Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 976–985
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Existence and nonexistence of solutions for nonlinear second order q-integro-difference equations with non-separated boundary conditions

Ravi P. Agarwal[a,c,*], Guotao Wang[b], Aatef Hobiny[c], Lihong Zhang[b], Bashir Ahmad[c]

[a] *Department of Mathematics, Texas A&M University, Kingsville, TX 78363-8202, USA.*
[b] *School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, People's Republic of China.*
[c] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.*

Abstract

In this paper, we investigate a nonlinear second order boundary value problem of q-integro-difference equations supplemented with non-separated boundary conditions. Sufficient conditions for the existence and nonexistence of solutions are presented. Examples are provided for explanation of the obtained work.

Keywords: q-integro-difference equations, non-separated boundary conditions, existence, nonexistence.
2010 MSC: 93C30, 93B18, 93A30, 39A99.

1. Introduction

Consider the following nonlinear second order q-integro-difference equation with non-separated boundary conditions:

$$\begin{cases} D_q^2 u(t) = f(t, u(t)) + I_q g(t, u(t)), & t \in I_q, \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T), \end{cases} \tag{1.1}$$

where $f, g \in C(I_q \times \mathbb{R}, \mathbb{R}),\ I_q = [0, T] \cap q^{\overline{N}},\ q^{\overline{N}} := \{q^n : n \in \mathbb{N}\} \cup \{0\},\ T \in q^{\overline{N}}$ is a fixed constant and $\eta \neq 1$ is a fixed real number.

*Corresponding author
Email addresses: ravi.agarwal@tamuk.edu (Ravi P. Agarwal), wgt2512@163.com (Guotao Wang), t.aatef@hotmail.com (Aatef Hobiny), zhanglih149@126.com (Lihong Zhang), bashirahmad_qau@yahoo.com (Bashir Ahmad)

Received 2015-05-29

The study of q-difference equations, initiated with the works of Jackson [18, 19], Carmichael [14], Mason [22] and Adams [1], has recently gained a considerable interest. The subject of q-calculus is also known as quantum calculus and distinguishes itself from the classical calculus in the sense that the notion of q-derivative is independent of the concept of limit and that q-difference equations are always completely controllable. The tools of q-calculus are found to be of a great value in studying q-optimal control problems [10]. The q-analogue of continuous variational calculus is variational q-calculus, where the extra-parameter q accounts for a physical or economical situation. In fact, the variational calculus on q-uniform lattice helps to find the extremum of the functional involved in Lagrange problems of q-Euler equations rather than solving the Euler-Lagrange equation itself [11]. The q-difference equations have potential applications in several fields such as special functions, super-symmetry, operator theory, combinatorics, etc. For examples and details, see a series of books ([7, 8, 16, 20]) and papers ([2, 12, 23]) and the references cited therein. Concerning the theory of initial and boundary value problems of q-difference equations, we refer the reader to the works obtained in papers ([3, 4, 5, 6, 9, 13, 15, 17, 24]).

In the sequel, we use the following conditions and notation:

(H_1) $\lim\limits_{|u|\to\infty}\dfrac{f(t,u)}{|u|}=a(t)$ and $\lim\limits_{|u|\to\infty}\dfrac{g(t,u)}{|u|}=b(t)$ uniformly on I_q.

(H_2) $\lim\limits_{|u|\to 0^+}\dfrac{f(t,u)}{|u|}=0$ and $\lim\limits_{|u|\to\infty}\dfrac{f(t,u)}{|u|}=0$ uniformly on I_q.

(H_3) $\lim\limits_{|u|\to 0^+}\dfrac{g(t,u)}{|u|}=0$ and $\lim\limits_{|u|\to\infty}\dfrac{g(t,u)}{|u|}=0$ uniformly on I_q.

