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A stronger inequality of Cîrtoaje's one with power exponential functions

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Abstract

In this paper, we will show that $a^{2b} + b^{2a} + r\,(ab(a-b))^2 \leq 1$ holds for all $0 \leq a$ and $0 \leq b$ with $a + b = 1$ and all $0 \leq r \leq 1/2$. This gives the first example of a stronger inequality of $a^{2b} + b^{2a} \leq 1$.

Keywords: Power-exponential function, monotonically decreasing function, monotonically increasing function.
2010 MSC: :26D10.

1. Introduction

The study of inequalities with power exponential functions is one of the active areas of research in the mathematical analysis. V. Cîrtoaje et al. [1, 2, 3, 4, 5, 6] studied some inequalities with power exponential functions. These problems of inequalities are very simple formula, but these proof are not as simple as it seems. It is noted that

$$a^{2b} + b^{2a} + \left(\frac{a-b}{2}\right)^2 \leq 2 \tag{1.1}$$

and

$$a^{3b} + b^{3a} + \left(\frac{a-b}{2}\right)^4 \leq 2 \tag{1.2}$$

holds for all $0 \leq a$ and $0 \leq b$ with $a + b = 2$. These inequalities (1.1) and (1.2) are proved by V. Cîrtoaje et al. [2, 6], respectively. In this paper, we will show that

$$a^{2b} + b^{2a} + r\,(ab(a-b))^2 \leq 1 \tag{1.3}$$

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holds for all $0 \le a$ and $0 \le b$ with $a+b=1$ and all $0 \le r \le 1/2$, which is a stronger inequality of

$$a^{2b} + b^{2a} \le 1. \tag{1.4}$$

The above inequality (1.4) is Conjecture 4.8 in [2] and proved by V. Cîrtoaje [3]. The following is our main theorem.

Theorem 1.1. *For all $0 \le a$ and $0 \le b$ with $a+b=1$ and all $0 \le r \le 1/2$, the inequality* (1.3) *holds.*

This gives the first example of a stronger inequality of (1.4).

2. Proof of Theorem 1.1

Proof. Without loss of generically, we assume that

$$0 \le b \le \frac{1}{2} \le a \le 1.$$

Applying Lemma 7.1 in [3], we have

$$a^{2b} \le 1 - 4ab^2 - 2ab(a-b)\ln a$$

and since the inequality (1.3) is strictly increasing for $0 \le r \le 1/2$, it suffices to show that

$$b^{2a} + \frac{1}{2}\left(ab(a-b)\right)^2 \le 4ab^2 + 2ab(a-b)\ln a. \tag{2.1}$$

We assume that $a = (1+t)/2$ and $b = (1-t)/2$, where $0 \le t \le 1$. Here, the inequality (2.1) is equivalent to

$$\left(\frac{1-t}{2}\right)^{t+1} + \frac{1}{32}(-1+t)^2(1+t)\left(-16+t^2+t^3\right) + \frac{1}{2}(1-t)t(1+t)(\ln(1+t) - \ln 2) \le 0.$$

Moreover, from Lemma 2.1 in [6], we have

$$(1-t)^{1+t} \le \frac{1}{4}(1-t)^2(2-t^2)(2+2t+t^2)$$

and by the well known fact we have

$$\begin{aligned}
2^{-t} &= e^{-t\ln 2} \\
&= 1 - (\ln 2)t + \frac{((\ln 2)t)^2}{2} - \frac{((\ln 2)t)^3}{3!} + \frac{((\ln 2)t)^4}{4!} - \cdots \\
&\le 1 - (\ln 2)t + \frac{((\ln 2)t)^2}{2} - \frac{((\ln 2)t)^3}{3!} + \frac{((\ln 2)t)^4}{4!}.
\end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned}
F(t) := \frac{1}{2}&\left(1 - (\ln 2)t + \frac{((\ln 2)t)^2}{2} - \frac{((\ln 2)t)^3}{3!} + \frac{((\ln 2)t)^4}{4!}\right) \\
&\times \frac{1}{4}(1-t)^2(2-t^2)(2+2t+t^2) + \frac{1}{32}(-1+t)^2(1+t)\left(-16+t^2+t^3\right) \\
&+ \frac{1}{2}(1-t)t(1+t)(\ln(1+t) - \ln 2) \le 0.
\end{aligned}$$

