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Gravity-capillary water waves generated by multiple pressure distributions

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Dedicated to the memory of Professor Viorel Radu

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Abstract

Steady two-dimensional free-surface flows subjected to multiple localised pressure distributions are considered. The fluid is bounded below by a rigid bottom, and above by a free-surface, and is assumed to be inviscid and incompressible. The flow is assumed irrotational, and the effects of both gravity and surface tension are taken into account. Forced solitary wave solutions are found numerically, using boundary integral equation techniques, based on Cauchy integral formula. The integrodifferential equations are solved iteratively by Newton's method. The behaviour of the forced waves is determined by the Froude number, the Bond number, and the coefficients of the pressure forcings. Multiple families of solutions are found to exist for particular values of the Froude number; perturbations from a uniform stream, and perturbations from pure solitary waves. Elevation waves are only obtained in the case of a negatively forced pressure distribution.

Keywords: Nonlinear waves, gravity-capillary waves
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1. Introduction

The study of free-surface flows past disturbances in water of both finite and infinite depth has been developed continuously in the last centuries. Such disturbances occur in many different physical situations; they can be in the form of localised pressure distributions, caused for example by atmospheric disturbances due to high

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winds, submerged obstacles such as rocks on a river bed, partially submerged obstacles (i.e. fully submerged obstacles which do not touch the bottom) for instance a submarine, or surface piercing objects such as a boat. A frame of reference moving with the disturbance(s) is chosen and steady solitary wave solutions are sought.

In the case of a single submerged obstruction, four classical free-surface flows, which are uniform far upstream and/or far downstream, are known to exist. Fully nonlinear solutions for gravity waves have been obtained numerically, for example by Forbes and Schwartz [6], Forbes [7], Dias and Vanden-Broeck [3], Dias and Vanden-Broeck [4] and Binder, Dias and Vanden-Broeck [2]. To classify the solutions, we introduce the Froude number

$$F = \frac{U}{\sqrt{gH}} \tag{1.1}$$

where g is the acceleration due to gravity, and U and H describe the height and velocity of the uniform flow, either up or downstream. A flow is said to be supercritical if $F > 1$ and subcritical if $F < 1$. The first type of solution are solitary waves. They consist of a uniform supercritical flow, both up and downstream of the obstruction, with an elevation or depression wave over the obstruction. The second type can be characterised by a train of waves downstream of the disturbance with a uniform subcritical flow upstream. Hydraulic falls are the third type of solution, and like solitary waves, these are uniform both upstream and downstream of the obstruction. However, for a hydraulic fall, the flow upstream is subcritical, and there exists a gradual change from subcritical to supercritical flow over the obstruction. Finally, there exist generalised hydraulic falls, see Dias and Vanden-Broeck [4], which differ from hydraulic falls as they have a train of waves upstream of the obstacle. However, this last type of classical solution is unphysical in the case of a single submerged obstruction, as it does not satisfy the radiation condition (requiring that there is no energy coming from infinity, i.e. that there are no waves upstream). Dias and Vanden-Broeck [5] have shown that these solutions can become physically relevant when considered as the local flow over an obstacle, in a configuration involving at least one other obstacle further upstream.

When the effects of surface tension are included, the linearised theory shows that the behaviour of the gravity-capillary waves depends not only on the Froude number, but also on the Bond number τ, given by

$$\tau = \frac{T}{\rho g H^2} \tag{1.2}$$

where T describes the constant tension on the free-surface and ρ is the density of the fluid. If $\tau > \frac{1}{3}$ then we need $F < 1$ to obtain only localised disturbances of the free surface such as free or forced solitary waves to exist. If $\tau < \frac{1}{3}$ we need $F < F_c$ for free and forced solitary waves to exist, where $F_c < 1$ is the critical value of the Froude number marking the turning point of the dispersion curve in the F-k plane. Multiple forced solutions are found to exist for the same values of the Froude and Bond numbers, and the same pressure distribution. One family of solutions is a perturbation from a uniform stream, and the other(s), a perturbation from a pure solitary wave. Maleewong, Asavanant and Grimshaw [8] found that for strong surface tension ($\tau > \frac{1}{3}$) an elevation solitary wave exists, only if the coefficient of the forcing term is negative. It is a perturbation from a uniform stream. If the forcing is positive, the perturbation from the uniform stream is a depression. Depression waves perturbating from a pure depression solitary wave exist for both positive and negative forcing.

