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Ulam-Hyers stability for coupled fixed points of contractive type operators

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`Dedicated to the memory of Professor Viorel Radu`

`Communicated by Adrian Petruşel`

Abstract

In this paper, we present existence, uniqueness and Ulam-Hyers stability results for the coupled fixed points of a pair of contractive type singlevalued and respectively multivalued operators on complete metric spaces. The approach is based on Perov type fixed point theorem for contractions in spaces endowed with vector-valued metrics.

Keywords: metric space, coupled fixed point, singlevalued operator, vector-valued metric, Perov type contraction.
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1. Introduction

The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to operatorial equations, fractal theory, optimization theory and other topics. Banach contraction principle was extended for singlevalued contraction on spaces endowed with vector-valued metrics by Perov in [10], while the case of multivalued contractions is treated in A. Petruşel [12].

In the study of the existence of fixed points for an operator, it is useful to consider a more general concept, namely coupled fixed points. The concept of coupled fixed point for continuous and discontinuous operators was introduced in 1987 by D. Guo and V. Lakshmikantham (see [6]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations.

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

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(a) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that we will make an identification between row and column vectors in $\mathbb{R}^m$.

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see [1], [14], [17]).

Theorem 1.1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalents:*
(i) A is convergent towards zero;
(ii) $A^n \to 0$ as $n \to \infty$;
(iii) The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
(iv) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + ... + A^n + ...; \tag{1.1}$$

(v) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;
(vi) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

We recall now Perov's fixed point theorem (see [10]).

Theorem 1.2 (Perov)**.** *Let (X, d) be a complete generalized metric space and the operator $f : X \to X$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that $d(f(x), f(y)) \leq A \cdot d(x, y)$ for all $x, y \in X$.*

If A is a matrix convergent towards zero, then:
(i) $Fix(f) = \{x^*\}$;
(ii) the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$ is convergent and has the limit x^*, for all $x_0 \in X$;
(iii) one has the following estimation

$$d(x_n, x^*) \leq A^n (I - A)^{-1} d(x_0, x_1); \tag{1.2}$$

(iv) if $g : X \to X$ is an operator such that there exist $y^* \in Fix(g)$ and $\eta \in \left(\mathbb{R}^m_+\right)^*$ with $d(f(x), g(x)) \leq \eta$, for each $x \in X$, then

$$d(x^*, y^*) \leq (I - A)^{-1} \eta;$$

(v) if $g : X \to X$ is an operator and there exists $\eta \in \left(\mathbb{R}^m_+\right)^*$ such that $d(f(x), g(x)) \leq \eta$, for all $x \in X$, then for the sequence $y_n := g^n(x_0)$ we have the following estimation

$$d(y_n, x^*) \leq (I - A)^{-1} \eta + A^n (I - A)^{-1} d(x_o, x_1). \tag{1.3}$$

Let (X, d) be a metric space. We will focus our attention to the following system of operatorial equations:

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases}$$

where $T_1, T_2 : X \times X \to X$ are two given operators.

By definition, a solution $(x, y) \in X \times X$ of the above system is called a coupled fixed point for the pair (T_1, T_2). In a similar way, the case of an operatorial inclusion (using the symbol $\in$ instead of $=$) could be considered.

In this paper we present some coupled fixed points results for contractive type singlevalued and multivalued operators on spaces endowed with vector-valued metrics. The approach is based on Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics. For related results to Perov's fixed point theorem and for some generalizations and applications of it we refer to [3], [4], [13].

2. Existence, uniqueness and stability results for coupled fixed points

For the proof of our main theorem we need the following notions and results.

Definition 2.1. Let (X, d) be a generalized metric space and $f : X \to X$ be an operator. Then, the fixed point equation

$$x = f(x) \tag{2.1}$$

is said to be generalized Ulam-Hyers stable if there exists an increasing function $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$, continuous in 0 with $\psi(0) = 0$, such that, for any $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$ with $\varepsilon_i > 0$ for $i \in \{1, ..., m\}$ and any solution $y^* \in X$ of the inequation

$$d(y, f(y)) \le \varepsilon \tag{2.2}$$

there exists a solution x^* of (2.1) such that

$$d(x^*, y^*) \le \psi(\varepsilon). \tag{2.3}$$

In particular, if $\psi(t) = C \cdot t$, $t \in \mathbb{R}^m_+$ (where $C \in M_{mm}(\mathbb{R}_+)$), then the fixed point equation (2.1) is called Ulam-Hyers stable.

