X$.

Then, $H(F_T, F_S) \leq \psi(l_G \eta)$.

Proof. We will prove that for each $x_T^* \in F_T$ there exists $x_S^* \in F_S$ such that $d(x_T^*, x_S^*) \leq \psi(L_g \eta)$ and, similarly, for each $y_S^* \in F_S$ there exists $y_T^* \in F_T$ such that $d(y_S^*, y_T^*) \leq \psi(L_g \eta)$.
From (i) we get that F_T, F_S are nonempty sets. Let $t : X \to X$ and respectively $s : X \to X$ be a selection of T, respectively of S. Using condition (i) and Lemma 2.2 we get that

$$H_d(F_T, F_S) \leq \max\{\sup_{x \in F_S} d(x, t^\infty(x, t(x))), \sup_{x \in F_T} d(x, s^\infty(x, s(x)))\}.$$

Let $q > 1$. Then, from Lemma 2.1, we can choose the operators t and s such that

$$d(x, t^\infty(x, t(x))) \leq q\psi(H(T(x), S(x)), \text{ for all } x \in F_S$$

and

$$d(x, s^\infty(x, s(x))) \leq q\psi(H(T(x), S(x)), \text{ for all } x \in F_T.$$

Thus, by Lemma 2.2 and (iii) we get that

$$H_d(F_T, F_S) \leq q\psi(\eta).$$

Letting $q \searrow 1$ we get the conclusion. □

Remark 7.3. For the case of Picard iteration see [24] and [41].

Problem 7.1 Study the data dependence of the fixed point sets in the case of GM, GH and G_1G_2I algorithms.

8. Stability of an iterative algorithm

There are several hypostasis of data dependence, some of them are called stability, see [3], [7], [21], [27], [29], [30], [31], [36], [41], [45], [6], [20], [9], etc.

In [40], the notion of stability of an iterative algorithm for singlevalued operators is given in terms of convergence and of limit shadowing property. Following this idea, we present this concept in the case of a multivalued operators.

Definition 8.1. Let (X,d) be a metric space. Then:

a) A multivalued opertor $T : X \to P(X)$ has the limit shadowing property with respect to the Picard iteration if for each sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $D_d(y_{n+1}, T(y_n)) \to 0$ as $n \to +\infty$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T such that $d(x_n, y_n) \to 0$ as $n \to +\infty$.

b) A multivalued opertor $T : X \to P(X)$ has the limit shadowing property with respect to the GK-algorithm if for each sequence $(y_n)_{n\in\mathbb{N}}$ in X such that $D_d(y_{n+1}, T(y_n)) \to 0$ as $n \to +\infty$ there exists a GK-sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $d(x_n, y_n) \to 0$ as $n \to +\infty$.

In a similar way we may define the shadowing property with repsect to GM-algorithm, to GH-algorithm and to G_1G_2I-algorithm.

Another important concept is the following.

Definition 8.2. Let (X,d) be a metric space. Then an iterative algorithm (the Picard algorithm, the GK-algorithm, the GM-algorithm, the GH-algorithm, the G_1G_2I-algorithm, etc.) is called stable with respect to a multivalued operator $T : X \to P(X)$ if it is convergent with respect to T and the multivalued operator T has the limit shadowing property with respect to this algorithm.

Remark 8.3. For the shadowing property see [21], [27], [31], [12], etc.

Remark 8.4. For the stability of an iterative algorithm see [47], [48], [22], [43], [32], [6], etc.

Problem 8.1 Study the stability of the above algorithms in terms of the operators T and G.

9. Gronwall lemmas

Let $(X, d, \leq)$ be an ordered metric space and $T : X \to P(X)$ be a multivalued operator. If $A, B \in P(X)$ then we denote

$$A \leq B \;\Leftrightarrow\; \text{ for all } a \in A \text{ there exists } b \in B \text{ such that } a \leq b.$$

By definition, the multivalued operator T is called increasing (see [46] and [16]; see also [5], [10]) if and only if

$$x \leq y \;\Rightarrow\; T(x) \leq T(y).$$

Let us consider now the following question.

Problem 9.1 Let $(X, d, \leq)$ be an ordered metric space and $T : X \to P(X)$ be a multivalued operator such that $F_T \neq \emptyset$. In which conditions there exists a set retraction $\Psi : X \to F_T$ such that the following implication holds:

$$x \leq T(x) \;\Rightarrow\; x \leq \Psi(x)?$$

Remark 9.1. In the case of a convergent algorithm (see Section 6) we can choose $\Psi(x) := x^*(x, x)$. In this situation, the Problem 9.1 takes the following form:

Problem 9.2 Let $(X, d, \leq)$ be an ordered metric space and $T : X \to P(X)$ be a multivalued operator such that $F_T \neq \emptyset$. Consider an iterative alkgorithm which is convergent with respect to T. In which conditions the following implication holds

$$x \leq T(x) \;\Rightarrow\; x \leq x^*(x, x)?$$

Remark 9.2. For the case of singlevalued operators see [40].

Some partial answers for the above problem are the following theorems.

Theorem 9.3. *Let $(X, d, \leq)$ be an ordered metric space and $T : X \to P(X)$ be a multivalued operator such that $F_T = (SF)_T = \{x^*\}$. Suppose that:*
(i) T is a multivalued Picard operator;
(ii) T is increasing.
Then

$$x \in X,\; x \leq T(x) \;\Rightarrow\; x \leq x^*.$$

Proof. Let $x \in X$ be such that $x \leq T(x)$. By (i) we have that

$$H_d(T^n(x), x^*) \to 0 \text{ as } n \to \infty.$$

Since $H_d(T^n(x), x^*) = \delta_d(T^n(x), x^*)$ we get that for each $y_n \in T^n(x)$ we have $(y_n)_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$.

On the other hand, by (ii), there exists an incresing sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x. Thus

$$x \leq x_n \to x^* \text{ as } n \to \infty.$$

Since $(X, d, \leq)$ is an ordered metric space, we have that $x \leq x^*$. The proof is complete. □

Let us consider now the GK algorithm. In this case, we have the following result.

Theorem 9.4. *Let* $(X, d, \leq)$ *be an ordered metric space and* $T : X \to P(X)$ *be a multivalued operator such that* $F_T = (SF)_T = \{x^*\}$*. Suppose that:*

(i) T_G *is a Picard operator;*

(ii) T *is increasing;*

(iii) $G : X \times X \to X$ *is increasing.*

Then

$$x \in X,\ x \leq T(x) \ \Rightarrow\ x \leq x^*.$$

Proof. From (ii) and (iii) it follows that T_G is increasing. Now the proof follows from Theorem 9.3.

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