We have the fourth derivated function

$$F^{(4)}(t) = \frac{d^4}{dt^4}F(t) = \frac{f(t)}{(t+1)^3}$$

of $F(t)$, where

$$\begin{aligned} f(t) =& 62 + 126t - 33t^2 - 375t^3 - 405t^4 - 135t^5 \\ & + 12(1+t)^3\left(-2 - 15t + 35t^3\right)(\ln 2) \\ & - 6(1+t)^3\left(4 - 10t - 45t^2 + 70t^4\right)(\ln 2)^2 \\ & + 2(1+t)^3\left(4 + 20t - 30t^2 - 105t^3 + 126t^5\right)(\ln 2)^3 \\ & - (1+t)^3\left(-2 + 10t + 30t^2 - 35t^3 - 105t^4 + 105t^6\right)(\ln 2)^4. \end{aligned}$$

Then, we have derivatives

$$\begin{aligned} f^{(6)}(t) =& -5040\big(-60 - 78(\ln 2)^2 - 756t(\ln 2)^2 - 1008t^2(\ln 2)^2 - 35(\ln 2)^3 \\ & + 180(\ln 2) + 420t(\ln 2) + 210t(\ln 2)^3 + 1260t^2(\ln 2)^3 + 1260t^3(\ln 2)^3\big) \end{aligned}$$

Since

$$\frac{69}{100} < \ln 2 < \frac{7}{10},$$

we have

$$\begin{aligned} & -60 - 78(\ln 2)^2 - 756t(\ln 2)^2 - 1008t^2(\ln 2)^2 - 35(\ln 2)^3 \\ & + 180(\ln 2) + 420t(\ln 2) + 210t(\ln 2)^3 + 1260t^2(\ln 2)^3 + 1260t^3(\ln 2)^3 \\ & > -60 - 78\left(\frac{7}{10}\right)^2 - 756t\left(\frac{7}{10}\right)^2 - 1008t^2\left(\frac{7}{10}\right)^2 - 35\left(\frac{7}{10}\right)^3 \\ & + 180\left(\frac{69}{100}\right) + 420t\left(\frac{69}{100}\right) + 210t\left(\frac{69}{100}\right)^3 + 1260t^2\left(\frac{69}{100}\right)^3 + 1260t^3\left(\frac{69}{100}\right)^3 \\ & = \frac{1}{100000}(1397500 - 1165311t - 7999866t^2 + 41392134t^3) \\ & > \frac{1}{100000}(1300000 - 1200000t - 8000000t^2 + 40000000t^3) \\ & = 13 - 12t - 80t^2 + 400t^3. \end{aligned}$$

We set

$$\tilde{f}(t) = 13 - 12t - 80t^2 + 400t^3$$

then we have

$$\tilde{f}'(t) = 4\left(-3 - 40t + 300t^2\right).$$

Since

$$\tilde{f}'\left(\frac{2-\sqrt{13}}{30}\right) = 0 \quad \text{and} \quad \tilde{f}'\left(\frac{2+\sqrt{13}}{30}\right) = 0,$$

we have

$$\tilde{f}(t) \geq \tilde{f}\left(\frac{2+\sqrt{13}}{30}\right) \cong 10.5742.$$

