Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 149–159
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point methods for solving solutions of a generalized equilibrium problem

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Communicated by Yeol Je Cho

Abstract

In this paper, a generalized equilibrium problem is investigated based on fixed point methods. Strong convergence theorems of solutions are established in the framework of Hilbert spaces.

Keywords: Equilibrium problem, fixed point, nonexpansive mapping, variational inequality.
2010 MSC: 47H05, 47H09.

1. Introduction and Preliminaries

Equilibrium problem which was introduced by Ky Fan [8] and further studied by Blum and Oettli [1] has been intensively investigated based on fixed point methods. The equilibrium problem have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, ecology, transportation, network, elasticity and optimization; see [9, 11, 14, 15, 24] and the references therein. It is known that the equilibrium problems cover variational inequality problems, saddle problems, inclusion problems, complementarity problem and minimization problem; see [3]-[7], [17]-[20] and the references therein.

In this paper, a fixed point method is investigated for solving solutions of a generalized equilibrium problem. The common solution is also a unique solution to another monotone variational inequality. Strong convergence theorems are established in the framework of Hilbert spaces.

From now on, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the projection of H onto C.

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Received 2015-07-19

Let $S : C \to C$ be a mapping. Throughout this paper, we use $F(S)$ to denote the fixed point set of S. Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we say that A is α-inverse-strongly monotone. It is known if $S : C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$-inverse-strongly monotone. Recall that a set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*, $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0, \ \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \ge 0$, for $\forall u \in C$; see [21] and the references therein

Recall that the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \quad \forall v \in C. \tag{1.1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of variational inequality (1.1). One can see that variational inequality (1.1) is equivalent to a fixed point problem. The element $u \in C$ is a solution of variational inequality (1.1) if and only if $u \in C$ is a fixed point of $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is an identity mapping. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Let A be an inverse-strongly monotone mapping and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We consider the following generalized equilibrium problem:

$$\text{Find } z \in C \text{ such that } F(z, y) + \langle Az, y - z\rangle \ge 0, \quad \forall y \in C. \tag{1.2}$$

In this paper, $EP(F, A)$ stands for the solution set of problem (1.2), *i.e.*,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z\rangle \ge 0, \quad \forall y \in C\}.$$

If $A \equiv 0$, the zero mapping, then problem (1.2) is reduced to

$$\text{Find } z \in C \text{ such that } F(z, y) \ge 0, \quad \forall y \in C. \tag{1.3}$$

In this paper, we use $EP(F)$ to denote the solution set of problem (1.3). Problem (1.3) first introduced by Fan [8]. In the terminology of Blum and Oettli [1], It is also said to an equilibrium problem. Since many real world problems, for example, signal processing, network traffic and intensity modulated radiation therapy, can be modelled as equilibrium problems (1.2) and (1.3), they have been investigated via fixed point methods by many authors; see, for example, [10, 12],[25]-[30] and the references therein.

If $F \equiv 0$, then problem (1.2) is reduced to variational inequality (1.1).

To study the equilibrium problems, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

In order to prove our main results, we also need the following tools.

A space X is said to satisfy the Opial's condition [16] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well-known that the above inequality is equivalent to

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is known that Hilbert spaces have the Opial's condition.

The following lemma can be found in [1].

Lemma 1.1. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$, then the following hold:

$$T_r(x) = \{z \in C : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Lemma 1.2 ([25])**.** *Let C, H, F and T_r be as in Lemma 1.1. Then the following holds:*

$$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle$$

for all $s, t > 0$ and $x \in H$.

Lemma 1.3 ([13])**.** *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Definition 1.4 ([22])**.** Let C be a nonempty closed convex subset of H. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \le \gamma_i < 1$, $\forall i \ge 1$. For $n \ge 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I,\\
&u_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I.
\end{aligned} \tag{1.4}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \dots, S_1$ and $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$.

Lemma 1.5 ([22])**.** *Let C be a nonempty closed convex subset of H, $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \ne \emptyset$, $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \le l < 1$, $\forall i \ge 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{\infty} F(S_i)$, for each $n \ge 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists.*

(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C,$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \dots$ and $\gamma_1, \gamma_2, \dots$.