If the effects of surface tension are weak ($\tau < \frac{1}{3}$), the behaviour of the flow changes. Solutions take the form of envelope solitary waves. In contrast to the strong tension case, Maleewong, Asavanant and Grimshaw [9] found that both elevation and depression waves perturbating from pure solitary waves exist, for both positive and negative forcing. As in the case for strong surface tension, elevation waves perturbating from a uniform stream exist for negative forcing. Perturbations from a uniform stream for positive forcing, are depressions.

In this paper, we extend the work of Maleewong, Asavanant and Grimshaw [9] by including multiple disturbances in the flow in the form of localised pressure distributions. Binder, Dias and Vanden-Broeck [1] considered the case of free-surface flow over two submerged obstructions in a channel, but in the absence of surface tension. Here, both the effects of gravity and surface tension are included, and we look for solitary wave solutions.

The problem is formulated in §2, the numerical method is described in §3 and the numerical results are presented and discussed in §4. Finally, we conclude with a summary of our findings in §5.

2. Formulation

We consider a steady two-dimensional flow of an incompressible, inviscid fluid of constant density ρ, bounded below by a horizontal impermeable bed at $y = -H$, and above by a free-surface. Cartesian coordinates (x, y) are introduced such that the x-axis is aligned on the undisturbed free-surface. The influences of both gravity and surface tension are taken into account. g is taken to be the acceleration due to gravity acting in the negative y-direction, and T to be the constant tension on the free-surface. The free-surface is given by $y = \eta(x)$, and the fluid is subjected to a localised pressure distribution $P(x)$ on the free-surface. We choose a frame of reference traveling with this pressure distribution. Far downstream of the pressure distribution (as $x \to \infty$), the flow is assumed to become a uniform stream of constant depth H and constant velocity U. We non-dimensionalise the problem by taking H and U as unit length and unit velocity, respectively.

The flow is assumed to be irrotational, so we can introduce the velocity potential $\phi(x, y)$ such that $\mathbf{u} = \nabla\phi$. Incompressibility of the flow then implies that the velocity potential satisfies Laplace's equation in the flow domain:

$$\nabla^2\phi = 0 \tag{2.1}$$

The kinematic boundary conditions on the bottom of the channel $y = -1$ and on the free-surface $y = \eta(x)$ are given, respectively, by

$$\phi_y = 0 \tag{2.2}$$

$$\phi_y = \phi_x\eta_x \tag{2.3}$$

The dynamic boundary condition on the free-surface $y = \eta(x)$ is obtained by applying the Bernoulli equation to the fluid at the free-surface:

$$\frac{1}{2}(\phi_x^2 + \phi_y^2 - 1) + \frac{1}{F^2}\eta = \beta\frac{\eta_{xx}}{(\eta_x^2 + 1)^{\frac{3}{2}}} - P(x). \tag{2.4}$$

Here F is the Froude number defined in (1.1), $\beta = \frac{\tau}{F^2}$ where τ is the Bond number given by (1.2), and $P(x)$ is the pressure distribution. To ensure a uniform stream as $x \to \pm\infty$ we impose the condition:

$$\phi_x \to 1, \;\; \eta \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.5}$$

The problem then becomes that of finding the unknown functions $\phi(x, y)$ and $\eta(x)$ satisfying (2.1)-(2.4) and (2.5).

3. Numerical Scheme

We solve the problem numerically by first reformulating it as a system of integro-differential equations (see for example Vanden-Broeck and Dias [11] and Părău and Vanden-Broeck [10]). The stream function $\psi(x, y)$ is defined, such that $\mathbf{u} = \nabla\times\psi$. This means we can introduce the complex velocity potential function $f(z) = \phi(x, y) + i\psi(x, y)$. We now map the problem into the inverse plane, so that ϕ and ψ become the independent variables, and without loss of generality, we choose $\phi = 0$ at $x = 0$, and take $\psi = 0$ on the free-surface so that $\psi = -1$ on the channel bottom. The fluid thus occupies the infinite strip $-\infty < \phi < \infty$, $-1 < \psi < 0$ in the complex f-plane. We apply Cauchy's integral formula to the function $x_\phi - 1 + iy_\phi$, with a contour consisting of the free-surface, its reflection in the channel bottom, and vertical lines joining them at $\pm\infty$. Taking the real part, with evaluation point ϕ_o on the free-surface, we obtain:

$$\begin{aligned} x_\phi(\phi_o) - 1 = &-\frac{1}{\pi}\mathrm{int}_{-\infty}^{+\infty}\frac{y_\phi}{\phi-\phi_o}d\phi \\ &+\frac{1}{\pi}\mathrm{int}_{-\infty}^{+\infty}\frac{(\phi-\phi_o)y_\phi+2(x_\phi-1)}{(\phi-\phi_o)^2+4}d\phi \end{aligned} \tag{3.1}$$

The first integral in this equation is evaluated as a Cauchy principal value.