Hence, we can get

$$f^{(6)}(t) < 0.$$

Thus, $f^{(5)}(t)$ is strictly decreasing for $0 < t < 1$. We have

$$\begin{aligned} f^{(5)}(t) =& -16200 + 151200(1+2t)(\ln 2) \\ & - 10800\left(11 + 84t + 98t^2\right)(\ln 2)^2 \\ & + 720\left(-73 + 546t + 2646t^2 + 2352t^3\right)(\ln 2)^3 \\ & - 3600\left(-13 - 49t + 147t^2 + 588t^3 + 441t^4\right)(\ln 2)^4, \end{aligned}$$

$$\begin{aligned} f^{(5)}(0) &= -16200 + 151200(\ln 2) - 118800(\ln 2)^2 - 52560(\ln 2)^3 + 46800(\ln 2)^4 \\ &\cong 24825.3, \end{aligned}$$

and

$$\begin{aligned} f^{(5)}(1) &= -16200 + 453600(\ln 2) - 2084400(\ln 2)^2 + 3939120(\ln 2)^3 - 4010400(\ln 2)^4 \\ &\cong -317162. \end{aligned}$$

Since $f^{(5)}(t)$ is strictly decreasing for $0 < t < 1$, there exists uniquely a real number $0 < t_1 < 1$ such that $f^{(5)}(t_1) = 0$. Since $f^{(5)}(t) > 0$ for $0 < t < t_1$ and $f^{(5)}(t) < 0$ for $t_1 < t < 1$, $f^{(4)}(t)$ is strictly increasing for $0 < t < t_1$ and $f^{(4)}(t)$ is strictly decreasing for $t_1 < t < 1$. We have

$$\begin{aligned} f^{(4)}(t) = &-9720 - 16200t \\ &+ 4320\left(6 + 35t + 35t^2\right)(\ln 2) \\ &- 3600\left(-3 + 33t + 126t^2 + 98t^3\right)(\ln 2)^2 \\ &+ 240\left(-77 - 219t + 819t^2 + 2646t^3 + 1764t^4\right)(\ln 2)^3 \\ &- 120\left(-22 - 390t - 735t^2 + 1470t^3 + 4410t^4 + 2646t^5\right)(\ln 2)^4, \end{aligned}$$

$$\begin{aligned} f^{(4)}(0) &= -9720 + 25920(\ln 2) + 10800(\ln 2)^2 - 18480(\ln 2)^3 + 2640(\ln 2)^4 \\ &\cong 7890.38 \end{aligned}$$

and

$$\begin{aligned} f^{(4)}(1) &= -25920 + 328320(\ln 2) - 914400(\ln 2)^2 + 1183920(\ln 2)^3 - 885480(\ln 2)^4 \\ &\cong -47797.5. \end{aligned}$$

Since $f^{(4)}(t)$ is strictly increasing for $0 < t < t_1$ and $f^{(4)}(t)$ is strictly decreasing for $t_1 < t < 1$, there exists uniquely a real number $t_1 < t_2 < 1$ such that $f^{(4)}(t_2) = 0$. Since $f^{(4)}(t) > 0$ for $0 < t < t_2$ and $f^{(4)}(t) < 0$ for $t_2 < t < 1$, $f^{(3)}(t)$ is strictly increasing for $0 < t < t_2$ and $f^{(3)}(t)$ is strictly decreasing for $t_2 < t < 1$. We have

$$\begin{aligned} f^{(3)}(t) = &-2250 - 9720t - 8100t^2 \\ &+ 144\left(-6 + 180t + 525t^2 + 350t^3\right)(\ln 2) \\ &- 36\left(-161 - 300t + 1650t^2 + 4200t^3 + 2450t^4\right)(\ln 2)^2 \\ &+ 12\left(-131 - 1540t - 2190t^2 + 5460t^3 + 13230t^4 + 7056t^5\right)(\ln 2)^3 \\ &- 6\left(83 - 440t - 3900t^2 - 4900t^3 + 7350t^4 + 17640t^5 + 8820t^6\right)(\ln 2)^4 \end{aligned}$$

$$\begin{aligned} f^{(3)}(0) &= -2250 - 864(\ln 2) + 5796(\ln 2)^2 - 1572(\ln 2)^3 - 498(\ln 2)^4 \\ &\cong -702.644 \end{aligned}$$

and

$$\begin{aligned} f^{(3)}(1) &= -20070 + 151056(\ln 2) - 282204(\ln 2)^2 + 262620(\ln 2)^3 - 147918(\ln 2)^4 \\ &\cong 2362.55. \end{aligned}$$