Lemma 1.6 ([2])**.** *Let C be a nonempty closed convex subset of H. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \ne \emptyset$ and let $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \le l < 1$, $\forall i \ge 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \le l < 1$, $\forall i \ge 1$.

Lemma 1.7 ([23])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Omega := \cap_{i=1}^{\infty} F(S_i) \cap EP(F,A) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with the constant $\kappa \in (0,1)$. Let $x_1 \in C$ be chosen arbitrarily and let $\{x_n\}$ be a sequence generated in the following process: $F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0$, $\forall y \in C$, $x_{n+1} = \beta_n x_n + (1-\beta_n)W_n\big(\alpha_n f(W_n x_n) + (1-\alpha_n)y_n\big)$, $\forall n \geq 1$, where $\{W_n\}$ is the mapping sequence defined by (1.4), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the following conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ converge strongly to a point $x \in \Omega$, where $x = P_\Omega f(x)$.*

Proof. First, we show that the sequence $\{x_n\}$ and $\{y_n\}$ are bounded. Fixing $x^* \in \Omega$, we find that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n Ax_n) - T_{r_n}(x^* - r_n Ax^*)\|^2 \\ &\leq \|(x_n - r_n Ax_n) - (x^* - r_n Ax^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, Ax_n - Ax^*\rangle + r_n^2\|Ax_n - Ax^*\|^2 \\ &\leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2. \end{aligned} \tag{2.1}$$

From the condition imposed on $\{r_n\}$, we find that

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{2.2}$$

In the same way, we find that $I - r_n A$ is also nonexpansive. Putting $z_n = \alpha_n f(W_n x_n) + (1-\alpha_n)y_n$, we find from (2.2) that

$$\begin{aligned} \|z_n - x^*\| &\leq \alpha_n\|f(W_n x_n) - x^*\| + (1-\alpha_n)\|y_n - x^*\| \\ &\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|. \end{aligned} \tag{2.3}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|W_n z_n - x^*\| \\ &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)\|z_n - x^*\| \\ &\leq \big(1 - \alpha_n(1-\beta_n)(1-\kappa)\big)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|f(x^*) - x^*\| \\ &\leq \cdots \\ &\leq \max\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\kappa}\}. \end{aligned}$$

This yields that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $x_n, y_n, z_n \in K$.

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|T_{r_{n+1}}(x_{n+1} - r_{n+1}Ax_{n+1}) - T_{r_{n+1}}(x_n - r_n Ax_n) \\ &\quad + T_{r_{n+1}}(x_n - r_n Ax_n) - T_{r_n}(x_n - r_n Ax_n)\| \\ &\leq \|(x_{n+1} - r_{n+1}Ax_{n+1}) - (x_n - r_n Ax_n)\| \\ &\quad + \|T_{r_{n+1}}(x_n - r_n Ax_n) - T_{r_n}(x_n - r_n Ax_n)\| \\ &\leq \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| \\ &\quad + \|T_{r_{n+1}}(x_n - r_n Ax_n) - T_{r_n}(x_n - r_n Ax_n)\|. \end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq \alpha_{n+1}\|f(W_{n+1}x_{n+1}) - f(W_n x_n)\| + |\alpha_{n+1} - \alpha_n|(\|f(W_{n+1}x_{n+1})\| + \|y_n\|) \\ &\quad + (1-\alpha_{n+1})\|y_{n+1} - y_n\| \\ &\leq \alpha_{n+1}\kappa\|W_{n+1}x_{n+1} - W_n x_n\| + |\alpha_{n+1} - \alpha_n|(\|f(W_{n+1}x_{n+1})\| + \|y_n\|) \\ &\quad + \|x_{n+1} - x_n\| + |r_{n+1} - r_n|\|Ax_n\| + \|T_{r_{n+1}}(x_n - r_n Ax_n) - T_{r_n}(x_n - r_n Ax_n)\|. \end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}\|W_{n+1}z_{n+1}-W_nz_n\| &= \|W_{n+1}z_{n+1}-Wz_{n+1}+Wz_{n+1}-Wz_n+Wz_n-W_nz_n\| \\ &\le \|W_{n+1}z_{n+1}-Wz_{n+1}\|+\|Wz_{n+1}-Wz_n\|+\|Wz_n-W_nz_n\| \\ &\le \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\|z_{n+1}-z_n\|.\end{aligned} \tag{2.6}$$