The dynamic condition (2.4) on the free-surface becomes:

$$\frac{1}{2}\left(\frac{1}{x_\phi^2+y_\phi^2}-1\right)+\frac{1}{F^2}y=\beta\frac{y_{\phi\phi}x_\phi-x_{\phi\phi}y_\phi}{(x_\phi^2+y_\phi^2)^{\frac{3}{2}}}-P \tag{3.2}$$

We choose

$$P(\phi)=\begin{cases}\epsilon_1\exp\frac{1}{(\phi-a)^2-1} & -1<|\phi+a|<1\\ \epsilon_2\exp\frac{1}{(\phi-b)^2-1} & -1<|\phi+b|<1\\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

where a and b give the position of the forcing on the ϕ-axis.

We solve the two equations (3.1) and (3.2) for the unknowns x_ϕ and y_ϕ. The potential function ϕ is discretized by introducing N equally spaced mesh-points

$$\phi(i)=-\frac{N}{2}e+(i-1)e \quad i=1,...,n \tag{3.4}$$

and the midpoints

$$\phi_m(i)=\phi(i)+e/2 \quad i=1,...,n-1 \tag{3.5}$$

where e is the interval of discretization. The integrals in (3.1) are truncated and approximated using the trapezoidal rule. Given a set of values for y_ϕ, the values of x_ϕ at the midpoints are then obtained by solving the discretized version of (3.1). A four point interpolation scheme is used to obtain the values of x_ϕ at the mesh points. The remaining unknowns in the problem are the values of y_ϕ. N equations are required to solve the problem, of which, $n-2$ are obtained by satisfying the dynamic condition (3.2) at the mesh-points (3.4) $i=2,...,n-1$. The remaining two equations come from enforcing

$$y(1)=y(n)=0 \tag{3.6}$$

which forces the elevation to be 0 as $x\to\pm\infty$, i.e. we enforce zero mean level. The system is solved iteratively using Newton's method. We denote $y_i'=y_\phi(\phi_i)$ for $i=1,...,n$ and all the nonlinear equations satisfied by $E_i(y_1',y_2',...,y_n')=0$ for $i=1,...,n$. We start the process with an initial guess for $\mathbf{y'}=[y_1',y_2',...,y_n']^T$ (usually $\mathbf{y'}=\mathbf{0}$). The vector $\mathbf{y'}$ is updated by adding a correction vector $\mathbf{\Delta}=[\Delta_1,\Delta_2,...,\Delta_n]^T$ to it at each iteration. The correction $\mathbf{\Delta}$ is the solution of of the matrix equation

$$\mathbf{J_E\Delta}=-\mathbf{E} \tag{3.7}$$

where $\mathbf{E}=[E_1,E_2,...,E_n]^T$ and $\mathbf{J_E}$ is the Jacobian of $\mathbf{E}$, $\mathbf{J_E}=\left(\frac{\partial E_i}{\partial y_j'}\right)_{i=1,...,n,j=1,...,n}$. The Jacobian matrix of partial derivatives $\mathbf{J_E}$ is evaluated numerically by forward differention and the matrix equation (3.7) is solved by LU-decomposition and backsubstitution.

To obtain the perturbations from a pure solitary wave, we first obtain a wave profile for the wave perturbating from the uniform stream, using the scheme as described above. Parameter continuation on the amplitude of the peak of one of the waves then allows us to follow the solution branch. After a critical value of the Froude number is reached, the perturbation from the uniform stream becomes a perturbation from a pure solitary wave. We check that decreasing the pressure forcing coefficient ϵ to zero after the turning point is reached, does result in a pure solitary wave.

4. Results

We restrict our attention to subcritical flows, $F < 1$, and concentrate on the case with strong surface tension $\tau > \frac{1}{3}$. Values for the parameters ϵ_1, ϵ_2, β and F are set, and the distance x_d between the pressure distributions is fixed. The solitary waves are calculated using the numerical scheme described in §3.