Our first abstract result is a direct consequence of Perov's fixed point theorem.

Theorem 2.2. *Let (X, d) be a generalized metric space and let $f : X \to X$ be an operator with the property that there exists a matrix $A \in M_{mm}(\mathbb{R})$ such that A converges to zero and*

$$d(f(x), f(y)) \le A \cdot d(x, y), \text{for all } x, y \in X.$$

Then the fixed point equation

$$x = f(x), \ x \in X$$

is Ulam-Hyers stable.

Proof. From Perov's fixed point theorem we get that $Fix(f) = \{x^*\}$. Let $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$, with $\varepsilon_i > 0$ for each $i \in \{1, ..., m\}$ and let y^* be a solution of the inequation

$$d(y, f(y)) \le \varepsilon.$$

Then we succesively have that $d(x^*, y^*) = d(f(x^*), y^*) \le d(f(x^*), f(y^*)) + d(f(y^*), y^*) \le Ad(x^*, y^*) + \varepsilon$. Thus, using Theorem 1.2, we get that

$$d(x^*, y^*) \le (I - A)^{-1}\varepsilon.$$

□

Definition 2.3. Let (X, d) be a metric space and let $T_1, T_2 : X \times X \to X$ be two operators. Then the operatorial equations system

$$\begin{cases} x = T_1(x, y) \\ y = T_2(x, y) \end{cases} \tag{2.4}$$

is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each pair $(u^*, v^*) \in X \times X$ such that

$$\begin{aligned} &d(u^*, T_1(u^*, v^*)) \leq \varepsilon_1 \\ &d(v^*, T_2(u^*, v^*)) \leq \varepsilon_2 \end{aligned} \tag{2.5}$$

there exists a solution $(x^*, y^*) \in X \times X$ of (2.4) such that

$$\begin{aligned} &d(u^*, x^*) \leq c_1\varepsilon_1 + c_2\varepsilon_2 \\ &d(v^*, y^*) \leq c_3\varepsilon_1 + c_4\varepsilon_2 \end{aligned} \tag{2.6}$$

For examples and other considerations regarding Ulam-Hyers stability and generalized Ulam-Hyers stability of the operatorial equations and inclusions see I.A. Rus [15], Bota-Petruşel [2], Petru-Petruşel-Yao [11].

Our first main result is the following existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the coupled fixed point of a pair of singlevalued operators (T_1, T_2). The conclusions (i)-(ii) are originally proved by R. Precup [13], but for the sake of completeness we recall here the whole proof.

Theorem 2.4. *Let (X, d) be a complete metric space, $T_1, T_2 : X \times X \to X$ be two operators such that*

$$\begin{aligned} &d(T_1(x, y), T_1(u, v)) \leq k_1 d(x, u) + k_2 d(y, v) \\ &d(T_2(x, y), T_2(u, v)) \leq k_3 d(x, u) + k_4 d(y, v) \end{aligned} \tag{2.7}$$

for all $(x, y), (u, v) \in X \times X$. We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:
(i) there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases} \tag{2.8}$$

(ii) the sequence $(T_1^n(x, y), T_2^n(x, y))_{n \in \mathbb{N}}$ converges to (x^*, y^*) as $n \to \infty$, where

$$\begin{aligned} T_1^{n+1}(x, y) &:= T_1^n(T_1(x, y), T_2(x, y)) \\ T_2^{n+1}(x, y) &:= T_2^n(T_1(x, y), T_2(x, y)) \end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}$.
(iii) we have the following estimation:

$$\begin{pmatrix} d(T_1^n(x_0, y_0), x^*) \\ d(T_2^n(x_0, y_0), y^*) \end{pmatrix} \leq A^n (I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0)) \\ d(y_0, T_2(x_0, y_0)) \end{pmatrix} \tag{2.10}$$

(iv) let $F_1, F_2 : X \times X \to X$ be two operators such that, there exist $\eta_1, \eta_2 > 0$ with

$$\begin{aligned} d(T_1(x, y), F_1(x, y)) \leq \eta_1 \\ d(T_2(x, y), F_2(x, y)) \leq \eta_2 \end{aligned}$$

for all $(x, y) \in X \times X$. If $(a^*, b^*) \in X \times X$ is such that

$$\begin{cases} a^* = F_1(a^*, b^*) \\ b^* = F_2(a^*, b^*) \end{cases} \tag{2.11}$$

then

$$\begin{pmatrix} d(a^*, x^*) \\ d(b^*, y^*) \end{pmatrix} \leq (I - A)^{-1} \eta \tag{2.12}$$

where $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$.