$$\begin{aligned}\Lambda=\sup_{t\in I_q}\Big\{&\int_0^t|t-qs||a(s)|d_qs+\int_0^t\Big|\frac{t^2}{1+q}-qrt+\frac{q^3r^2}{1+q}\Big||b(r)|d_qr\\&+\frac{|\eta|}{(\eta-1)^2}\int_0^T|T+(1-\eta)(t-qs)|a(s)|d_qs\\&+\frac{|\eta|}{(\eta-1)^2}\int_0^T\Big|[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}\Big||b(r)|d_qr\Big\},\\\mathcal{M}=&\frac{T^2}{1+q}+\frac{T^3}{(1+q)(1+q+q^2)}+\frac{|\eta|T^2}{(\eta-1)^2}+\frac{|\eta|T^2}{|\eta-1|(1+q)}\\&+\frac{|\eta|(T+|1-\eta|T)T^2}{(\eta-1)^2(1+q)}+\frac{|\eta|q}{|\eta-1|(1+q+q^2)}.\end{aligned}$$

The main objective of the present paper is to establish the following results which deal with the existence and nonexistence of solutions for the problem (1.1).

Theorem 1.1. *Assume that the condition* (H_1) *holds. Then the problem* (1.1) *has at least one solution provided that* $0<\Lambda<1$.

Theorem 1.2. *Assume that the conditions* (H_2) *and* (H_3) *hold. If there exists a constant* $\mathcal{B}>0$ *such that* $\mathcal{M}<\mathcal{B}$, *then the non-separated boundary value problem* (1.1) *has no solution.*

2. Preliminaries

Let us describe some basic concepts of q-calculus [7, 16].

For $0<q<1$, the q-derivative of a real valued function f is defined as

$$D_qf(t)=\frac{f(t)-f(qt)}{(1-q)t},\quad D_qf(0)=\lim_{t\to 0}D_qf(t).$$

The q-integral of a function f is defined as

$$\int_a^x f(t)d_qt := \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n) - a(1-q)q^n f(aq^n), \quad x \in [a,b],$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t)d_qt = \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n),$$

provided the series converges. If $a \in [0,b]$ and f is defined on the interval $[0,b]$, then

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.1}$$

and if f is continuous at $x = 0$, then $I_q D_q f(x) = f(x) - f(0)$. In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t)h(t) + g(qt)D_q h(t), \tag{2.2}$$

$$\int_0^x f(t)D_q g(t)dqt = \Big[f(t)g(t)\Big]_0^x - \int_0^x D_q f(t)g(qt)d_qt. \tag{2.3}$$

We introduce the Banach space $X = C(I_q, \mathbb{R}) = \{u : I_q \to \mathbb{R} \mid u \in C(I_q)\}$ equipped with a topology of uniform convergence with respect to the norm $\|u\| = \sup_{t\in I_q} |u(t)|$.

Lemma 2.1 ([4])**.** *The linear problem of a second order q-difference equation supplemented with non-separated boundary conditions:*

$$\begin{cases} D_q^2 u(t) = y(t), & t \in I_q, \\ u(0) = \eta u(T), \quad D_q u(0) = \eta D_q u(T), \end{cases} \tag{2.4}$$

has a unique solution given by

$$u(t) = \int_0^t (t-qs)y(s)d_qs + \frac{\eta}{(\eta-1)^2}\int_0^T [T + (1-\eta)(t-qs)]y(s)d_qs. \tag{2.5}$$

To transform the problem (1.1) into a fixed point problem we use Lemma 2.1 to define an operator $T : X \to X$ as

$$\begin{aligned} (Tu)(t) &= \int_0^t (t-qs)[f(s,u(s)) + I_q g(s,u(s))]d_qs \\ &\quad + \frac{\eta}{(\eta-1)^2}\int_0^T [T + (1-\eta)(t-qs)][f(s,u(s)) + I_q g(s,u(s))]d_qs \\ &= \int_0^t (t-qs)f(s,u(s))d_qs + \int_0^t (t-qs)\int_0^s g(r,u(r))d_qrd_qs \\ &\quad + \frac{\eta}{(\eta-1)^2}\int_0^T [T + (1-\eta)(t-qs)]f(s,u(s))d_qs \\ &\quad + \frac{\eta}{(\eta-1)^2}\int_0^T [T + (1-\eta)(t-qs)]\int_0^s g(r,u(r))d_qrd_qs, \end{aligned} \tag{2.6}$$

which can alternatively be written as

$$
\begin{aligned}
(Tu)(t) = & \int_0^t (t-qs)f(s,u(s))d_qs + \int_0^t (\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q})g(r,u(r))d_qr \\
& + \frac{\eta}{(\eta-1)^2}\int_0^T [T+(1-\eta)(t-qs)]f(s,u(s))d_qs \\
& + \frac{\eta}{(\eta-1)^2}\int_0^T \Big[[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}\Big]g(r,u(r))d_qr.
\end{aligned} \tag{2.7}
$$

Observe that the problem (1.1) has a solution if and only if the operator T has a fixed point.