Combing (2.5) with (2.6) yields that

$$\begin{aligned}&\|W_{n+1}z_{n+1}-W_nz_n\|-\|x_{n+1}-x_n\| \\ &\quad\le \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\alpha_{n+1}\kappa\|W_{n+1}x_{n+1}-W_nx_n\| \\ &\qquad+|\alpha_{n+1}-\alpha_n|(\|f(W_{n+1}x_{n+1})\|+\|y_n\|) \\ &\qquad+|r_{n+1}-r_n|\|Ax_n\|+\|T_{r_{n+1}}(x_n-r_nAx_n)-T_{r_n}(x_n-r_nAx_n)\|.\end{aligned}$$

By using Lemma 1.6, we find that

$$\limsup_{n\to\infty}\{\|W_{n+1}z_{n+1}-W_nz_n\|-\|x_{n+1}-x_n\|\}\le 0.$$

It follows from Lemma that

$$\lim_{n\to\infty}\|W_nz_n-x_n\|=0. \tag{2.7}$$

Hence, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.8}$$

In view of (2.1), we find that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_nx_n+(1-\beta_n)W_nz_n-x^*\|^2 \\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2 \\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\big(\alpha_n\|f(W_nx_n)-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2\big) \\ &\le \|x_n-x^*\|^2+\alpha_n\|f(W_nx_n)-x^*\|^2 \\ &\quad+r_n(r_n-2\alpha)(1-\alpha_n)(1-\beta_n)\|Ax_n-Ax^*\|^2,\end{aligned}$$

which in turn leads to

$$\begin{aligned}&r_n(2\alpha-r_n)(1-\alpha_n)(1-\beta_n)\|Ax_n-Ax^*\|^2 \\ &\qquad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|f(W_nx_n)-x^*\|^2 \\ &\qquad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|f(W_nx_n)-x^*\|^2.\end{aligned}$$

By using (2.8), we find that

$$\lim_{n\to\infty}\|Ax_n-Ax^*\|=0. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned}\|y_n-x^*\|^2 &= \|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)x^*\|^2 \\ &\le \langle (I-r_nA)x_n-(I-r_nA)x^*,y_n-x^*\rangle \\ &\le \frac{1}{2}(\|x_n-x^*\|^2+\|y_n-x^*\|^2-\|(x_n-y_n)-r_n(Ax_n-Ax^*)\|^2) \\ &= \frac{1}{2}(\|x_n-x^*\|^2+\|y_n-x^*\|^2-\|x_n-y_n\|^2 \\ &\quad+2r_n\langle x_n-y_n,Ax_n-Ax^*\rangle-r_n^2\|Ax_n-Ax^*\|^2).\end{aligned}$$

Hence, we have

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + 2r_n\|x_n - y_n\|\|Ax_n - Ax^*\|.$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + \alpha_n\|f(W_nx_n) - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - y_n\|^2 \\ &\quad + 2r_n(1-\alpha_n)(1-\beta_n)\|x_n - y_n\|\|Ax_n - Ax^*\| \\ &\leq \|x_n - x^*\|^2 + \alpha_n\|f(W_nx_n) - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - y_n\|^2 \\ &\quad + 2r_n\|x_n - y_n\|\|Ax_n - Ax^*\|.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\|x_n - y_n\|^2 &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|f(W_nx_n) - x^*\|^2 \\ &\quad + 2r_n\|x_n - y_n\|\|Ax_n - Ax^*\|.\end{aligned}$$

By using (2.8) and (2.9), we find that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.10}$$