(v) let $F_1, F_2 : X \times X \to X$ *be two operators such that, there exist* $\eta_1, \eta_2 > 0$ *with*

$$\begin{array}{l} d(T_1(x,y), F_1(x,y)) \leq \eta_1 \\ d(T_2(x,y), F_2(x,y)) \leq \eta_2 \end{array} \tag{2.13}$$

for all $(x,y) \in X \times X$. *If we consider the sequence* $(F_1^n(x,y), F_2^n(x,y))_{n\in\mathbb{N}}$, *given by*

$$\begin{array}{l} F_1^{n+1}(x,y) := F_1^n(F_1(x,y), F_2(x,y)) \\ F_2^{n+1}(x,y) := F_2^n(F_1(x,y), F_2(x,y)) \end{array} \tag{2.14}$$

for all $n \in \mathbb{N}^*$ *and* $\eta := \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, *then*

$$\begin{pmatrix} d(F_1^n(x_0, y_0), x^*) \\ d(F_2^n(x_0, y_0), y^*) \end{pmatrix} \leq (I - A)^{-1} \eta + A^n (I - A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0, y_0)) \\ d(y_0, T_2(x_0, y_0)) \end{pmatrix}$$

(vi) the operatorial equations system

$$\begin{cases} x = T_1(x,y) \\ y = T_2(x,y) \end{cases} \tag{2.15}$$

is Ulam-Hyers stable.

Proof. $(i) - (ii)$ Let us define $T : X \times X \to X \times X$ by

$$T(x,y) = \begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix} = (T_1(x,y), T_2(x,y))$$

Denote $Z := X \times X$ and consider $\widetilde{d} : Z \times Z \to \mathbb{R}_+^2$,

$$\widetilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}.$$

Then we have

$$\begin{aligned} \widetilde{d}(T(x,y), T(u,v)) &= \widetilde{d}\left(\begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix}\begin{pmatrix} T_1(u,v) \\ T_2(u,v) \end{pmatrix}\right) \\ &= \begin{pmatrix} d(T_1(x,y), T_1(u,v)) \\ d(T_2(x,y), T_2(u,v)) \end{pmatrix} \\ &\leq \begin{pmatrix} k_1 d(x,u) + k_2 d(y,v) \\ k_3 d(x,u) + k_4 d(y,v) \end{pmatrix} \\ &= \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = A \cdot \widetilde{d}((x,y),(u,v)) \end{aligned} \tag{2.16}$$

If we denote $(x,y) := z, (u,v) := w$, we get that

$$\widetilde{d}(T(z), T(w)) \leq A \cdot \widetilde{d}(z,w).$$

Applying Perov's fixed point Theorem 1.2 (i), we get that there exists a unique element $(x^*, y^*) \in X \times X$ such that

$$(x^*, y^*) = T(x^*, y^*)$$

and is equivalent with

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases}$$

Moreover, for each $z \in X \times X$, we have that $T^n(z) \to z^*$ as $n \to \infty$, where

$$\begin{aligned} T^0(z) &:= z, T^1(z) = T(x,y) = (T_1(x,y), T_2(x,y)) \\ T^2(z) &= T(T_1(x,y), T_2(x,y)) = \left(T_1^2(x,y), T_2^2(x,y)\right) \end{aligned}$$

and, in generally

$$\begin{aligned} T_1^{n+1}(x,y) &:= T_1^n(T_1(x,y), T_2(x,y)) \\ T_2^{n+1}(x,y) &:= T_2^n(T_1(x,y), T_2(x,y)) \end{aligned} \tag{2.17}$$

We obtain that $T^n(z) = (T_1^n(z), T_2^n(z)) \to z^* := (x^*, y^*)$ as $n \to \infty$, for all $z := (x,y) \in X \times X$.
So, for all $(x,y) \in X \times X$, we have that

$$\begin{aligned} T_1^n(x,y) &\to x^* \text{ as } n \to \infty \\ T_2^n(x,y) &\to y^* \text{ as } n \to \infty. \end{aligned} \tag{2.18}$$