Since $f^{(3)}(t)$ is strictly decreasing for $0 < t < t_2$ and $f^{(3)}(t)$ is strictly decreasing for $t_2 < t < 1$, there exists uniquely a real number $0 < t_3 < t_2$ such that $f^{(3)}(t_3) = 0$. Since $f^{(3)}(t) < 0$ for $0 < t < t_3$ and $f^{(3)}(t) > 0$

for $t_3 < t < 1$, $f^{(2)}(t)$ is strictly decreasing for $0 < t < t_3$ and $f^{(2)}(t)$ is strictly increasing for $t_3 < t < 1$. We have

$$\begin{aligned} f^{(2)}(t) &= -66 - 2250t - 4860t^2 - 2700t^3 \\ &\quad + 72(1+t)\left(-17 + 5t + 175t^2 + 175t^3\right)(\ln 2) \\ &\quad - 36(1+t)\left(-21 - 140t - 10t^2 + 560t^3 + 490t^4\right)(\ln 2)^2 \\ &\quad + 12(1+t)\left(14 - 145t - 625t^2 - 105t^3 + 1470t^4 + 1176t^5\right)(\ln 2)^3 \\ &\quad - 6(1+t)\left(18 + 65t - 285t^2 - 1015t^3 - 210t^4 + 1680t^5 + 1260t^6\right)(\ln 2)^4 \end{aligned}$$

$$\begin{aligned} f^{(2)}(0) &= -66 - 1224(\ln 2) + 756(\ln 2)^2 + 168(\ln 2)^3 - 108(\ln 2)^4 \\ &\cong -520.172 \end{aligned}$$

and

$$\begin{aligned} f^{(2)}(1) &= -9876 + 48672(\ln 2) - 63288(\ln 2)^2 + 42840(\ln 2)^3 - 18156(\ln 2)^4 \\ &\cong 3529.68 \end{aligned}$$

Since $f^{(2)}(t)$ is strictly decreasing for $0 < t < t_3$ and $f^{(2)}(t)$ is strictly increasing for $t_3 < t < 1$, there exists uniquely a real number $t_3 < t_4 < 1$ such that $f^{(2)}(t_4) = 0$. Since $f^{(2)}(t) < 0$ for $0 < t < t_4$ and $f^{(2)}(t) > 0$ for $t_4 < t < 1$, $f'(t)$ is strictly decreasing for $0 < t < t_4$ and $f'(t)$ is strictly increasing for $t_4 < t < 1$. We have

$$\begin{aligned} f'(t) &= 126 - 66t - 1125t^2 - 1620t^3 - 675t^4 \\ &\quad + 36(1+t)^2\left(-7 - 20t + 35t^2 + 70t^3\right)(\ln 2) \\ &\quad - 6(1+t)^2\left(2 - 130t - 225t^2 + 280t^3 + 490t^4\right)(\ln 2)^2 \\ &\quad + 2(1+t)^2\left(32 + 20t - 465t^2 - 630t^3 + 630t^4 + 1008t^5\right)(\ln 2)^3 \\ &\quad - (1+t)^2\left(4 + 100t + 45t^2 - 630t^3 - 735t^4 + 630t^5 + 945t^6\right)(\ln 2)^4, \end{aligned}$$

$$\begin{aligned} f'(0) &= 126 - 252(\ln 2) - 12(\ln 2)^2 + 64(\ln 2)^3 - 4(\ln 2)^4 \\ &\cong -34.0483, \end{aligned}$$

and

$$\begin{aligned} f'(1) &= -3360 + 11232(\ln 2) - 10008(\ln 2)^2 + 4760(\ln 2)^3 - 1436(\ln 2)^4 \\ &\cong 870.774. \end{aligned}$$

