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A fixed point theorem in generalized ordered metric spaces with application

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Abstract

In this paper, we consider the concept of Ω-distance on a complete, partially ordered G-metric space and prove a fixed point theorem for (ψ, ϕ)-Weak contraction. Then, we present some applications in integral equations.

Keywords: Ω-distance; fixed point; G-metric space; (ψ, ϕ)-Weak contraction.
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1. Introduction and Preliminaries

The Banach fixed point theorem for contraction mapping has been generalized and extended in many direction [[3]-[11]],[18],[20],[27]. Nieto and Rodriguez-Lopez [18], Ran and Reurings [23] and Petrusel and Rus [21] presented some new results for contractions in partially ordered metric spaces. The main idea in [18, 19, 23] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. In [7], Dutte, presented the concept of (ψ, ϕ)-Weak contraction which includes the generalizations Theorem (1.2) in [13] and Theorem (1.4) in [24]. Also, Mustafa and sims [15] introduced the concept of G-metric. Some authors [2, 14, 16, 26] have proved some fixed point theorems in these spaces. Aage [1], proved a fixed point theorem for weak contraction in G-metric space. Recently, Saadati et al. [25], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of ω-distance due to Kada et al. [12].
In this paper, inspire of [12] we prove a fixed point theorem for (ψ, ϕ)-Weak contraction in generalized partially ordered metric spaces.
At first we recall some definitions and lemmas. For more information see [2, 7, 14, 15, 17, 22].

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Definition 1.1. ([15]) Let X be a non-empty set. A function $G : X \times X \times X \longrightarrow [0, \infty)$ is called a G-metric if the following conditions are satisfied:

(i) $G(x, y, z) = 0$ if $x = y = z$ (coincidence),
(ii) $G(x, x, y) > 0$ for all $x, y \in X$, where $x \neq y$,
(iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,
(iv) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 1.2. ([15]) Let (X, G) be a G-metric space,

(1) a sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$.
(2) a sequence $\{x_n\}$ in X is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, \geq n_0$, $G(x_m, x_n, x) < \varepsilon$.

Definition 1.3. ([25]) Let (X, G) be a G-metric space. Then a function $\Omega : X \times X \times X \longrightarrow [0, \infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z)$ for all $x, y, z, a \in X$,
(b) for any $x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \to [0, \infty)$ are lower semi-continuous,
(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \varepsilon$.

Example 1 : Let (X, d) be a metric space and $G : X^3 \longrightarrow [0, \infty)$ defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},$$

for all $x, y, z \in X$. Then $\Omega = G$ is an Ω-distance on X.

Example 2 : Let $X = \mathbb{R}$ and consider the G-metric G defined by

$$G(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid y - z \mid + \mid x - z \mid),$$

for all $x, y, z \in \mathbb{R}$. Then $\Omega : \mathbb{R}^3 \longrightarrow [0, \infty)$ defined by

$$\Omega(x, y, z) = \frac{1}{3}(\mid x - y \mid + \mid z - x \mid),$$

for all $x, y, z \in \mathbb{R}$ is an Ω-distance on $\mathbb{R}$.

For more examples see [25].

Lemma 1.4. *([25]) Let X be a metric space with metric G and Ω be an Ω-distance on X. Let x_n, y_n be sequences in X, α_n, β_n be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:*

(1) If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$;
(2) If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$ then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$;
(3) If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then x_n is a G-Cauchy sequence;
(4) If $\Omega(x_n, a, a) \leq \alpha_n$ for any $n \in \mathbb{N}$ then x_n is a G-Cauchy sequence.

2. Main results

Definition 2.1. Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of X into itself. We say that T is non-decreasing if for $x, y \in X$,

$$x \leq y \Longrightarrow T(x) \leq T(y).$$

Definition 2.2. Let $\Phi = \{\phi | \phi : [0, \infty) \to [0, \infty)\}$ and $\Psi = \{\psi | \psi : [0, \infty) \to [0, \infty)\}$ be the set of continuous, non-decreasing functions with $\phi^{-1}(0) = \psi^{-1}(0) = 0$.