Since $z_n = \alpha_n f(W_nx_n) + (1-\alpha_n)y_n$, we find that

$$\lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{2.11}$$

Notice that $\|x_{n+1} - x_n\| = (1-\beta_n)\|W_nz_n - x_n\|$. This implies from (2.8) gives that

$$\lim_{n\to\infty} \|W_nz_n - x_n\| = 0. \tag{2.12}$$

Note that $\|W_nz_n - z_n\| \leq \|z_n - y_n\| + \|y_n - x_n\| + \|x_n - W_nz_n\|$. From (2.10), (2.11) and (2.12), we obtain that

$$\lim_{n\to\infty} \|W_nz_n - z_n\| = 0. \tag{2.13}$$

Since the mapping $P_\Omega f$ is contractive, we denote the unique fixed point by x. Next, we prove that $\limsup_{n\to\infty}\langle f(x) - x, z_n - x\rangle \leq 0$. To see this, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x) - x, z_n - x\rangle = \lim_{i\to\infty}\langle f(x) - x, z_{n_i} - x\rangle.$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to z. Without loss of generality, we may assume that $z_{n_i} \rightharpoonup z$. Indeed, we also have $y_{n_i} \rightharpoonup f$.

First, we show that $z \in \cap_{i=1}^{\infty} F(S_i)$. Suppose the contrary, $Wz \neq z$. Note that

$$\begin{aligned}\|z_n - Wz_n\| &\leq \|Wz_n - W_nz_n\| + \|W_nz_n - z_n\| \\ &\leq \sup_{x\in K}\{\|Wx - W_nx\|\} + \|W_nz_n - z_n\|.\end{aligned}$$

In view of Lemma 1.6, we obtain from (2.13) that $\lim_{n\to\infty}\|z_n - Wz_n\| = 0$. By using the Opial's condition, we see that

$$\begin{aligned}\liminf_{i\to\infty}\|z_{n_i} - z\| &< \liminf_{i\to\infty}\|z_{n_i} - Wz\| \\ &\leq \liminf_{i\to\infty}\{\|z_{n_i} - Wz_{n_i}\| + \|Wz_{n_i} - Wz\|\} \\ &\leq \liminf_{i\to\infty}\{\|z_{n_i} - Wz_{n_i}\| + \|z_{n_i} - z\|\}.\end{aligned}$$

This implies that $\liminf_{i\to\infty}\|z_{n_i} - z\| < \liminf_{i\to\infty}\|z_{n_i} - z\|$, which derives a contradiction. Hence, we have $z \in \cap_{i=1}^{\infty} F(S_i)$.

Next, we show that $f \in EP(F, A)$. Note that $y_n \rightharpoonup z$. Since $y_n = T_{r_n}(x_n - r_n Ax_n)$, we have

$$F(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

By using condition (A2), we see that

$$\langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq F(y, y_n), \quad \forall y \in C.$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, y - y_{n_i}\rangle + \langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, y_{n_i}), \quad \forall y \in C. \tag{2.14}$$

For t with $0 < t \leq 1$ and $\rho \in C$, let $\rho_t = t\rho + (1-t)z$. Since $\rho \in C$ and $z \in C$, we have $\rho_t \in C$. It follows from (2.14) that

$$\begin{aligned}\langle \rho_t - y_{n_i}, A\rho_t\rangle &\geq \langle \rho_t - y_{n_i}, A\rho_t\rangle - \langle Ax_{n_i}, \rho_t - y_{n_i}\rangle - \langle \rho_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(\rho_t, y_{n_i})\\ &= \langle \rho_t - y_{n_i}, A\rho_t - Ay_{n,i}\rangle + \langle \rho_t - y_{n_i}, Ay_{n,i} - Ax_{n_i}\rangle\\ &\quad - \langle \rho_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle + F(\rho_t, y_{n_i}).\end{aligned} \tag{2.15}$$

