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Difference equations involving causal operators with nonlinear boundary conditions

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Abstract

In this paper, we investigate nonlinear boundary problems for difference equations with causal operators. Our boundary condition is given by a nonlinear function, and more general than ones given before. By using the method of upper and lower solutions coupled with the monotone iterative technique, criteria on the existence of extremal solutions are obtained, an example is also presented.

Keywords: Causal operators, monotone iterative technique, upper and lower solutions, extremal solutions.
2010 MSC: 39A10, 34A34.

1. Introduction

The qualitative properties of solutions for difference problems have attracted lots of researchers because it arises frequently in many fields such as computing, electrical circuit analysis, numerous settings and forms, biology, readers can refer [1, 7, 13, 14, 16, 17, 20] for details. As an important branch, boundary value problems (see [5, 11, 12, 19]), especially, difference equations with nonlinear boundary value problems have drawn much attention. In 2008, Wang [15] investigated the first-order functional difference problems with nonlinear boundary value conditions, obtained the existence of extremal solutions by using the monotone iterative method. Immediately after it, he discussed in [18] the initial-value problems of nonlinear singular discrete systems. However, difference equations with causal operators have not much studied and many aspects of these equations are yet to be explored, see [8, 9]. A causal operator is a non-anticipative operator. Its theory

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has the powerful quality of unifying ordinary differential equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations and neutral functional equations. Readers can refer to the monograph [10], papers [2, 3, 4, 6] for more details.

Motivated by the some recent work on difference equations involving causal operators and nonlinear boundary value problems, this paper is devoted to investigate the following boundary value problems with causal operators

$$\begin{cases} \Delta y(k-1) = (Qy)(k), \quad k \in Z[1,T] = \{1,2,\cdots,T\}, \\ B(y(0),y) = 0, \end{cases} \tag{1.1}$$

where $\Delta y(k-1) = y(k)-y(k-1)$, $E_1 = C(Z[1,T],\mathbb{R})$, $Q \in C(E_1,E_1)$ is a causal operator, $B \in C(\mathbb{R}\times E_1,\mathbb{R})$, Q is bounded, and the next type of equations

$$\begin{cases} \Delta y(k) = (Qy)(k), \quad k \in Z[0,T-1] = \{0,1,\cdots,T-1\}, \\ B(y(0),y) = 0, \end{cases} \tag{1.2}$$

where $\Delta y(k) = y(k+1) - y(k)$, $E_0 = C(Z[0,T-1],\mathbb{R})$, $Q \in C(E_0,E_0)$, $B \in C(\mathbb{R} \times E_0,\mathbb{R})$, Q is bounded.

The main interest of the paper lies in the fact that we consider nonlinear boundary conditions which, of course, includes the usual linear boundary conditions (such as initial and periodic) and other general conditions such as $y(0) = \max\limits_{j\in Z[0,T]} y(j)$ and $y(0) = \sum\limits_{j\in Z[0,T]} y(j)$.

The paper is organized as follows. In Section 2, some comparison principles are established. In Sections 3, after introducing the definition of upper and lower solutions, we obtain existence of solutions for (1.1) and (1.2) by Schauder fixed point. Moreover, the existence of extremal solutions for (1.1) and (1.2) are established by utilizing the monotone iterative technique. At last, an example is given to illustrate the results.

2. Comparison results

Definition 2.1. Suppose that $Q \in C(E,E)$, then Q is said to be a causal map or a nonanticipative map if $u(s) = v(s)$, $t_0 \le s \le t \le T$, where $u, v \in E$, then

$$(Qu)(s) = (Qv)(s), \quad t_0 \le s \le t.$$

Lemma 2.2 ([8])**.** *Assume that $\mathcal{L} \in C(E_1,E_1)$ is a positive linear operator, $M \in C(Z[1,T],\mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$, $y \in C(Z[0,T],\mathbb{R})$ and*

$$\begin{cases} \Delta y(k-1) + M(k)y(k) + (\mathcal{L}y)(k) \le 0, \quad k \in Z[1,T] \\ y(0) \le 0, \end{cases}$$

and

$$\sum_{i=1}^{T} (\mathcal{L}1)(i) \prod_{j=1}^{i-1} (1+M(j)) < 1, \quad \textit{with } 1(k) = 1, \quad k \in Z[1,T]. \tag{2.1}$$

Then $y(k) \le 0$, $k \in Z[0,T]$.