(iii) By Perov's Theorem (iii) we successively have

$$\begin{aligned} \begin{pmatrix} d(T_1^n(x_0,y_0), x^*) \\ dT_2^n(x_0,y_0), y^*) \end{pmatrix} &= \widetilde{d}((T^n(x_0,y_0)), (x^*, y^*)) \\ &\leq A^n (I-A)^{-1} \widetilde{d}((x_0,y_0), (T_1(x_0,y_0), T_2(x_0,y_0))) \\ &= A^n (I-A)^{-1} \begin{pmatrix} d(x_0, T_1(x_0,y_0)) \\ d(y_0, T_2(x_0,y_0)) \end{pmatrix}. \end{aligned}$$

(iv) If we consider $F : X \times X \to X \times X$ such that

$$F(x,y) = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} \tag{2.19}$$

and

$$\begin{aligned} \widetilde{d}(T(x,y), F(x,y)) &= \widetilde{d}\left(\begin{pmatrix} T_1(x,y) \\ T_2(x,y) \end{pmatrix} \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}\right) \\ &= \begin{pmatrix} d(T_1(x,y), F_1(x,y)) \\ d(T_2(x,y), F_2(x,y)) \end{pmatrix} \leq \eta \end{aligned} \tag{2.20}$$

then, applying Perov‘s fixed point Theorem 1.2 (iv) we get

$$\widetilde{d}((x^*, y^*), (a^*, b^*)) \leq (I-A)^{-1} \eta \tag{2.21}$$

(v) By (2.20) we get that

$$\widetilde{d}(T(x,y), F(x,y)) \leq \eta.$$

Notice that $F^n(x,y) = F\left(F^{n-1}(x,y)\right)$, for all $(x,y) \in X \times X$.
Using the assertion (iii) of this theorem, we can successively write:

$$\begin{aligned} \widetilde{d}(F^n(x_0,y_0), (x^*, y^*)) &\leq \widetilde{d}(F^n(x_0,y_0), T^n(x_0,y_0)) + \widetilde{d}(T^n(x_0,y_0), (x^*, y^*)) \\ &\leq \widetilde{d}(F^n(x_0,y_0), T^n(x_0,y_0)) + A^n (I-A)^{-1} \widetilde{d}(T(x_0,y_0), (x_0,y_0)) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \widetilde{d}\left(F^n\left(x_0,y_0\right),T^n\left(x_0,y_0\right)\right) &= \widetilde{d}\left(F\left(F^{n-1}\left(x_0,y_0\right)\right),T\left(T^{n-1}\left(x_0,y_0\right)\right)\right) \\ &\le \widetilde{d}\left(F\left(F^{n-1}\left(x_0,y_0\right)\right),T\left(F^{n-1}\left(x_0,y_0\right)\right)\right) \\ &\quad +\widetilde{d}\left(T\left(F^{n-1}\left(x_0,y_0\right)\right),T\left(T^{n-1}\left(x_0,y_0\right)\right)\right) \\ &\le \eta + A\widetilde{d}\left(F^{n-1}\left(x_0,y_0\right),T^{n-1}\left(x_0,y_0\right)\right) \\ &\le \eta + A\left(\eta+\widetilde{d}\left(F^{n-2}\left(x_0,y_0\right),T^{n-2}\left(x_0,y_0\right)\right)\right) \\ &\le \ldots \le \eta\left(I+A+\ldots+A^n+\ldots\right) = \eta\left(I-A\right)^{-1} \end{aligned} \tag{2.22}$$

Thus, we finally get the conclusion

$$\widetilde{d}\left(F^n\left(x_0,y_0\right),\left(x^*,y^*\right)\right) \le \eta\left(I-A\right)^{-1} + A^n\left(I-A\right)^{-1}\widetilde{d}\left(T\left(x_0,y_0\right),\left(x_0,y_0\right)\right).$$

(vi) By (i) and (ii) there exists a unique element $(x^*,y^*)\in X\times X$ such that (x^*,y^*) is a solution for (2.15) and the sequence $\left(T_1^n(x,y),T_2^n(x,y)\right)$ converges to (x^*,y^*) as $n\to\infty$. Let $\varepsilon_1,\varepsilon_2>0$ and $(u^*,v^*)\in X\times X$ such that

$$\begin{aligned} d\left(u^*,T_1\left(u^*,v^*\right)\right) \le \varepsilon_1 \\ d\left(v^*,T_2\left(u^*,v^*\right)\right) \le \varepsilon_2 \end{aligned} \tag{2.23}$$