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered space. Suppose there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that*

$$\psi(\Omega(Tx, Ty, Tz)) \leq \psi(\Omega(x, y, z)) - \phi(\Omega(x, y, z)), \qquad \forall x \leq y, z \in X$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Also, for every $x \in X$

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, Tx, y) : x \leq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point. Moreover, if $v = Tv$, then $\Omega(v, v, v) = 0$.

Proof. If $x_0 = Tx_0$, then the proof is finished. Suppose that $x_0 \neq Tx_0$. Since $x_0 \leq Tx_0$ and T is non-decreasing, we obtain

$$x_0 \leq Tx_0 \leq T^2x_0 \leq ... \leq T^{n+1}x_0 \leq ... .$$

Now if for some $n \in \mathbb{N}$, $\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = 0$ then,

$$\begin{aligned} \psi(\Omega(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0)) &\leq \psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)) \\ &- \phi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)), \end{aligned}$$

therefore, $\Omega(T^{n+1}x_0, T^{n+2}x_0, T^{n+2}x_0) = 0$, and by Part (c) of Definition (1.3), $G(T^nx_0, T^{n+2}x_0, T^{n+2}x_0) = 0$ and consequently $T^nx_0 = T^{n+2}x_0$, which implies T^nx_0 is a fixed point of T If n is even, and T^2x_0 is a fixed point of T if n is odd, then proof is complete.
Otherwise $\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) > 0$, for all $n \in \mathbb{N}$ and we have

$$\begin{aligned} \psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)) &\leq \psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)) \\ &- \phi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)). \end{aligned} \tag{2.1}$$

Then,

$$\psi(\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)) \leq \psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)).$$

Similarly,

$$\psi(\Omega(T^{n-1}x_0, T^nx_0, T^nx_0)) \leq \psi(\Omega(T^{n-2}x_0, T^{n-1}x_0, T^{n-1}x_0)).$$

This shows that $\{\Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0)\}$ is non-increasing. Then, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = r.$$

If $r > 0$, then $\phi(r) > 0$ and by taking $n \to \infty$ on (2.1), we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contraction. So,

$$\lim_{n\to\infty} \Omega(T^nx_0, T^{n+1}x_0, T^{n+1}x_0) = 0.$$

We claim that $\{T^n x_0\}$ is a G-Cauchy sequence. Suppose $\{T^n x_0\}$ is not a G-Cauchy sequence. Then, there exists $\varepsilon > 0$ and subsequences $\{T^{n_k} x_0\}$ and $\{T^{m_k} x_0\}$ such that n_k is the smallest integer with $n_k > m_k > k$ and

$$\Omega(T^{m_k} x_0, T^{n_k} x_0, T^{n_k} x_0) > \varepsilon.$$

Then,

$$\Omega(T^{m_k} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0) \leq \varepsilon.$$

By Part (a) of Definition (1.3), we obtain

$$\begin{aligned} \varepsilon &< \Omega(T^{m_k} x_0, T^{n_k} x_0, T^{n_k} x_0) \\ &\leq \Omega(T^{m_k} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0) + \Omega(T^{n_k-1} x_0, T^{n_k} x_0, T^{n_k} x_0) \\ &\leq \varepsilon + \Omega(T^{n_k-1} x_0, T^{n_k} x_0, T^{n_k} x_0). \end{aligned}$$

Thus,

$$\lim_{k\to\infty} \Omega(T^{m_k} x_0, T^{n_k} x_0, T^{n_k} x_0) = \varepsilon.$$

Since,

$$\begin{aligned} \Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0) &\leq \Omega(T^{m_k-1} x_0, T^{m_k} x_0, T^{m_k} x_0) \\ &+ \Omega(T^{m_k} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0), \end{aligned}$$

and,

$$\begin{aligned} \psi(\varepsilon) &< \psi(\Omega(T^{m_k} x_0, T^{n_k} x_0, T^{n_k} x_0)) \\ &\leq \psi(\Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0)) - \phi(\Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0)) \\ &< \psi(\Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0)), \end{aligned}$$

then, we obtain

$$\lim_{k\to\infty} \Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0) = \varepsilon.$$