Lemma 2.3 ([8])**.** *Assume that $\mathcal{L} \in C(E_0,E_0)$ is a positive linear operator, $M \in C(Z[0,T-1],\mathbb{R}_+)$ with $\mathbb{R}_+ = [0,\infty)$, $y \in C(Z[0,T],\mathbb{R})$ and*

$$\begin{cases} \Delta y(k) + M(k)y(k) + (\mathcal{L}y)(k) \le 0, \quad k \in Z[0,T-1] \\ y(0) \le 0, \end{cases}$$

and

$$\sum_{i=1}^{T} \frac{(\mathcal{L}1)(i)}{\prod_{j=1}^{i-1}\big(1-M(j)\big)} < 1 \quad \textit{with } 1(k)=1,\ \ k\in Z[0,T-1]. \tag{2.2}$$

Then $y(k)\le 0$, $k\in Z[0,T]$.

3. Existence results

Definition 3.1. Functions $\alpha,\beta\in E_1$ are said to be lower and upper solutions of problem (1.1), respectively, if

$$\Delta\alpha(k-1)\le (Q\alpha)(k),\quad B(\alpha(0),\alpha)\le 0$$

and

$$\Delta\beta(k-1)\ge (Q\beta)(k),\quad B(\beta(0),\beta)\ge 0.$$

Similarly, a function $\alpha\in E_0$ is said to be a lower solution of (1.2) if it satisfies

$$\begin{cases}\Delta\alpha(k)\le (Q\alpha)(k),\\ B(\alpha(0),\alpha)\le 0,\end{cases}$$

and an upper solution of (1.2) is defined analogously by reversing the inequalities of above.

For α,β, we write $\alpha\le\beta$ if $\alpha(k)\le\beta(k)$ for all $k\in Z[1,T]$. Also, we denote $[\alpha,\beta]=\{y,\alpha(k)\le y(k)\le \beta(k)\}$.

Theorem 3.2 (Discrete Arzela-Ascoli Theorem [1])**.** *Let $\mathcal{A}$ be a closed subset of C. If $\mathcal{A}$ is uniformly boundary and the set $\{y(k): y\in\mathcal{A}\}$ is relatively compact for each $k\in Z[0,T]$, then $\mathcal{A}$ is compact.*

Theorem 3.3. *Let* (2.1) *hold. Assume the following conditions hold,*

(H_0) the functions α, β are lower and upper solutions of (1.1)*, respectively, such that $\alpha\le\beta$;*
(H_1) there exist the positive linear operator $\mathcal{L}\in C(E_1,E_1)$ and $M\in C(Z[1,T],\mathbb{R}_+)$ and

$$(Qu)(k)-(Qv)(k)\ge -M(k)(u-v)-(\mathcal{L}(u-v))(k),\ \ \textit{for}\ \ \alpha\le v\le u\le\beta;$$

(H_2) $B(u,\cdot)$ is a nonincreasing function for each $u\in[\alpha(0),\beta(0)]$.

Then (1.1) *has at least one solution $u\in[\alpha,\beta]$.*

Proof. Let $P\in C(Z[1,T],\mathbb{R})$ be defined by $(Py)(k)=\max[\alpha(k),\min[y(k),\beta(k)]]$. Then $(QPy)(k)$ defines a continuous extension of Q on E_1 which is also bounded since Q is assumed to be bounded.

We consider the following modified problem:

$$\begin{cases}\Delta y(k-1)+M(k)y(k)+(\mathcal{L}y)(k)=\sigma(k),\\ y(0)=(P\bar{y})(0),\end{cases} \tag{3.1}$$

where $\sigma(k)=(QPy)(k)+M(k)(Py)(k)+(\mathcal{L}Py)(k)$ and $\bar{y}(0)=y(0)-B(y(0),y)$.