When $\epsilon_1 = \epsilon_2 = 0.05$, $x_d = 6$, $\beta = 0.49$, and $F = 0.886$, typical free-surface solitary wave profiles are shown in Figure 1. This result is similar to the wave profile found by Binder, Vanden-Broeck and Dias [1] for supercritical flow in the absence of surface tension, over two submerged triangular obstructions. The gravity-capillary waves obtained here are depression waves, whereas the gravity waves they obtained were elevation waves. The result is also similar to the classical gravity-capillary solitary wave profile over a single disturbance in the flow. As $\epsilon_1 \to 0, \epsilon_2 \to 0$ the broken curve solution reduces to a uniform stream. The solid curve solution tends toward the pure solitary wave solution.

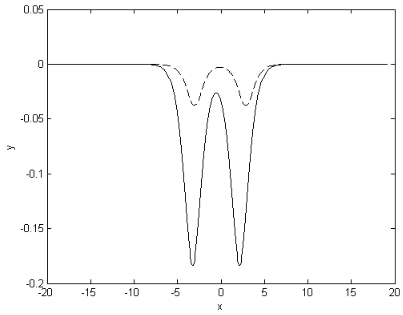

Figure 1: Typical fully nonlinear free-surface profiles for $\epsilon_1 = \epsilon_2 = 0.05$, $x_d = 6$, $\beta = 0.49$, and $F = 0.886$. The pressures are centred at $a = 3$,$b = -3$. The broken curve is a perturbation from a uniform stream and has maximum depression -0.037. The solid curve is a perturbation from a depression solitary wave solution, and has maximum amplitude -0.184.

Only depression waves are found for $\epsilon > 0$, and a turning point is obtained on the solution branch in the amplitude vs F plane. When $F > F^*$, the critical value of the Froude number corresponding to the turning point, no solutions are found to exist. For $F < F^{**}$, just one family of solutions is found; perturbations from a uniform stream. For $F^{**} < F < F^*$ we find two families of solution; perturbations from the uniform stream, and perturbations from a pure solitary wave. This is in agreement with the findings of Maleewong, Asavanant and Grimshaw [8] for flow disturbed by a single localised pressure distribution.

As the distance x_d between the obstacles decreases, the wave profile of the perturbation from pure depression solitary waves reduces to the classical free-surface profile obtained for flow over a single disturbance. See the solid curve in Figure 2. As the distance x_d increases, the wave profile becomes that of two classical solutions over two separate disturbances; the depressions resulting from the localised pressure distributions do not influence one another.

Typical negatively forced wave profiles, with $\epsilon_1 = \epsilon_2 = -0.01$ can be seen in Figure 3 for $F = 0.895$, $\beta = 0.49$. Both elevation and depression waves are obtained. The elevation waves are perturbations from a uniform stream, and are of much smaller amplitude than the depression waves, perturbating from a pure solitary wave solution. We obtain the depression waves by firstly finding the pure depression solitary wave curve, using parameter continuation on the amplitude of a positively forced flat stream depression perturbation. The coefficients ϵ_1 and ϵ_2 of the pressure distributions are then gradually decreased from zero, and parameter continuation on the amplitude of the depression is used to move along the negatively forced solution branch.

As $F \to 1$, the amplitude of the peak between the two depressions increases, and a flat region between the depressions' troughs is approached. This is illustrated in Figure 4 for $F = 0.958$, with pressure distributions centred at $a = 4$ and $b = -4$. As the distance x_d between the pressure distributions increases, the region of free-surface between the two troughs steepens, and approaches the level of the undisturbed flat stream far upstream. As x_d decreases, the region between the troughs flattens more quickly, and the amplitude of the

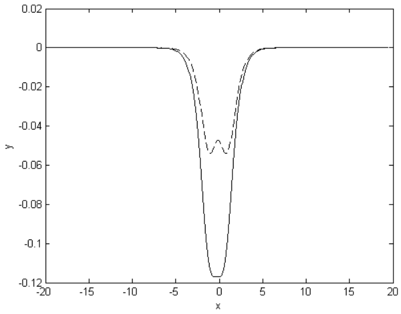

Figure 2: Fully nonlinear wave profiles for $\epsilon_1 = \epsilon_2 = 0.05$, $x_d = 2.4$, $\beta = 0.49$, and $F = 0.898$. The pressures are centred at $a = 1.2$,$b = -1.2$. The broken curve is a perturbation from a uniform stream and has maximum amplitude -0.0569. The solid curve is a perturbation from a classical depression solitary wave and has maximum amplitude -0.117.