Then we have

$$\begin{aligned} \widetilde{d}\left(\left(u^*,v^*\right),\left(x^*,y^*\right)\right) &\le \widetilde{d}\left(\left(u^*,v^*\right),\left(T_1\left(u^*,v^*\right),T_2\left(u^*,v^*\right)\right)\right) \\ &\quad +\widetilde{d}\left(\left(T_1\left(u^*,v^*\right),T_2\left(u^*,v^*\right)\right),\left(x^*,y^*\right)\right) \\ &= \widetilde{d}((u^*,v^*),(T_1\left(u^*,v^*\right),T_2\left(u^*,v^*\right))) \\ &\quad +\widetilde{d}\left(\left(T_1\left(u^*,v^*\right),T_2\left(u^*,v^*\right)\right),\left(T_1\left(x^*,y^*\right),T_2\left(x^*,y^*\right)\right)\right) \\ &= \begin{pmatrix} d(u^*,T_1\left(u^*,v^*\right)) \\ d(v^*,T_2\left(u^*,v^*\right)) \end{pmatrix} + \begin{pmatrix} d(T_1\left(u^*,v^*\right),T_1\left(x^*,y^*\right)) \\ d(T_2\left(u^*,v^*\right),T_2\left(x^*,y^*\right)) \end{pmatrix} \\ &\le \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} + \widetilde{d}\left(T\left(u^*,v^*\right),T\left(x^*,y^*\right)\right) \\ &\le \varepsilon + A\widetilde{d}\left(\left(u^*,v^*\right),\left(x^*,y^*\right)\right) \end{aligned}$$

Since $(I-A)$ is inversable and $(I-A)^{-1}$has positive elements, we imediately obtain

$$\widetilde{d}\left(\left(u^*,v^*\right),\left(x^*,y^*\right)\right) \le \left(I-A\right)^{-1}\varepsilon$$

or equivalently

$$\begin{pmatrix} d\left(u^*,x^*\right) \\ d\left(v^*,y^*\right) \end{pmatrix} \le \left(I-A\right)^{-1}\varepsilon$$

If we denote $(I-A)^{-1} := \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, then we obtain

$$\begin{aligned} d\left(u^*,x^*\right) &\le c_1\varepsilon_1 + c_2\varepsilon_2 \\ d\left(v^*,y^*\right) &\le c_3\varepsilon_1 + c_4\varepsilon_2 \end{aligned} \tag{2.24}$$

proving that the operatorial system (2.15) is Ulam-Hyers stable. □

Remark 2.5. Notice that, if (X,d) is a metric space and $T: X\times X\to X$ is an operator and we define

$$T_1(x,y) := T(x,y) \text{ and } T_2(x,y) := T(y,x),$$

then the above approach leads to some well-known coupled fixed point theorems, see [5], [6]. Moreover, in a forthcoming paper, the same approach will be applied for the case of coupled fixed points for mixed monotone operators, see, for example, [7], [8], [9], [16].

We will consider now the case of multivalued operators. We need first some notations.

Let (X, d) be a generalized metric space with $d : X \times X \to \mathbb{R}^m_+$ given by

$$d(x,y) = \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_m(x,y) \end{pmatrix}$$

Then, for $x \in X$ and $A \subseteq X$ we denote:

$$D_d(x,A) = \begin{pmatrix} D_{d_1}(x,A) \\ \vdots \\ D_{d_m}(x,A) \end{pmatrix} := \begin{pmatrix} \inf\limits_{a\in A} d_1(x,a) \\ \vdots \\ \inf\limits_{a\in A} d_m(x,a) \end{pmatrix}$$

$$\begin{aligned} P(X) \;&:\; = \{Y \subseteq X \mid Y \text{ is nonempty}\}, \\ P_{cl}(X) \;&:\; = \{Y \in P(X) \mid Y \text{ is closed}\}. \end{aligned}$$

We also denote $D((x,y), A \times B) := \begin{pmatrix} D_d(x,A) \\ D_d(y,B) \end{pmatrix}$.

Our second main result is an existence, uniqueness, data dependence and Ulam-Hyers stability theorem for the coupled fixed point of a pair of multivalued operators (T_1, T_2). For the proof of our main result, we give the following theorem.