Let y be any solution of the problem (3.1). Note that problem (3.1) can be also written in the following form

$$\Delta\Big[y(k-1)\prod_{i=1}^{k-1}\big(1+M(i)\big)\Big]=\big[-(Ly)(k)+\sigma(k)\big]\prod_{i=1}^{k-1}\big(1+M(i)\big)$$

for $k\in Z[1,T]$. Summing it from 1 to k gives

$$y(k)=\Big[y(0)+\sum_{i=1}^{k}\big(-(Ly)(k)+\sigma(k)\big)\prod_{i=1}^{k-1}\big(1+M(i)\big)\Big]\Big(\prod_{j=1}^{k}\big(1+M(j)\big)\Big)^{-1}\equiv(\phi_\sigma y)(k)$$

for $k \in Z[0,T]$. Similarly, it is easy to see that if y is any solution of $y = \phi_\sigma y$, then y is a solution of problem (3.1).

The continuity of M, $\mathcal{L}$, σ imply that $\phi : E \to E$ is continuous and bounded. This and Theorem 3.2 imply that ϕ is compact. Now, Schauder′s fixed point theorem implies that ϕ has a fixed point, and problem (3.1) has a solution.

We can prove $y \in [\alpha, \beta]$. Firstly, we prove $\alpha \le y$, if $p(k) = \alpha(k) - y(k)$, $k \in Z[0,T]$, from the definition of lower solution, we get

$$\begin{aligned}\Delta p(k-1) &= \Delta\alpha(k-1) - \Delta y(k-1) \le (Q\alpha)(k) - [\sigma(k) - M(k)y(k) - (\mathcal{L}y)(k)]\\ &\le (Q\alpha)(k) - (QPy)(k) - M(k)\big((Py)(k) - y(k)\big) - \big(\mathcal{L}(Py-y)\big)(k)\\ &\le -M(k)p(k) - (\mathcal{L}p)(k).\end{aligned}$$

By Lemma 2.2, we have $\alpha \le y$. Similarly, we conclude that $y \in [\alpha, \beta]$.

Next, we shall prove that $\alpha(0) \le y(0) - B(y(0), y) \le \beta(0)$.

If $y(0) - B(y(0), y) < \alpha(0)$, we obtain that $y(0) = \alpha(0)$, In consequence $\alpha(0) > \alpha(0) - B(y(0), y)$. Since $B(\alpha(0), \cdot)$ is nonincreasing in $[\alpha, \beta]$, and we know that $y \in [\alpha, \beta]$, we obtain that $B(\alpha(0), \alpha) > 0$, which contradicts with the definition of the lower solution. Analogously, we can prove that $y(0) - B(y(0), y) \le \beta(0)$.

Thus every solution u of (3.1)is a solution of (1.1), and it belongs to $[\alpha, \beta]$, and the proof is complete. □

4. Main results

Theorem 4.1. *Assume that the hypotheses of Theorem 3.3 are satisfied.*
Then there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ *with* $\alpha_0 = \alpha$, $\beta_0 = \beta$, *such that* $\lim\limits_{n\to\infty} \alpha_n(k) = \rho(k)$, $\lim\limits_{n\to\infty} \beta_n(k) = \gamma(k)$ *uniformly on* J, *and* ρ, γ *are the minimal and maximal solutions of* (1.1), *respectively.*

Proof. Let $\eta \in [\alpha, \beta]$, we consider the following problem:

$$(P_\eta)\begin{cases}\Delta y(k-1) + M(k)y(k) + (\mathcal{L}y)(k) = (Q\eta)(k) + M(k)\eta(k) + (\mathcal{L}\eta)(k),\\ B(y(0), y) = 0,\end{cases}$$

since $\alpha \le \eta \le \beta$, we have using (H_1) and the defining of lower and upper solutions, that

$$\begin{aligned}\Delta\alpha(k-1) + M(k)\alpha(k) + (\mathcal{L}\alpha)(k) &\le (Q\alpha)(k) + M(k)\alpha(k) + (\mathcal{L}\alpha)(k)\\ &\le (Q\eta)(k) + M(k)\eta(k) + (\mathcal{L}\eta)(k)\end{aligned}$$

and

$$\begin{aligned}\Delta\beta(k-1) + M(k)\beta(k) + (\mathcal{L}\beta)(k) &\ge (Q\beta)(k) + M(k)\beta(k) + (\mathcal{L}\beta)(k)\\ &\ge (Q\eta)(k) + M(k)\eta(k) + (\mathcal{L}\eta)(k).\end{aligned}$$

Thus α is a lower solution for (P_η). Analogously, β is an upper solution for (P_η).