3. The Proof of Main Results

In order to obtain the proof of Theorem 1.1, we need the following fixed point theorem.

Theorem 3.1 ([21]). *Let X be a Banach Space. Let $T : X \to X$ be a completely continuous mapping and let $L : X \to X$ be a bounded linear mapping such that 1 is not an eigenvalue of L. Suppose that*

$$\lim_{\|u\|\to\infty} \frac{\|Tu - Lu\|}{\|u\|} = 0.$$

Then T has a fixed point in X.

Proof of Theorem 1.1.

In the first step, we show that T is a completely continuous operator. Obviously, the operator T is continuous in view of continuity of functions f and g.

Let $\Omega \subset C(I_q,\mathbb{R})$ be bounded. Then, for any $t \in I_q$, there exist positive constants L_1 and L_2 such that $|f(t,u)| \le L_1$ and $|g(t,u)| \le L_2, \forall u \in \Omega$. Then, we have

$$
\begin{aligned}
|(Tu)(t)| = & \Big| \int_0^t (t-qs)f(s,u(s))d_qs + \int_0^t (\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q})g(r,u(r))d_qr \\
& + \frac{\eta}{(\eta-1)^2}\int_0^T [T+(1-\eta)(t-qs)]f(s,u(s))d_qs \\
& + \frac{\eta}{(\eta-1)^2}\int_0^T \Big[[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}\Big]g(r,u(r))d_qr\Big| \\
\le & \int_0^t |t-qs||f(s,u(s))|d_qs + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q}||g(r,u(r))|d_qr \\
& + \frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)||f(s,u(s))|d_qs \\
& + \frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}||g(r,u(r))|d_qr \\
\le & \int_0^t |t-qs|L_1d_qs + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q}|L_2d_qr \\
& + \frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)|L_1d_qs \\
& + \frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}|L_2d_qr \\
\le & \sup_{t\in I_q}\Big\{\frac{t^2}{1+q}L_1 + \frac{t^3}{(1+q)(1+q+q^2)}L_2 + \frac{|\eta|T^2}{(\eta-1)^2}L_1 + \frac{|\eta|T^2}{|\eta-1|(1+q)}L_1 \\
& + \frac{|\eta|(T+|1-\eta|t)T^2}{(\eta-1)^2(1+q)}L_2 + \frac{|\eta|q}{|\eta-1|(1+q+q^2)}L_2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{T^2}{1+q}L_1 + \frac{T^3}{(1+q)(1+q+q^2)}L_2 + \frac{|\eta|T^2}{(\eta-1)^2}L_1 + \frac{|\eta|T^2}{|\eta-1|(1+q)}L_1 \\
&\quad + \frac{|\eta|(T+|1-\eta|T)T^2}{(\eta-1)^2(1+q)}L_2 + \frac{|\eta|q}{|\eta-1|(1+q+q^2)}L_2 := L,
\end{aligned}
$$

which implies that $\|Tu\| \leq L$. Moreover, for $\forall t_1, t_2 \in I_q,\ t_1 < t_2$, we obtain