Theorem 2.6. *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued A-contraction, i.e. there exists $A \in M_{mm}(\mathbb{R}_+)$ which converges towards zero as $n \to \infty$ and for each $x, y \in X$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $d(u,v) \le A \cdot d(x,y)$. Then T is a MWP-operator, i.e. $FixT \neq \emptyset$ and for each $(x,y) \in Graph(T)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of succesive approximations for T starting from (x,y) which converges to a fixed point x^* of T. Moreover $d(x,x^*) \le (I-A)^{-1} d(x,y)$, for all $(x,y) \in Graph(T)$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then by the A-contraction condition, there exists $x_2 \in T(x_1)$ such that $d(x_1,x_2) \le A \cdot d(x_0,x_1)$. Now, for $x_2 \in T(x_1)$ there exists $x_3 \in T(x_2)$ such that $d(x_2,x_3) \le A \cdot d(x_1,x_2) \le A^2 \cdot d(x_0,x_1)$.

In this way, by an iterative construction, we get a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} x_0 \in X \\ x_{n+1} \in T(x_n) \\ d(x_n,x_{n+1}) \le A^n d(x_0,x_1) \end{cases}$$

for all $n \in \mathbb{N}$.

Thus, by the above relation, we get

$$\begin{aligned} d(x_n,x_{n+1}) &\le d(x_n,x_{n+1}) + \ldots + d(x_{n+p+1},x_{n+p}) \\ &\le A^n d(x_0,x_1) + \ldots + A^{n+p-1} d(x_0,x_1) \\ &= A^n (I + A + \ldots + A^{p-1}) d(x_0,x_1) \end{aligned}$$

Leting $n \to \infty$ we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Hence there exist $x^* \in X$ such that $x^* = \lim\limits_{n\to\infty} x_n$.

We prove that $x^* \in T(x^*)$. Indeed, for $x_n \in T(x_{n-1})$ there exist $u_n \in T(x^*)$ such that

$$d(x_n,u_n) \le A d(x_{n-1},x^*), \text{ for all } n \in \mathbb{N}^*.$$

On the other side

$$d(x^*,u_n) \le d(x^*,x_n) + d(x_n,u_n) \le d(x^*,x_n) + A d(x_{n-1},x^*) \to 0, \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} u_n = x^*$. But $u_n \in T(x^*)$, for $n \in \mathbb{N}$ and because $T(x^*)$ is closed, we have that $x^* \in T(x^*)$.

Moreover we can write

$$d(x_n, x_{n+p}) \leq A^n(I + A + ... + A^{p-1} + ...)d(x_0, x_1) = A^n(I - A)^{-1}d(x_0, x_1).$$

Letting $p \to \infty$ we get that

$$d(x_n, x^*) \leq A^n(I - A)^{-1}d(x_0, x_1), \text{ for all } n \geq 1.$$

Thus

$$\begin{aligned} d(x_0, x^*) &\leq d(x_0, x_1) + d(x_1, x^*) \leq d(x_0, x_1) + A(I - A)^{-1}d(x_0, x_1) \\ &= (I + A(I - A)^{-1})d(x_0, x_1) = (I + A + A^2 + ...)d(x_0, x_1) \\ &= (I - A)^{-1}d(x_0, x_1). \end{aligned}$$

□

Definition 2.7. Let (X, d) generalized metric space and $F : X \to P(X)$. The fixed point inclusion

$$x \in F(x), x \in X \tag{2.25}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ increasing, continuous in o with $\psi(0) = 0$ such that for each $\varepsilon := (\varepsilon_1, ..., \varepsilon_m) > 0$ and for each ε-solution y^* of (2.25), i.e.

$$D_d(y^*, F(y^*)) \leq \varepsilon$$

there exists a solution x^* of the fixed point inclusion (2.25) such that

$$d(y^*, x^*) \leq \psi(\varepsilon)$$

In particular, if $\psi(t) = C \cdot t$, for each $t \in \mathbb{R}^m_+$ (where $C \in M_{mm}(\mathbb{R}_+)$), then (2.25) is said to be Ulam-Hyers stable.

Definition 2.8. A subset U of a generalized metric space (X, d) is called proximinal if for each $x \in X$ there exists $u \in U$ such that $d(x, u) = D_d(x, U)$.

Theorem 2.9. *Let (X, d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued A-contraction with proximinal values. Then, the fixed point inclusion (2.25) is Ulam-Hyers stable.*

Proof. Let $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$ with $\varepsilon_i > 0$, for each $i \in \{1, 2, \cdots m\}$ and let $y^* \in X$ an ε-solution of (2.25), i.e.,

$$D_d(y^*, F(y^*)) \leq \varepsilon$$

By the second conclusion of Theorem 2.6 we have that for any $(x, y) \in Graph(T)$

$$d(x, x^*(x, y)) \leq (I - A)^{-1}d(x, y), \tag{2.26}$$

where $x^*(x, y)$ denotes the fixed point of F which is obtained by Theorem 2.6 by successive approximations starting from (x, y).