Let ξ be a solution of problem (1.1) in $[\alpha, \beta]$. Such a solution exists by Theorem 3.3. Clearly, ξ is a solution of (P_ξ). Furthermore, if $\eta \le \xi$, then

$$\begin{aligned}\Delta\xi(k-1) + M(k)\xi(k) + (\mathcal{L}\xi)(k) &= (Q\xi)(k) + M(k)\xi(k) + (\mathcal{L}\xi)(k)\\ &\ge (Q\eta)(k) + M(k)\eta(k) + (\mathcal{L}\eta)(k).\end{aligned}$$

Thus, Theorem 3.3 assures that there exists at least one solution of the problem (P_η) on $[\alpha, \xi]$.

Analogously, we can show that if $\eta \ge \xi$, and ξ is a solution of (P_ξ), then (P_η) admits one solution on $[\xi, \beta]$.

Observe that for every solution τ of (1.1), τ is a solution of (P_τ) and then if we take $\eta \le \tau$ (respectively, $\eta \ge \tau$), the problem (P_η) admits one solution $y \le \tau$ (Resp. $y \ge \tau$)

Let $\alpha_0 = \alpha$. The previous arguments prove that the problem (P_α) has at least one solution on $[\alpha, \beta]$. Moreover, we choose

$$\alpha_1 = \min\{\xi \in [\alpha, \beta] : \xi \text{ is a solution of}(P_\alpha)\},$$

we have that $\alpha \le \alpha_1 \le y$ for each solution y of (1.1).

Inductively, we define for each $n \in \mathbb{N}$

$$\alpha_{n+1} = \min\{\xi \in [\alpha_n, \beta] : \xi \text{ is a solution of}(P_{\alpha_n})\}$$

Thus we obtain a nondecreasing sequence α_n, with

$$\alpha = \alpha_0 \le \cdots \le \alpha_n \le \beta, \quad \forall n \in \mathbb{N}.$$

Since α_n is increasing and bounded, we have $\{\alpha_n\}$ converging to ρ uniformly. The continuity of B implies that ρ is a solution of (1.1).

Reasoning analogously, we take $\beta_0 = \beta$ and define

$$\beta_{n+1} = \max\{\omega \in [\alpha, \beta_n] : \omega \text{ is a solution of}(P_{\beta_n})\}.$$

So we construct a nonincreasing sequence $\{\beta_n\}$ which converges to γ uniformly.

Finally, as $\alpha_n \le y \le \beta_n$ for each solution of (1.1), it clear that ρ and γ are the minimal and the maximal solutions of (1.1) on $[\alpha, \beta]$, respectively. The proof is then finished. □

We note that the difficulty in the construction of the monotone sequences is solving the nonlinear equation $B(y(0), y) = 0$. In the following, we present a second way to construct the sequences, this method is theoretically worse than first one, but in practical situations ($B(u, v) = u-g(v)$) would be more convenient.

Proof. Let $\eta \in [\alpha, \beta]$, we consider the following problem:

$$(Q_\eta) \begin{cases} \Delta y(k-1) + M(k)y(k) + (\mathcal{L}y)(k) = (Q\eta)(k) + M(k)\eta(k) + (\mathcal{L}\eta)(k), \\ y(0) = \varsigma_\eta, \end{cases}$$

where ς_η is the minimal solution in $[\alpha_0, \beta_0]$ of the equations $B(\varsigma_\eta, \eta) = 0$. Since B is continuous and $B(\alpha_0, \eta) \le B(\alpha_0, \alpha) \le 0$ and $0 \le B(\beta_0, \beta) \le B(\beta_0, \eta)$, ς_η is well defined.

Noticing that the hypotheses of Theorem 3.3 hold, problem (Q_η) has at least one solutions (defining $B(u, v) = u - \varsigma_\eta$). Next we prove the uniqueness of solution to this problem. If y_1, y_2 are solutions of (Q_η), set $v_1 = y_1 - y_2$, and $v_2 = y_2 - y_1$, then

$$v_1(0) = 0, \quad \Delta v_1(k-1) + M(k)v_1(k) + (\mathcal{L}v_1)(k) = 0,$$

and

$$v_2(0) = 0, \quad \Delta v_2(k-1) + M(k)v_2(k) + (\mathcal{L}v_2)(k) = 0,$$

from Lemma 2.2, we have that $v_1 = y_1 - y_2 \le 0$, $v_2 = y_2 - y_1 \le 0$, and so $y_1 = y_2$. Then problem (Q_η) has exactly on solution.