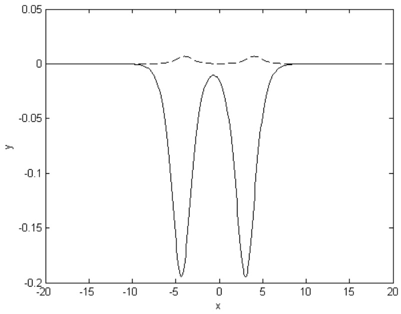

Figure 3: Typical negatively forced fully nonlinear wave profiles with $\epsilon_1 = \epsilon_2 = -0.01$, $x_d = 8$, $\beta = 0.49$, and $F = 0.895$. The pressures are centred at $a = 4$,$b = -4$. The broken curve is a perturbation from a uniform stream, with maximum elevation 0.0069. The depressions shown with a solid curve, are perturbations from a pure solitary wave solution. They have maximum amplitude -0.1946.

region gradually increases.

A non-symmetric solution can be found by taking $\epsilon_1 \neq \epsilon_2$. Setting $\epsilon_1 > 0$ and $\epsilon_2 < 0$ we obtain a depression wave near the positively forced pressure distribution, and an elevation wave near the negatively forced distribution, see Figure 5. This solution is similar, but of opposite orientation, to the solution obtained by Binder, Vanden-Broeck and Dias [1].

Using parameter continuation on the amplitude of the depression wave, we find a turning point for the curve in the amplitude vs F plane. The solution for this configuration is therefore also multivalued for some particular values of the Froude number. The wave profile given by the broken curve in Figure 5 is a perturbation from the uniform stream. The solid curve solution is more complex. After the turning point on the solution branch is reached, decreasing the pressure forcings to zero reduces the amplitude of the elevation to zero, but the depression wave reduces to a pure depression solitary wave. The elevation is therefore a perturbation from the uniform stream, but the depression is a perturbation from a pure depression solitary wave.

5. Conclusions

We have considered subcritical free-surface flows with a strong surface tension, subjected to two localised pressure distributions. Fully nonlinear numerical results, calculated using a boundary integral equation method based on Cauchy integral formula, have been presented. Multiple families of solitary wave solutions were found to exist for particular values of the Froude number. Depression waves perturbating from both a uniform stream and pure solitary waves have been found. Elevation waves were only found to exist in the case of a negatively forced pressure distribution.

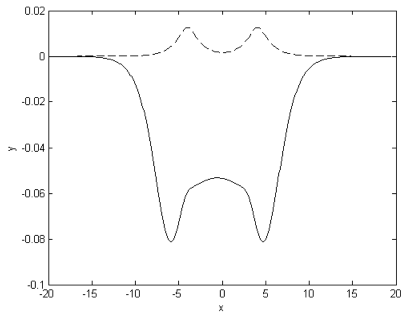

Figure 4: Negatively forced fully nonlinear free-surface profile with $\epsilon_1 = \epsilon_2 = -0.01$, $x_d = 8$, $\beta = 0.49$, and $F = 0.958$. The pressures are centred at $a = 4, b = -4$. The broken curve is a perturbation from a uniform stream, with maximum elevation 0.0126. The solid curve is a perturbation from a pure solitary wave solution. It has maximum amplitude -0.081.

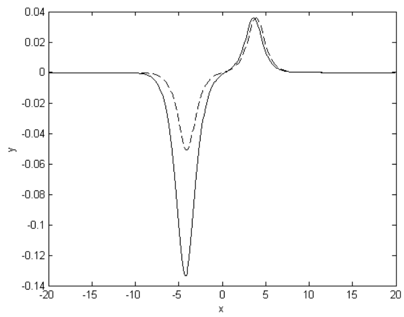

Figure 5: Fully nonlinear wave profiles for $\epsilon_1 = 0.05, \epsilon_2 = -0.05$, $x_d = 8$, $\beta = 0.49$, and $F = 0.91$. The pressures are centred at $a = 4, b = -4$. The broken curve is a perturbation from a uniform stream and has maximum amplitude -0.0509 and maximum elevation 0.0359. The depression wave on the solid curve is a perturbation from a pure depression solitary wave and has maximum amplitude -0.1337. The elevation wave is a perturbation from the uniform stream, with maximum elevation 0.0359.

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