Since $T(y^*)$ is proximinal there exists $u^* \in T(y^*)$ such that

$$d(y^*, u^*) = D_d(y^*, T(y^*))$$

Hence, by (2.26)

$$d(y^*, x^*(y^*, u^*)) \leq (I - A)^{-1}d(y^*, u^*) \leq (I - A)^{-1}\varepsilon.$$

□

Theorem 2.10. *Let (X,d) be a complete generalized metric space and let $T : X \to P_{cl}(X)$ be a multivalued A-contraction such that there exists $x^* \in X$ with $T(x^*) = \{x^*\}$. Then the fixed point inclusion (2.25) is Ulam-Hyers stable.*

Proof. Let $\varepsilon := (\varepsilon_1, ..., \varepsilon_m)$ with $\varepsilon_i > 0$, for each $i \in \{1, 2, \cdots m\}$ and let $y^* \in X$ an ε-solution of (2.25), i.e.,

$$D_d(y^*, T(y^*)) \leq \varepsilon$$

By the A-contraction condition, for $x := y^*, y := x^*$ and $u \in T(y^*)$ we get that

$$d(u, x^*) \leq Ad(y^*, x^*).$$

Then, for any $u \in T(y^*)$ we have

$$d(y^*, x^*) \leq d(y^*, u) + d(u, x^*) \leq d(y^*, u) + Ad(y^*, x^*)$$

Hence

$$d(y^*, x^*) \leq (I - A)^{-1} d(y^*, u), \text{ for any } u \in T(y^*)$$

Thus

$$d(y^*, x^*) \leq (I - A)^{-1} D_d(y^*, T(y^*)) \leq (I - A)^{-1}\varepsilon.$$

□

Let (X,d) be a metric space. We will focus our attention to the following system of operatorial inclusions:

$$\begin{cases} x \in T_1(x, y) \\ y \in T_2(x, y) \end{cases} \tag{2.27}$$

where $T_1, T_2 : X \times X \to P(X)$ are two given multivalued operators.

By definition, a solution $(x, y) \in X \times X$ of the above system is called a coupled fixed point for the pair (T_1, T_2).

Definition 2.11. Let (X,d) be a metric space and let $T_1, T_2 : X \times X \to P(X)$ are two multivalued operators. Then the operatorial inclusions system (2.27) is said to be Ulam-Hyers stable if there exist $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each pair $(u^*, v^*) \in X \times X$ which satisfies the relations

$$\begin{aligned} &d(u^*, w) \leq \varepsilon_1, \text{ for all } w \in T_1(u^*, v^*) \\ &d(v^*, z) \leq \varepsilon_2, \text{ for all } z \in T_2(u^*, v^*) \end{aligned} \tag{2.28}$$

there exists a solution $(x^*, y^*) \in X \times X$ of (2.27) such that

$$\begin{aligned} &d(u^*, x^*) \leq c_1\varepsilon_1 + c_2\varepsilon_2 \\ &d(v^*, y^*) \leq c_3\varepsilon_1 + c_4\varepsilon_2 \end{aligned} \tag{2.29}$$

Definition 2.12. Let (X,d) be a metric space. By definition, we say that $S : X \times X \to P(X)$ has proximinal values with respect to the first variable if for any $x, y \in X$ there exists $u \in S(x, y)$ such that

$$d(x, u) = D_d(x, S(x, y))$$

Definition 2.13. Let (X,d) be a metric space. By definition we say that $S : X \times X \to P(X)$ has proximinal values with respect to the second variable if for any $x, y \in X$ there exists $v \in S(x, y)$ such that

$$d(y, v) = D_d(y, S(x, y))$$

Now we are in the position to give our next main results.

Theorem 2.14. *Let (X, d) be a complete metric space and let $T_1, T_2 : X \times X \to P_{cl}(X)$ be two multivalued operators. Suppose that T_1 has proximinal values with respect to the first variable and T_2 with respect to the second one. For each $(x, y), (u, v) \in X \times X$ and each $z_1 \in T_1(x, y)$, $z_2 \in T_2(x, y)$ there exist $w_1 \in T_1(u, v)$, $w_2 \in T_2(u, v)$ satisfying*

$$\begin{array}{rcl} d(z_1, w_1) & \leq & k_1 d(x, u) + k_2 d(y, v) \\ d(z_2, w_2) & \leq & k_3 d(x, u) + k_4 d(y, v) \end{array}$$

We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:

(i) there exists $(x^*, y^*) \in X \times X$ a solution for (2.27).