It follows from (2.10) that $Ay_{n,i} - Ax_{n_i} \to 0$ as $i \to \infty$. On the other hand, we get from the monotonicity of A that $\langle \rho_t - y_{n_i}, A\rho_t - Ay_{n,i}\rangle \geq 0$. It follows from (A4) and (2.15) that

$$\langle \rho_t - z, A\rho_t\rangle \geq F(\rho_t, z). \tag{2.16}$$

From (A1) and (A4), we see from (2.16) that

$$\begin{aligned}0 = F(\rho_t, \rho_t) &\leq tF(\rho_t, \rho) + (1-t)F(\rho_t, z)\\ &\leq tF(\rho_t, \rho) + (1-t)\langle \rho_t - z, A\rho_t\rangle\\ &= tF(\rho_t, \rho) + (1-t)t\langle \rho - z, A\rho_t\rangle,\end{aligned}$$

which leads to $F(\rho_t, \rho) + (1-t)\langle \rho - f, A_3\rho_t\rangle \geq 0$. Letting $t \to 0$ in the above inequality, we arrive at $F(z, \rho) + \langle \rho - z, Az\rangle \geq 0$. This shows that $f \in EP(F, A)$. It follows that

$$\limsup_{n\to\infty}\langle f(x) - x, z_n - x\rangle \leq 0. \tag{2.17}$$

Finally, we show that $x_n \to x$, as $n \to \infty$. Note that

$$\begin{aligned}\|z_n - x\|^2 &= \alpha_n\langle f(W_n x_n) - x, z_n - x\rangle + (1-\alpha_n)\langle y_n - x, z_n - x\rangle\\ &\leq \big(1 - \alpha_n(1-\kappa)\big)\|x_n - x\|\|z_n - x\| + \alpha_n\langle f(x) - x, z_n - x\rangle\\ &\leq \frac{1 - \alpha_n(1-\kappa)}{2}(\|x_n - x\|^2 + \|z_n - x\|^2) + \alpha_n\langle f(x) - x, z_n - x\rangle.\end{aligned}$$

Hence, we have

$$\|z_n - x\|^2 \leq (1 - \alpha_n(1-\kappa))\|x_n - x\|^2 + 2\alpha_n\langle f(x) - x, z_n - x\rangle.$$

This implies that

$$\begin{aligned}\|x_{n+1} - x\|^2 &= \|\beta_n x_n + (1-\beta_n)W_n z_n - x\|^2\\ &\leq \beta_n\|x_n - x\|^2 + (1-\beta_n)\|z_n - x\|^2\\ &\leq \big(1 - \alpha_n(1-\beta_n)(1-\kappa)\big)\|x_n - x\|^2 + 2\alpha_n(1-\beta_n)\langle f(x) - x, z_n - x\rangle.\end{aligned}$$

By using Lemma 1.3 and (2.17), we find that $\lim_{n\to\infty}\|x_n - x\| = 0$. This completes the proof. □

Remark 2.2. The well known convex feasibility problem which captures applications in various disciplines such as image restoration, and radiation therapy treatment planning is to find a point in the intersection of convex set. In Theorem 2.1, a generalized equilibrium problem is investigated via fixed point methods. An infinite family of nonexpansive mappings are considered. To complement of algorithm, there is no metric projection involved in the strong convergence theorem. It also deserve mentioning that the common solution is also another monotone variational inequality.

3. Applications

For a single mapping, we find from Theorem 2.1 the following result.

Theorem 3.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let S be a nonexpansive mapping.*
Assume that $\Omega := F(S) \cap EP(F, A) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with the constant $\kappa \in (0, 1)$. Let $x_1 \in C$ be chosen arbitrarily and let $\{x_n\}$ be a sequence generated in the following process: $F(y_n, y)+\langle Ax_n, y-y_n\rangle+\frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \geq 0$, $\forall y \in C$, $x_{n+1} = \beta_n x_n+(1-\beta_n)W_n\big(\alpha_n f(Sx_n)+(1-\alpha_n)y_n\big)$, $\forall n \geq 1$, where $\{W_n\}$ is the mapping sequence defined by (1.4), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ converge strongly to a point $x \in \Omega$, where $x = P_\Omega f(x)$.

If S is the identity, we find the following result on the generalized equilibrium problem.