Since $f'(t)$ is strictly decreasing for $0 < t < t_4$ and $f'(t)$ is strictly increasing for $t_4 < t < 1$, there exists uniquely a real number $t_4 < t_5 < 1$ such that $f'(t_5) = 0$. Since, $f'(t) < 0$ for $0 < t < t_5$ and $f'(t) > 0$ for $t_5 < t < 1$, $f(t)$ is strictly decreasing for $0 < t < t_5$ and $f(t)$ is strictly increasing for $t_5 < t < 1$. Since

$$f(0) = 2\left(31 - 12(\ln 2) - 12(\ln 2)^2 + 4(\ln 2)^3 + (\ln 2)^4\right) \cong 36.9595,$$

$$f(1) = -8\left(95 - 216(\ln 2) + 114(\ln 2)^2 - 30(\ln 2)^3 + 3(\ln 2)^4\right) \cong 73.9711,$$

and

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \frac{1}{512}\left(20656 - 106272(\ln 2) + 81648(\ln 2)^2 - 9288(\ln 2)^3 - 2079(\ln 2)^4\right) \\ &\cong -33.889. \end{aligned}$$

Since $f(t)$ is strictly decreasing for $0 < t < t_5$ and $f(t)$ is strictly increasing for $t_5 < t < 1$, we have only two real numbers a_1 and a_2 with $0 < a_1 < 1/2 < a_2 < 1$ such that $f(a_1) = 0$ and $f(a_2) = 0$. Since $f(t) > 0$ for all $0 < t < a_1$, $a_2 < t < 1$ and $f(t) < 0$ for all $a_1 < t < a_2$, $F^{(3)}(t)$ is strictly increasing for $0 < t < a_1$, $a_2 < t < 1$ and $F^{(3)}(t)$ is strictly decreasing for $a_1 < t < a_2$. We have

$$F^{(3)}(t) = \frac{g(t)}{(t+1)^2},$$

where

$$\begin{aligned} g(t) =& 200t + 304t^2 - 60t^3 - 360t^4 - 180t^5 \\ & + 12t(1+t)^2 \left(-8 - 30t + 35t^3\right) (\ln 2) \\ & - 24(1+t)^2 \left(1 + 4t - 5t^2 - 15t^3 + 14t^5\right) (\ln 2)^2 \\ & + 2(1+t)^2 \left(-4 + 16t + 40t^2 - 40t^3 - 105t^4 + 84t^6\right) (\ln 2)^3 \\ & - t(1+t)^2 \left(-8 + 20t + 40t^2 - 35t^3 - 84t^4 + 60t^6\right) (\ln 2)^4 \\ & + 48(1+t)^2 \ln(1+t). \end{aligned}$$

We have

$$F^{(3)}(0) = -\frac{1}{2}(\ln 2)^2(3 + \ln 2) \cong -0.887192,$$

$$F^{(3)}(1) = \frac{1}{16}\left(-24 + 12(\ln 2) + 24(\ln 2)^2 - 18(\ln 2)^3 + 7(\ln 2)^4\right) \cong -0.533122$$

and

$$\begin{aligned} F^{(3)}\left(\frac{1}{2}\right) = & \frac{1}{4608}(17968 - 46008(\ln 2) - 2160(\ln 2)^2 \\ & + 2160(\ln 2)^3 - 477(\ln 2)^4 + 13824(\ln 3)) \cong 0.181499. \end{aligned}$$