(ii) the operatorial system (2.27) is Ulam-Hyers stable.

Proof. (i)-(ii) Let us define $T : X \times X \to P_{cl}(X) \times P_{cl}(X)$ by

$$T(x, y) := T_1(x, y) \times T_2(x, y).$$

Denote $Z := X \times X$ and consider $\widetilde{d} : Z \times Z \to \mathbb{R}^2_+$,

$$\widetilde{d}((x, y), (u, v)) := \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix}.$$

Then, from the hypotheses of the theorem, we get that for each $s := (x, y), t := (u, v) \in X \times X$ and each $z := (z_1, z_2) \in T(x, y)$, there exists $w := (w_1, w_2) \in T(u, v)$ satisfying the relation

$$\widetilde{d}(z, w) \leq A \cdot \widetilde{d}(s, t),$$

which proves that T is a multivalued A-contraction.
Since $T_1(x, y) \subset X$ is proximinal with respect to the first variable we have that, for any $(x, y) \in X$ there exists $u \in T_1(x, y)$ such that

$$d(x, u) = D_d(x, T_1(x, y))$$

Since $T_2(x, y) \subset X$ is proximinal with respect to the second variable we get that, for any $(x, y) \in X$ there exists $v \in T_2(x, y)$ such that

$$d(y, v) = D_d(y, T_2(x, y))$$

Then the set $T(x, y) := T_1(x, y) \times T_2(x, y)$ is proximinal, since for any $(x, y) \in X$ there exists $(u, v) \in T(x, y)$ such that

$$\widetilde{d}((x, y), (u, v)) = D_{\widetilde{d}}((x, y), T(x, y)).$$

The conclusions follow now from Theorem 2.6 and Theorem 2.9. □

Theorem 2.15. *Let (X, d) be a complete metric space and let $T_1, T_2 : X \times X \to P_{cl}(X)$ be two multivalued operators. Suppose there exist $x_1^*, x_2^* \in X$ such that*

$$T_1(x^*, y^*) = \{x^*\}, \quad T_2(x^*, y^*) = \{y^*\}. \tag{2.30}$$

For each $(x, y), (u, v) \in X \times X$ and each $z_1 \in T_1(x, y)$, $z_2 \in T_2(x, y)$ there exist $w_1 \in T_1(u, v)$, $w_2 \in T_2(u, v)$ satisfying

$$\begin{array}{rcl} d(z_1, w_1) & \leq & k_1 d(x, u) + k_2 d(y, v) \\ d(z_2, w_2) & \leq & k_3 d(x, u) + k_4 d(y, v). \end{array}$$

We suppose that $A := \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}$ converges to zero. Then:

(i) there exists $(x^*, y^*) \in X \times X$ a solution for (2.27).

(ii) the operatorial system (2.27) is Ulam-Hyers stable.

Proof. (i)-(ii) Let us define $T : X \times X \to P_{cl}(X) \times P_{cl}(X)$ by

$$T(x, y) := T_1(x, y) \times T_2(x, y).$$

Then from the hypotheses of the theorem we get that

$$T(x^*, y^*) = T_1(x^*, y^*) \times T_2(x^*, y^*) = \{(x^*, y^*)\}.$$

So, T has at least one strict fixed point.
We denote $Z := X \times X$ and consider $\widetilde{d} : Z \times Z \to \mathbb{R}^2_+$,

$$\widetilde{d}((x, y), (u, v)) := \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix}.$$

Then from the hypotheses of the theorem, we have that for each $s := (x, y), t := (u, v) \in X \times X$ and each $z := (z_1, z_2) \in T(x, y)$, there exists $w := (w_1, w_2) \in T(u, v)$ satisfying the relation

$$\widetilde{d}(z, w) \le A \cdot \widetilde{d}(s, t),$$

which proves that T is a multivalued A-contraction.
The conclusions follow now from Theorem 2.6 and Theorem 2.9. □

Remark 2.16. Notice again that, if (X, d) is a metric space and $T : X \times X \to P(X)$ is a multivalued operator and we define

$$T_1(x, y) := T(x, y) \text{ and } T_2(x, y) := T(y, x),$$

then the above approach leads to some coupled fixed point theorems in the classical sense.

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