Corollary 3.2. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Assume that $EP(F, A) \neq \emptyset$. Let $f : C \to C$ be a contractive mapping with the constant $\kappa \in (0, 1)$. Let $x_1 \in C$ be chosen arbitrarily and let $\{x_n\}$ be a sequence generated in the following process:*
$F(y_n, y)+\langle Ax_n, y-y_n\rangle+\frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \geq 0$, $\forall y \in C$, $x_{n+1} = \beta_n x_n+(1-\beta_n)W_n\big(\alpha_n f(x_n)+(1-\alpha_n)y_n\big)$, $\forall n \geq 1$, where $\{W_n\}$ is the mapping sequence defined by (1.4), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the conditions: $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ converge strongly to a point $x \in EP(F, A)$, where $x = P_{EP(F,A)} f(x)$.

Next, we give a result on equilibrium problem (1.3).

Corollary 3.3. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with a nonempty common fixed point set. Let $f : C \to C$ be a contractive mapping with the constant $\kappa \in (0, 1)$. Let $x_1 \in C$ be chosen arbitrarily and let $\{x_n\}$ be a sequence generated in the following process: $F(y_n, y)+\frac{1}{r_n}\langle y-y_n, y_n-x_n\rangle \geq 0$, $\forall y \in C$, $x_{n+1} = \beta_n x_n+(1-\beta_n)W_n\big(\alpha_n f(W_n x_n)+(1-\alpha_n)y_n\big)$, $\forall n \geq 1$, where $\{W_n\}$ is the mapping sequence defined by (1.4), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the conditions $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < +\infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. Then $\{x_n\}$ converge strongly to a point $x \in \cap_{i=1}^{\infty} F(S_i)$, where $x = P_{\cap_{i=1}^{\infty} F(S_i)} f(x)$.*

Proof. By putting $A_3 \equiv 0$, the zero operator, we find the desired conclusion easily. This completes the proof. □

Next, we give a result on the classical variational inequality.

Theorem 3.4. *Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that*

$\Omega := \cap_{i=1}^{\infty} F(S_i) \cap VI(C, A) \neq \emptyset$. *Let* $f : C \to C$ *be a contractive mapping with the constant* $\kappa \in (0, 1)$. *Let* $x_1 \in C$ *be chosen arbitrarily and let* $\{x_n\}$ *be a sequence generated in the following process:*
$x_{n+1} = \beta_n x_n + (1 - \beta_n) W_n \big(\alpha_n f(W_n x_n) + (1 - \alpha_n) y_n\big),$, *where* $y_n = P_C(x_n - r_n A x_n)$, $\forall n \geq 1$, *where* $\{W_n\}$ *is the mapping sequence defined by* (1.4), $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *and* $\{r_n\}$ *is a positive number sequence. Assume that the above control sequences satisfy the conditions* $0 < a \leq \beta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\alpha$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$. *Then* $\{x_n\}$ *converge strongly to a point* $x \in \Omega$, *where* $x = P_\Omega f(x)$.

Proof. Putting $F \equiv 0$, we see from Theorem 2.1 that

$$\langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \quad \forall y \in C, \forall n \geq 1.$$

This implies that

$$\langle y - y_n, x_n - r_n A x_n - y_n\rangle \leq 0, \quad \forall y \in C.$$

It follows that

$$y_n = P_C(x_n - r_n A x_n).$$

This completes the proof. □

Remark 3.5. The convex feasibility problem finds a lot applications in diverse areas of mathematics and physical sciences. It consists of trying to a solution satisfying certain constraints. In this paper, we investigate a generalized equilibrium problem via a fixed point method. A viscosity algorithm is proposed and strong convergence is obtained. Consider the following optimization problem:

$$\min_{x \in C} h(x),$$

where C is a nonempty closed convex subset of H, and $h : C \to \mathbb{R}$ is a convex and lower semi-continuous functional. Let $F : C \times C \to \mathbb{R}$ be a bifunction defined by $F(x, y) = h(y) - h(x)$. We consider the following equilibrium problem, that is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C.$$

It is easy to see that the bifunction F satisfies conditions $(A1)$-$(A4)$ and $EP(F) = \Omega$, the solution set.

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