$$
\begin{aligned}
&|(Tu)(t_2) - (Tu)(t_1)| \\
&= \Big| \int_0^{t_2} (t_2 - qs) f(s, u(s)) d_q s - \int_0^{t_1} (t_1 - qs) f(s, u(s)) d_q s \\
&\quad + \int_0^{t_2} (\frac{t_2^2}{1+q} - qrt_2 + \frac{q^3 r^2}{1+q}) g(r, u(r)) d_q r - \int_0^{t_1} (\frac{t_1^2}{1+q} - qrt_1 + \frac{q^3 r^2}{1+q}) g(r, u(r)) d_q r \\
&\quad + \frac{\eta(t_2 - t_1)}{(1-\eta)} \int_0^T f(s, u(s)) d_q s + \frac{\eta(t_2 - t_1)}{(1-\eta)} \int_0^T (T - qr) g(r, u(r)) d_q r \Big|, \\
&\leq \Big| (t_2 - t_1) \int_0^{t_1} f(s, u(s)) d_q s + \int_{t_1}^{t_2} (t_2 - qs) f(s, u(s)) d_q s \\
&\quad + (t_2 - t_1) \int_0^{t_1} (\frac{t_1 + t_2}{1+q} - qr) g(r.u(r)) d_q r + \int_{t_1}^{t_2} (\frac{t_2^2}{1+q} - qrt_2 + \frac{q^3 r^2}{1+q}) g(r, u(r)) d_q r \\
&\quad + \frac{\eta(t_2 - t_1)}{(1-\eta)} \int_0^T f(s, u(s)) d_q s + \frac{\eta(t_2 - t_1)}{(1-\eta)} \int_0^T (T - qr) g(r, u(r)) d_q r \Big|,
\end{aligned}
$$

which tends to zero independent of u as $t_2 - t_1 \to 0$. Thus, the operator T is relatively compact on Ω. Hence, by the Arzela-Ascoli Theorem, the operator T is compact on Ω. Hence, the operator T is completely continuous.

Next, we show that T has a fixed point. Define an operator $L : X \to X$ as

$$
\begin{aligned}
(Lu)(t) &= \int_0^t (t - qs) a(s) |u(s)| d_q s + \int_0^t (\frac{t^2}{1+q} - qrt + \frac{q^3 r^2}{1+q}) b(r) |u(r)| d_q r \\
&\quad + \frac{\eta}{(\eta-1)^2} \int_0^T [T + (1-\eta)(t - qs)] a(s) |u(s)| d_q s \\
&\quad + \frac{\eta}{(\eta-1)^2} \int_0^T \Big[[T + (1-\eta)t](T - qr) + (1-\eta) \frac{q^3 r^2 - qT^2}{1+q} \Big] b(r) |u(r)| d_q r.
\end{aligned}
\tag{3.1}
$$

Thus L is a bounded linear operator. In addition, we have

$$
\begin{aligned}
|(Lu)(t)| &= \Big| \int_0^t (t - qs) a(s) |u(s)| d_q s + \int_0^t (\frac{t^2}{1+q} - qrt + \frac{q^3 r^2}{1+q}) b(r) |u(r)| d_q r \\
&\quad + \frac{\eta}{(\eta-1)^2} \int_0^T [T + (1-\eta)(t - qs)] a(s) |u(s)| d_q s \\
&\quad + \frac{\eta}{(\eta-1)^2} \int_0^T \Big[[T + (1-\eta)t](T - qr) + (1-\eta) \frac{q^3 r^2 - qT^2}{1+q} \Big] b(r) |u(r)| d_q r \Big| \\
&\leq \int_0^t |t - qs| |a(s)| d_q s + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3 r^2}{1+q}| |b(r)| d_q r \\
&\quad + \frac{|\eta|}{(\eta-1)^2} \int_0^T |T + (1-\eta)(t - qs)| |a(s)| d_q s \\
&\quad + \frac{|\eta|}{(\eta-1)^2} \int_0^T |[T + (1-\eta)t](T - qr) + (1-\eta) \frac{q^3 r^2 - qT^2}{1+q}| |b(r)| d_q r
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\le \sup_{t\in I_q}\Big\{\int_0^t |t-qs||a(s)|d_qs+\int_0^t |\frac{t^2}{1+q}-qrt+\frac{q^3r^2}{1+q}||b(r)|d_qr \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)||a(s)|d_qs \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}||b(r)|d_qr\Big\}\|u\| \\
&=\Lambda\|u\|.
\end{aligned}
$$

This, together with $0<\Lambda<1$, for any u such that $u=Lu$, implies that

$$\|u\|=\|Lu\|\le\Lambda\|u\|<\|u\|,$$

which is a contradiction. This means that $\lambda=1$ is not an eigenvalue of the linear operator L.