Again, we have

$$\begin{aligned} \psi(\varepsilon) &< \psi(\Omega(T^{m_k} x_0, T^{n_k} x_0, T^{n_k} x_0)) \\ &\leq \psi(\Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0)) - \phi(\Omega(T^{m_k-1} x_0, T^{n_k-1} x_0, T^{n_k-1} x_0)). \end{aligned}$$

So, $\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon)$, which is a contradiction. Therefore $\{T^n x_0\}$ is a G-Cauchy sequence. Since X is G-complete, $\{T^n x_0\}$ converges to a point $u \in X$. Now, for $\varepsilon > 0$ and by lower semi-continuity of Ω,

$$\Omega(T^n x_0, T^m x_0, u) \leq \liminf_{p\to\infty} \Omega(T^n x_0, T^m x_0, T^p x_0) \leq \varepsilon, \qquad m \geq n$$

and,

$$\Omega(T^n x_0, u, T^l x_0) \leq \liminf_{p\to\infty} \Omega(T^n x_0, T^p x_0, T^l x_0) \leq \varepsilon, \qquad l \geq n.$$

Assume that $u \neq Tu$. Since $T^n x_0 \leq T^{n+1} x_0$,

$$0 < \inf\{\Omega(T^n x_0, u, T^n x_0) + \Omega(T^n x_0, u, T^{n+1} x_0) + \Omega(T^n x_0, T^{n+1} x_0, u) : n \in \mathbb{N}\} \leq 3\varepsilon,$$

which is a contraction. Therefore, we have $u = Tu$.
To prove the uniqueness, let v be another fixed point of T, then

$$\begin{aligned} \psi(\Omega(u, u, v)) &= \psi(\Omega(Tu, Tu, Tv)) \\ &\leq \psi(\Omega(u, u, v)) - \phi(\Omega(u, u, v)) \\ &< \psi(\Omega(u, u, v)), \end{aligned}$$

which is a contraction. Therefore, the fixed point u is unique. Now, if $v = Tv$, we have,

$$\begin{aligned}\psi(\Omega(v,v,v)) &= \psi(\Omega(Tv,Tv,Tv)) \\ &\leq \psi(\Omega(v,v,v)) - \phi(\Omega(v,v,v)).\end{aligned}$$

So, $\Omega(v,v,v) = 0$. □

Example 2.4. Let $X = [0,1]$ and $G(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid y-z \mid + \mid x-z \mid)$. Then (X, G) is a complete G-metric space. Suppose $\Omega(x,y,z) = \frac{1}{3}(\mid x-y \mid + \mid z-x \mid)$, $T(x) = \frac{x}{3}$, $\phi(t) = 3t$ and $\psi(t) = 9t$. Then,

$$\begin{aligned}\psi(\Omega(Tx,Ty,Tz)) &= \psi(\frac{1}{3}(\mid Tx-Ty \mid + \mid Tz-Tx \mid)) \\ &= \psi(\frac{1}{3}(\mid \frac{x}{3}-\frac{y}{3} \mid + \mid \frac{z}{3}-\frac{x}{3} \mid)) \\ &= \mid x-y \mid + \mid z-x \mid \\ &\leq \psi(\frac{1}{3}(\mid x-y \mid + \mid z-x \mid)) - \phi(\frac{1}{3}(\mid x-y \mid + \mid z-x \mid)) \\ &= \psi(\Omega(x,y,z)) - \phi(\Omega(x,y,z)),\end{aligned}$$

also, for every $x \in X$

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. So, by Theorem 2.3, T has a unique fixed point that is 0.