Define a mapping A by $A\eta = y$, then the operator A has the following properties:

(a) $\alpha \le A\alpha$, $\beta \ge A\beta$;

(b) A is monotonically nondecreasing in $[\alpha, \beta]$, i.e., for any $\eta_1, \eta_2 \in [\alpha, \beta]$, $\eta_1 \le \eta_2$ implies $A\eta_1 \le A\eta_2$.

To prove (a), set $m = \alpha - \alpha_1$, where $\alpha_1 = A\alpha$. Employing (H_0), we have

$$\Delta m(k-1) + M(k)m(k) + (\mathcal{L}m)(k) \le 0,$$

note that $m(0) \le 0$, then based on Lemma 2.2, we get $\alpha \le \alpha_1$.

Analogously, we have $\beta \ge A\beta$.

To prove (b), let $\eta_1,\ \eta_2 \in [\alpha, \beta]$ such that $\eta_1 \le \eta_2$. Suppose that $v_1 = A\eta_1$, $v_2 = A\eta_2$ and $m = v_1 - v_2$, we acquire

$$\begin{aligned} \Delta m(k-1) + M(k)m(k) + (\mathcal{L}m)(k) &= (Q\eta_1)(k) + M(k)\eta_1(k) + (\mathcal{L}\eta_1)(k) \\ &\quad -(Q\eta_2)(k) - M(k)\eta_2(k) - (\mathcal{L}\eta_2)(k) \\ &\le 0, \end{aligned}$$

and

$$B(x, \eta_1) \geq B(x, \eta_2), \quad \text{for all } x \in [\alpha_0, \beta_0],$$

then $A\eta_1(0) = \varsigma_{\eta_1} \leq \varsigma_{\eta_2} = A\eta_2(0)$, thus $m(0) \leq 0$. Based on Lemma 2.2 we have $m(k) \leq 0$ on $Z[1, T]$, which implies $A\eta_1 \leq A\eta_2$.

Now define the sequences $\{\alpha_n\}$, $\{\beta_n\}$ by $\alpha_n = A\alpha_{n-1}$, $\beta_n = \beta_{n-1}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$. Owing to (a) and (b), one attains

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \ldots \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0.$$

Consequently, there exist ρ and r such that $\lim_{n\to\infty} \alpha_n(k) = \rho(k)$ and $\lim_{j\to\infty} \beta_n(k) = r(k)$ uniformly and monotonically on $Z[1, T]$.

Using the definition of (Q_η) and taking the limit as $n \to \infty$, we arrive that ρ and r are the solutions of problem (1.1).

To prove that ρ, r are extremal solutions of problem (1.1), let y be any solution of (1.1) such that $\alpha \leq y \leq \beta$. Based on the monotonically nondecreasing property of A, we can easily see that $\alpha_{n+1} \leq y$ by $\alpha_{n+1} = A\alpha_n \leq Ay = y$. Similarly, we can get $y \leq \beta_n$ of $Z[1, T]$. Since $\alpha_0 \leq y \leq \beta_0$, by induction we derives $\alpha_n \leq y \leq \beta_n$ for every n. Taking the limit as $n \to \infty$, we conclude $\rho(k) \leq y(k) \leq r(k)$, and the proof is then finished. □

Next, we compare the two different ways to approximate the extremal solutions.

Suppose that lower solutions α_n and $\bar{\alpha}_n$ are obtained by the two different methods, respectively. First we show that $\alpha_1 \geq \bar{\alpha}_1$, set $v = \bar{\alpha}_1 - \alpha_1$, we have

$$\Delta v(k-1) + M(k)v(k) + (\mathcal{L}v)(k) = 0,$$

since $\bar{\alpha}_1(0)$ is the minimal solution of $B(\cdot, \alpha)$ in $[\alpha_0, \beta_0]$ and

$$B(x, \alpha) \geq B(x, \alpha_1), \quad \text{for all } x \in [\alpha_0, \beta_0],$$

then $\bar{\alpha}_1(0) \leq \alpha_1(0)$, thus $v(0) \leq 0$. Based on Lemma 2.2 we have $v(k) \leq 0$ on $Z[1, T]$, which implies $\bar{\alpha}_1 \leq \alpha_1$.