In view of (H_1), for any $\varepsilon>0$, there exists a positive constant M such that for any $|u|>M$, we have $\Big|\frac{f(t,u)}{|u|}-a(t)\Big|<\varepsilon$ and $\Big|\frac{g(t,u)}{|u|}-b(t)\Big|<\varepsilon$ for any $t\in I_q$.
Thus $|f(t,u)-a(t)|u||<\varepsilon|u|$ and $|g(t,u)-b(t)|u||<\varepsilon|u|$ for any $t\in I_q$.

Thus, for $\varepsilon>0$ and $|u|>M$, we have

$$
\begin{aligned}
&|(Tu)(t)-(Lu)(t)| \\
&\le \int_0^t |t-qs||f(s,u(s))-a(s)|u(s)||d_qs \\
&\quad+\int_0^t |\frac{t^2}{1+q}-qrt+\frac{q^3r^2}{1+q}||g(r,u(r))-b(r)|u(r)||d_qr \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)||f(s,u(s))-a(s)|u(s)||d_qs \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}||g(r,u(r))-b(r)|u(r)||d_qr \\
&\le \Big\{\int_0^t |t-qs|d_qs+\int_0^t |\frac{t^2}{1+q}-qrt+\frac{q^3r^2}{1+q}|d_qr \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)|d_qs \\
&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}|d_qr\Big\}\varepsilon\|u\| \\
&\le \sup_{t\in I_q}\Big\{\frac{t^2}{1+q}+\frac{t^3}{(1+q)(1+q+q^2)}+\frac{|\eta|T^2}{(\eta-1)^2}+\frac{|\eta|T^2}{|\eta-1|(1+q)} \\
&\quad+\frac{|\eta|(T+|1-\eta|t)T^2}{(\eta-1)^2(1+q)}+\frac{|\eta|q}{|\eta-1|(1+q+q^2)}\Big\}\varepsilon\|u\| \\
&=\Big\{\frac{T^2}{1+q}+\frac{T^3}{(1+q)(1+q+q^2)}+\frac{|\eta|T^2}{(\eta-1)^2}+\frac{|\eta|T^2}{|\eta-1|(1+q)} \\
&\quad+\frac{|\eta|(T+|1-\eta|T)T^2}{(\eta-1)^2(1+q)}+\frac{|\eta|q}{|\eta-1|(1+q+q^2)}\Big\}\varepsilon\|u\| \\
&=\mathcal{M}\varepsilon\|u\|.
\end{aligned}
$$

This yields $\|Tu-Lu\|\le\mathcal{M}\varepsilon\|u\|$, that is, $\lim\limits_{\|u\|\to\infty}\frac{\|Tu-Lu\|}{\|u\|}=0$. Therefore, all the conditions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, there exists a fixed point u of the operator T

which corresponds to a solution the problem (1.1). This completes the proof.

The Proof of Theorem 1.2.

Proof. In view of (H_2), we have

$$\begin{cases} \forall\varepsilon_1 > 0 \text{ there is } \delta_1 > 0 \text{ such that if } 0 < |u| < \delta_1 \text{ then } \left|\dfrac{f(t,u)}{|u|} - 0\right| < \varepsilon_1, \text{ i.e. } |f(t,u)| < \varepsilon_1|u|, \\ \forall\varepsilon_2 > 0 \text{ there is } N_1 > 0 \text{ such that if } |u| > N_1 \text{ then } \left|\dfrac{f(t,u)}{|u|} - 0\right| < \varepsilon_2, \text{ i.e. } |f(t,u)| < \varepsilon_2|u|. \end{cases} \tag{3.3}$$

Without loss of generality, let $\delta_1 < N_1$ and

$$\varepsilon' = \max\Big\{\varepsilon_1, \varepsilon_2, \max\{\frac{f(t,u)}{|u|} : \delta_1 \le |u| \le N_1, t \in I_q\}\Big\}.$$

Then, for all $t \in I_q$ and $u \in \mathbb{R}$, we have $|f(t,u)| < \varepsilon'|u|$.