Denote by Λ the set all functions $\lambda : [0,+\infty) \to [0,+\infty)$ satisfying the following hypotheses:

(i) λ is a Lebesgue-integrable mapping on each compact subset of $[0,+\infty)$,
(ii) for every $\varepsilon > 0$, we have $\int_0^\varepsilon \lambda(s)ds > 0$,
(iii) $\|\lambda\| < 1$, where $\|\lambda\|$ denotes to the norm of λ.

Now, we have the following corollary.

Corollary 2.5. *Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on X such that (X, G) is a complete G-metric space and Ω is an Ω-distance on X and T is a non-decreasing mapping from X into itself. Suppose that for all $x \leq y, z \in X$,*

$$\int_0^{\psi(\Omega(Tx,Ty,Tz))} \lambda(s)ds \leq \int_0^{\psi(\Omega(x,y,z))} \lambda(s)ds - \int_0^{\phi(\Omega(x,y,z))} \lambda(s)ds, \tag{3.1}$$

where $\lambda \in \Lambda$. Also, for every $x \in X$

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\} > 0,$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a unique fixed point.

Proof. Define $\gamma : [0,+\infty) \to [0,+\infty)$ by $\gamma(t) = \int_0^t \lambda(s)ds$, then from inequality (3.1), we have

$$\gamma(\psi(\Omega(Tx,Ty,Tz))) \leq \gamma(\psi(\Omega(x,y,z))) - \gamma(\phi(\Omega(x,y,z))),$$

which can be written as

$$\psi_1(\Omega(Tx,Ty,Tz)) \leq \psi_1(\Omega(x,y,z)) - \phi_1(\Omega(x,y,z)),$$

where $\psi_1 = \gamma \circ \psi$ and $\phi_1 = \gamma \circ \phi$. Since the functions ψ_1 and ϕ_1 satisfy the properties of ψ and ϕ, by Theorem 2.3, T has a unique fixed point. □

3. Application

In this section, we give an existence theorem for a solution of the following integral equations:

$$x(t) = \int_0^1 K(t,s,x(s))ds + g(t), \qquad t \in [0,1]. \tag{3.1}$$

Let $X = C([0,1])$ be the set all continuous functions defined on $[0,1]$. Define $G : X \times X \times X \to \mathbb{R}$ by

$$G(x,y,z) = \|x-y\| + \|y-z\| + \|z-x\|,$$

where $\|x\| = \sup\{|\, x(t)\,| : t \in [0,1]\}$. Then (X, G) is a complete G-metric space. Let $\Omega = G$. Then Ω is an Ω-distance on X. Define an ordered relation $\leq$ on X by

$$x \leq y \quad iff \quad x(t) \leq y(t), \qquad \forall t \in [0,1].$$

Then $(X, \leq)$ is a partially ordered set. Now, we prove the following result.

Theorem 3.1. *Suppose the following hypotheses hold:*

(1) $K : [0,1] \times [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : [0,1] \to \mathbb{R}$ *are continuous mappings,*
(2) K *is nondecreasing in its first coordinate and* g *is nondecreasing,*
(3) There exists a continuous function $G : [0,1] \times [0,1] \to [0,+\infty)$ *such that*

$$|\, K(t,s,u) - K(t,s,v)\,| \leq G(t,s)\,|\, u - v\,|,$$

for every comparable $u, v \in \mathbb{R}^+$ *and* $s, t \in [0,1]$ *with* $\sup_{t\in[0,1]} \int_0^1 G(t,s)ds \leq \frac{1}{2}$,
(4) There exist continuous, non-decreasing functions $\phi, \psi : [0,\infty) \to (0,\infty)$ *with* $\psi^{-1}(0) = \phi^{-1}(0) = 0$ *and* $\psi(r) \leq \psi(2r) - \phi(2r)$ *for all* $r \in [0,\infty)$.

Then the integral equation has a solution in $C([0,1])$.