Now, assume for $n \geq 2$, the following inequalities hold: $\bar{\alpha}_{n-1} \leq \alpha_{n-1}$, taking $v = \bar{\alpha}_n - \alpha_n$, we may get

$$B(x, \bar{\alpha}_{n-1}) \geq B(x, \alpha_{n-1}), \quad \text{for all } x \in [\alpha_0, \beta_0],$$

and owe to the fact $B(\alpha_{n-1}(0), \alpha_{n-1}) = 0$, we derive

$$\bar{\alpha}_n(0) \leq \alpha_{n-1}(0) \leq \alpha_n(0),$$

then $v(0) \leq 0$, and using the definition of $\bar{\alpha}_n$ and α_n, we attain

$$\Delta v(k-1) + M(k)v(k) + (\mathcal{L}v)(k) \leq 0.$$

By employing Lemma 2.2 we have $v(k) \leq 0$ on $Z[1, T]$, which implies $\bar{\alpha}_n \leq \alpha_n$. Using the mathematical induction we get $\bar{\alpha}_n \leq \alpha_n$, for all $n \in \mathbb{N}$.

The same arguments prove that $\beta_n \leq \bar{\beta}_n$, for all $n \in \mathbb{N}$.

So we can see, the second method is theoretically worse than first one, but in practical case of $B(u, v) = u - g(v)$, monotone sequences are obtained more explicitly.

The following two theorems we formulate without any proof since they are similar to Theorem 3.3 and Theorem 4.1.

Theorem 4.2. *Suppose that assumption* (2.2) *and* (H_2) *of Theorem 3.3 hold. Let* α, β *be lower and upper solutions of* (1.2), *respectively, with* $\alpha \leq \beta$, *and assume that*

(H_3) there exist the positive linear operator $\mathcal{L} \in C(E_0, E_0)$ and $M \in C(Z[0, T-1], \mathbb{R}_+)$ and

$$(Qu)(k) - (Qv)(k) \geq -M(k)(u - v) - (\mathcal{L}(u - v))(k), \ \ for\ \alpha \leq u \leq v \leq \beta.$$

Then (1.2) *has at least one solution $u \in [\alpha, \beta]$.*

Theorem 4.3. *Assume that the hypotheses of Theorem 4.2 are satisfied.*
Then there exist two monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$ such that $\alpha = \alpha_0 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_0 = \beta$ which converge uniformly to the minimal and maximal solutions of (1.2)*, respectively.*

5. Example

Consider the following problem:

$$\begin{cases} \Delta y(k-1) = -\dfrac{3k}{100} y(k) + \dfrac{3k}{100} y^2(k) - \dfrac{k}{100} y(k-1), \ \ k \in Z[1, T], \\ B(y(0), y) = 2y(0) - y(k) = 0. \end{cases} \tag{5.1}$$

Now, define $M(k) = \frac{3k}{100}$, $(\mathcal{L}y)(k) = \frac{k}{100} y(k-1)$, it is easy to verify that the conditions (H_1) and (H_2) of Theorem 3.3 are satisfied, then we assume that

$$\begin{aligned} u = \sum_{i=1}^{T} (\mathcal{L}1)(i) \prod_{j=1}^{i-1} \big(1 + M(j)\big) &= \frac{1}{100} \sum_{i=1}^{T} i \prod_{j=1}^{i-1} \left(1 + \frac{3j}{100}\right) \leq \frac{T}{100} \sum_{i=1}^{T} \left(1 + \frac{3T}{100}\right)^{i-1} \\ &= \frac{1}{3} \left[(1 + \frac{3T}{100})^T - 1\right] < 1, \end{aligned}$$

this provided that $T \leq 7$. Now we put $\alpha_0 = -\frac{1}{2}$, $\beta_0 = \frac{1}{2}$, it is easy to show that α_0, β_0 are lower and upper solutions of (5.1), respectively, then problem has extremal solutions by Theorem 4.1.

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