Similarly, using (H_3), we get

$$\begin{cases} \forall\varepsilon_3 > 0 \text{ there is } \delta_2 > 0 \text{ such that if } 0 < |u| < \delta_2 \text{ then } \left|\dfrac{g(t,u)}{|u|} - 0\right| < \varepsilon_3, \text{ i.e. } |g(t,u)| < \varepsilon_3|u|, \\ \forall\varepsilon_4 > 0 \text{ there is } N_2 > 0 \text{ such that if } |u| > N_2 \text{ then } \left|\dfrac{g(t,u)}{|u|} - 0\right| < \varepsilon_4, \text{ i.e. } |g(t,u)| < \varepsilon_4|u|. \end{cases} \tag{3.4}$$

Let $\delta_2 < N_2$ and

$$\varepsilon'' = \max\Big\{\varepsilon_3, \varepsilon_4, \max\{\frac{g(t,u)}{|u|} : \delta_2 \le |u| \le N_2, t \in I_q\}\Big\}.$$

Thus, for all $t \in I_q$ and $u \in \mathbb{R}$, we have $|g(t,u)| < \varepsilon''|u|$.

Let us pick $\varepsilon = \max\{\varepsilon', \varepsilon''\}$. Then, for any $t \in I_q$ and $u \in \mathbb{R}$, we obtain $|f(t,u)| < \varepsilon|u|,\ |g(t,u)| < \varepsilon|u|$ and

$$\begin{aligned} |(Tu)(t)| \le& \int_0^t |t-qs||f(s,u(s))|d_qs + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q}||g(r,u(r))|d_qr \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)||f(s,u(s))|d_qs \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr) + (1-\eta)\frac{q^3r^2-qT^2}{1+q}||g(r,u(r))|d_qr \\ \le& \int_0^t |t-qs|\varepsilon|u(s)|d_qs + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q}|\varepsilon|u(r)|d_qr \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)|\varepsilon|u(s)|d_qs \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr) + (1-\eta)\frac{q^3r^2-qT^2}{1+q}|\varepsilon|u(r)|d_qr \\ \le& \Big\{\int_0^t |t-qs|d_qs + \int_0^t |\frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q}|d_qr \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |T+(1-\eta)(t-qs)d_qs \\ &+ \frac{|\eta|}{(\eta-1)^2}\int_0^T |[T+(1-\eta)t](T-qr) + (1-\eta)\frac{q^3r^2-qT^2}{1+q}|d_qr\Big\}\varepsilon\|u\| \end{aligned}$$

$$\begin{aligned}&\leq \sup_{t\in I_q}\Big\{\frac{t^2}{1+q}+\frac{t^3}{(1+q)(1+q+q^2)}+\frac{|\eta|T^2}{(\eta-1)^2}+\frac{|\eta|T^2}{|\eta-1|(1+q)}\\&\quad+\frac{|\eta|(T+|1-\eta|t)T^2}{(\eta-1)^2(1+q)}+\frac{|\eta|q}{|\eta-1|(1+q+q^2)}\Big\}\varepsilon\|u\|\\&=\mathcal{M}\varepsilon\|u\|,\end{aligned}$$

which implies that

$$\|Tu\|\leq \mathcal{M}\varepsilon\|u\|. \tag{3.5}$$

Take $\mathcal{B}=\frac{1}{\varepsilon}(\mathcal{B}>\mathcal{M})$. If u is a solution of the non-separated boundary value problem (1.1), then u is a fixed point of operator T. Thus, $\|u\|=\|Tu\|$. This, together with (3.5) and $\mathcal{M}<\mathcal{B}$, yields

$$\|u\|=\|Tu\|\leq \mathcal{M}\varepsilon\|u\|<\|u\|,$$

which is a contradiction. That is, u is not a solution of the non-separated boundary value problem (1.1). This completes the proof. □

4. Examples

Example 4.1. Consider the following non-separated boundary value problem given by

$$\begin{cases}D_{\frac{1}{2}}u(t)=3t^3+t^2\sin u(t)+\frac{t^2}{7}|u(t)|+\int_0^t(2s^3+\frac{s^5}{u^2(s)}+\frac{s^2}{4}|u(s)|)d_{\frac{1}{2}}s, & t\in[0,1]_{\frac{1}{2}},\\ u(0)=-2u(1), \qquad D_{\frac{1}{2}}u(0)=-2D_{\frac{1}{2}}u(1),\end{cases} \tag{4.1}$$

Here, $q=\frac{1}{2}$, $\eta=-2$, $T=1$, $f(t,u)=3t^3+t^2\sin u+\frac{t^2}{7}|u|$ and $g(t,u)=2t^3+\frac{t^5}{u^2}+\frac{t^2}{4}|u|$.