Proof. Define $Tx(t) = \int_0^1 K(t,s,x(s))ds + g(t)$. By hypothesis (2), we have that T is nondecreasing. Now, if

$$\inf\{\Omega(x,y,x) + \Omega(x,y,Tx) + \Omega(x,Tx,y) : x \leq Tx\} = 0,$$

for every $y \in X$ with $y \neq Ty$, then for each $n \in \mathbb{N}$, there exists $x_n \in C([0,1])$ with $x_n \leq Tx_n$ such that

$$\Omega(x_n,y,x_n) + \Omega(x_n,y,Tx_n) + \Omega(x_n,Tx_n,y) \leq \frac{1}{n}.$$

Then, we have

$$\Omega(x_n,y,Tx_n) = \sup_{t\in[0,1]} |\, x_n - y\,| + \sup_{t\in[0,1]} |\, y - Tx_n\,| + \sup_{t\in[0,1]} |\, Tx_n - x_n\,| \leq \frac{1}{n}.$$

Thus,

$$\lim_{n\to\infty} x_n(t) = y(t),$$
$$\lim_{n\to\infty} Tx_n(t) = y(t).$$

By the continuity of K, we have

$$\begin{aligned} y(t) &= \lim_{n\to\infty} Tx_n(t) = \int_0^1 K(t,s,\lim_{n\to\infty} x_n(s))ds + g(t) \\ &= \int_0^1 K(t,s,y(s))ds + g(t) = Ty(t). \end{aligned}$$

Which is a contradiction. Therefore,

$$\inf\{\Omega(x,y,x)+\Omega(x,y,Tx)+\Omega(x,Tx,y): x\leq Tx\}>0.$$

Now, for $x,y,z\in X$ with $x\leq y$, we have

$$\begin{aligned}
\psi(\Omega(Tx,Ty,Tz)) &= \psi(\sup_{t\in[0,1]} \mid Tx(t)-Ty(t)\mid + \sup_{t\in[0,1]} \mid Ty(t)-Tz(t)\mid \\
&+ \sup_{t\in[0,1]} \mid Tz(t)-Tx(t)\mid) \\
&\leq \psi(\sup_{t\in[0,1]}\int_0^1 \mid K(t,s,x(s))-K(t,s,y(s))\mid ds \\
&+ \sup_{t\in[0,1]}\int_0^1 \mid K(t,s,y(s))-K(t,s,z(s))\mid ds \\
&+ \sup_{t\in[0,1]}\int_0^1 \mid K(t,s,z(s))-K(t,s,x(s))\mid ds) \\
&\leq \psi(\sup_{t\in[0,1]}(\int_0^1 G(t,s)\mid x(s)-y(s)\mid ds)+\sup_{t\in[0,1]}(\int_0^1 G(t,s)\mid y(s)-z(s)\mid ds) \\
&+ \sup_{t\in[0,1]}(\int_0^1 G(t,s)\mid z(s)-x(s)\mid ds)) \\
&\leq \psi(\sup_{t\in[0,1]}(\mid x(t)-y(t)\mid)\sup_{t\in[0,1]}\int_0^1 G(t,s)ds \\
&+ \sup_{t\in[0,1]}(\mid y(t)-z(t)\mid)\sup_{t\in[0,1]}\int_0^1 G(t,s)ds \\
&+ \sup_{t\in[0,1]}(\mid z(t)-x(t)\mid)\sup_{t\in[0,1]}\int_0^1 G(t,s)ds) \\
&\leq \psi(\frac{1}{2}\sup_{t\in[0,1]}(\mid x(t)-y(t)\mid)+\frac{1}{2}\sup_{t\in[0,1]}(\mid y(t)-z(t)\mid)+\frac{1}{2}\sup_{t\in[0,1]}(\mid z(t)-x(t)\mid)) \\
&\leq \psi(\frac{1}{2}\Omega(x,y,z))\leq\psi(\Omega(x,y,z))-\phi(\Omega(x,y,z)).
\end{aligned}$$

Thus, by Theorem 2.3, there exists a solution $u\in C[0,1]$ of the integral equation (3.1). □

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