Since we have only two real numbers a_3 and a_4 with $0 < a_3 < 1/2$ and $1/2 < a_4 < 1$ such that $F^{(3)}(a_3) = 0$ and $F^{(3)}(a_4) = 0$, $F^{(3)}(t) < 0$ for all $0 < t < a_3$, $a_4 < t < 1$ and $F^{(3)}(t) > 0$ for all $a_3 < t < a_4$. Therefore, $F^{(2)}(t)$ is strictly decreasing for $0 < t < a_3$, $a_4 < t < 1$ and $F^{(2)}(t)$ is strictly increasings for $a_3 < t < a_4$. We have

$$F^{(2)}(t) = \frac{h(t)}{96(t+1)},$$

where

$$\begin{aligned} h(t) =& -6\left(15 + 15t - 76t^2 - 60t^3 + 45t^4 + 45t^5\right) \\ & + 24(1+t)\left(4 - 12t^2 - 30t^3 + 21t^5\right) (\ln 2) \\ & - 12(1+t)\left(-4 + 12t + 24t^2 - 20t^3 - 45t^4 + 28t^6\right) (\ln 2)^2 \\ & + 4t(1+t)\left(-12 + 24t + 40t^2 - 30t^3 - 63t^4 + 36t^6\right) (\ln 2)^3 \\ & - t^2(1+t)\left(-24 + 40t + 60t^2 - 42t^3 - 84t^4 + 45t^6\right) (\ln 2)^4 \\ & + 288t(1+t)\ln(1+t). \end{aligned}$$

We have

$$F^{(2)}(0) = \frac{1}{16}\left(-15 + 16(\ln 2) + 8(\ln 2)^2\right) \cong -0.00412631,$$

$$F^{(2)}(1) = \frac{1}{96}\left(48 - 120(\ln 2) + 60(\ln 2)^2 - 20(\ln 2)^3 + 5(\ln 2)^4\right) \cong -0.123508$$

and

$$F^{(2)}\left(\frac{1}{2}\right) = \frac{1}{24576}(-224 - 49728(\ln 2) - 9600(\ln 2)^2 + 1472(\ln 2)^3 - 77(\ln 2)^4 + 36864(\ln 3)) \cong 0.0678104.$$

Since we have only two real numbers a_5 and a_6 with $0 < a_5 < 1/2$ and $1/2 < a_6 < 1$ such that $F^{(2)}(a_5) = 0$ and $F^{(2)}(a_6) = 0$, $F^{(2)}(t) < 0$ for all $0 < t < a_5$, $a_6 < t < 1$ and $F^{(2)}(t) > 0$ for all $a_5 < t < a_6$. Therefore, $F'(t)$ is strictly decreasing for $0 < t < a_5$, $a_6 < t < 1$ and $F'(t)$ is strictly increasing for $a_5 < t < a_6$. We have

$$F'(t) = \frac{p(t)}{96},$$

where

$$\begin{aligned} p(t) = &- 6(-1+t)^2 t\left(7 + 18t + 9t^2\right) \\ &+ 12t(8 - 8t^2 - 15t^3 + 7t^5)(\ln 2) \\ &- 12(-1+t)t\left(4 - 2t - 10t^2 - 5t^3 + 4t^4 + 4t^5\right)(\ln 2)^2 \\ &+ 2(-1+t)t^2\left(12 - 4t - 24t^2 - 12t^3 + 9t^4 + 9t^5\right)(\ln 2)^3 \\ &- (-1+t)t^3\left(8 - 2t - 14t^2 - 7t^3 + 5t^4 + 5t^5\right)(\ln 2)^4 \\ &+ 48\left(-1 + 3t^2\right)\ln(1+t) \end{aligned}$$

We have

$$F'(0) = 0$$

and

$$F'(1) = 0.$$

Since there exists uniquely a real number a_7 with $0 < a_7 < 1$ such that $F'(a_7) = 0$, $F(t)$ is strictly decreasing for $0 < t < a_7$ and $F(t)$ is strictly increasing for $a_7 < t < 1$ Hence, we can get

$$F(t) \leq \max\{F(0), F(1)\}.$$

Since $F(0) = F(1) = 0$, we have $F(t) \leq 0$ for all $0 \leq t \leq 1$. Therefore, the proof of Theorem 1.1 is completed. □

Problem 2.1. What is the maximum value of a nonnegative real number r in the inequality $a^{2b} + b^{2a} + r\left(ab(a-b)\right)^2 \leq 1$ for all nonnegative real numbers a and b with $a + b = 1$?

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