Obviously,

$$\lim_{|u|\to\infty}\frac{f(t,u)}{|u|}=\lim_{|u|\to\infty}\frac{3t^3+t^2\sin u+\frac{t^2}{7}|u|}{|u|}=\frac{t^2}{7},$$

$$\lim_{|u|\to\infty}\frac{g(t,u)}{|u|}=\lim_{|u|\to\infty}\frac{2t^3+\frac{t^5}{u^2}+\frac{t^2}{4}|u|}{|u|}=\frac{t^2}{4}.$$

For $a(t)=\frac{t^2}{7}, b(t)=\frac{t^2}{4}$, we have

$$\begin{aligned}\Lambda&=\sup_{t\in I_q}\Big\{\int_0^t|t-qs|\frac{s^2}{7}d_qs+\int_0^t|\frac{t^2}{1+q}-qrt+\frac{q^3r^2}{1+q}|\frac{r^2}{4}d_qr\\&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T|T+(1-\eta)(t-qs)|\frac{s^2}{7}d_qs\\&\quad+\frac{|\eta|}{(\eta-1)^2}\int_0^T|[T+(1-\eta)t](T-qr)+(1-\eta)\frac{q^3r^2-qT^2}{1+q}|\frac{r^2}{4}d_qr\Big\}\\&\leq\sup_{t\in[0,1]_{\frac{1}{2}}}\Big\{\frac{1}{7}\int_0^t|t-\frac{1}{2}s|s^2d_{\frac{1}{2}}s+\frac{1}{4}\int_0^t|\frac{2t^2}{3}-\frac{1}{2}rt+\frac{r^2}{12}|r^2d_{\frac{1}{2}}r\\&\quad+\frac{2}{63}\int_0^1|1+3(t-\frac{1}{2}s)|s^2d_{\frac{1}{2}}s+\frac{1}{18}\int_0^1|3(1-\frac{r}{2})t+\frac{r}{2}(\frac{r}{2}-1)|r^2d_{\frac{1}{2}}r\Big\}\\&<\frac{2}{21}+\frac{1}{7}+\frac{2}{21}+\frac{8}{21}=\frac{15}{21}.\end{aligned}$$

Thus, all the conditions of Theorem 1.1 are satisfied. Hence, we conclude that the problem (4.1) has at least one solution.

Example 4.2. Consider the following non-separated boundary value problem

$$\begin{cases} D_{\frac{1}{3}}u(t) = t^3 \sin u(t)(1-\cos u(t)) + \int_0^t (s^2u^2(s)e^{-|u(s)|})d_{\frac{1}{3}}s, \quad t \in [0,1]_{\frac{1}{3}}, \\ u(0) = \frac{1}{5}u(1), \qquad D_{\frac{1}{3}}u(0) = \frac{1}{5}D_{\frac{1}{3}}u(1). \end{cases} \tag{4.2}$$

Here, $q = \frac{1}{3}$, $\eta = \frac{1}{5}$, $T = 1$, $f(t,u) = t^3(1-\cos u)\sin u$ and $g(t,u) = t^2u^2e^{-|u|}$.

Note that

$$\begin{cases} \lim_{|u|\to 0^+} \frac{f(t,u)}{|u|} = \lim_{|u|\to 0^+} \frac{t^3(1-\cos u)\sin u}{|u|} = 0, \\ \lim_{|u|\to \infty} \frac{f(t,u)}{|u|} = \lim_{|u|\to \infty} \frac{t^3(1-\cos u)\sin u}{|u|} = 0, \\ \lim_{|u|\to 0^+} \frac{g(t,u)}{|u|} = \lim_{|u|\to 0^+} \frac{t^2|u|}{e^{|u|}} = 0, \\ \lim_{|u|\to \infty} \frac{g(t,u)}{|u|} = \lim_{|u|\to \infty} \frac{t^2|u|}{e^{|u|}} = 0. \end{cases} \tag{4.3}$$

Clearly all the conditions of Theorem 1.2 hold. Consequently, the non-separated boundary value problem (4.2) has no solution.

Acknowledgment

The project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, under grant no. 65-130-35-HiCi. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] C. R. Adams, *On the linear ordinary q-difference equation*, Ann. Math., **30** (1928), 195–205. 1
[2] O. Agrawal, *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Calc. Appl. Anal., **15** (2012), 700–711. 1
[3] B. Ahmad, S. K. Ntouyas, *Boundary value problems for q-difference inclusions*, Abstr. Appl. Anal., **2011** (2011), 15 pages. 1
[4] B. Ahmad, A. Alsaedi, S. K. Ntouyas, *A study of second-order q-difference equations with boundary conditions*, Adv. Difference Equ., **2012** (2012), 10 pages. 1, 2.1
[5] B. Ahmad, J. J. Nieto, *Basic theory of nonlinear third-order q-difference equations and inclusions*, Math. Model. Anal., **18** (2013), 122–135. 1
[6] B. Ahmad, S. K. Ntouyas, *Boundary value problems for q-difference equations and inclusions with nonlocal and integral boundary conditions*, Math. Model. Anal., **19** (2014), 647–663. 1
[7] M. H. Annaby, Z. S. Mansour, *q-Fractional Calculus and Equations*, Lecture Notes in Mathematics 2056, Springer-Verlag, (2012). 1, 2
[8] A. Aral, V. Gupta, R .P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York, (2013). 1
[9] I. Area, N. Atakishiyev, E. Godoy, J. Rodal, *Linear partial q-difference equations on q-linear lattices and their bivariate q-orthogonal polynomial solutions*, Appl. Math. Comput., **223** (2013), 520–536. 1
[10] G. Bangerezako, *q-difference linear control systems*, J. Difference Equ. Appl., **17** (2011), 1229–1249. 1
[11] G. Bangerezako, *Variational q-calculus*, J. Math. Anal. Appl., **289** (2004), 650–665. 1
[12] Z. Bartosiewicz, E. Pawluszewicz, *Realizations of linear control systems on time scales*, Control Cybernet., **35** (2006), 769–786. 1
[13] M. Bohner, R. Chieochan, *Floquet theory for q-difference equations*, Sarajevo J. Math., **8** (2012), 355–366. 1
[14] R. D. Carmichael, *The general theory of linear q-difference equations*, Amer. J. Math., **34** (1912), 147–168. 1

[15] M. El-Shahed, H. A. Hassan, *Positive solutions of q-difference equation*, Proc. Amer. Math. Soc., **138** (2010), 1733–1738. 1

[16] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, (1990). 1, 2

[17] M. E. H. Ismail, P. Simeonov, *q-difference operators for orthogonal polynomials*, J. Comput. Appl. Math., **233** (2009), 749–761. 1

[18] F. H. Jackson, *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinburgh, **46** (1909), 253–281. 1

[19] F. H. Jackson, *On q-difference equations*, Amee. J. Math., **32** (1910), 305–314. 1

[20] V. Kac, P. Cheung, *Quantum Calculus*, Springer, New York, (2002). 1

[21] M. A. Krasnosel'ski, P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, Berlin, (1984). 3.1

[22] T. E. Mason, *On properties of the solutions of linear q-difference equations with entire function coefficients*, Amer. J. Math., **37** (1915), 439–444. 1

[23] D. Mozyrska, Z. Bartosiewicz, *On Observability of Nonlinear Discrete-Time Fractional-Order Control Systems*, New Trends in Nanotechnology and Fractional Calculus Applications, (2010), 305–312. 1

[24] N. Pongarm, S. Asawasamrit, J. Tariboon, *Sequential derivatives of nonlinear q-difference equations with three-point q-integral boundary conditions*, J. Appl. Math., **2013** (2013), 9 pages. 1