



Positive solutions for a second-order delay p -Laplacian boundary value problem

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Abstract

This paper investigates the existence and multiplicity of positive solutions for a second-order delay p -Laplacian boundary value problem. By using fixed point index theory, some new existence results are established. ©2015 All rights reserved.

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1. Introduction

In this paper, we are mainly concerned with the existence and multiplicity of positive solutions for the following second-order delay p -Laplacian boundary value problem

$$\begin{cases} (\phi_p(u'(t)))' + f(t, u(t - \tau)) = 0, & t \in (0, 1), \tau \in (0, 1), \\ u(t) = \varphi(t), & t \in [-\tau, 0], \\ u(0) = u'(1) = 0, \end{cases} \quad (1.1)$$

where $\phi_p(s) = |s|^{p-2}s$, $\phi_p^{-1} = \phi_q$, $p^{-1} + q^{-1} = 1$, $p > 1, q > 1$, $\varphi \in C([-\tau, 0], \mathbb{R}^+)$, $\varphi(0) = 0$, and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0, \infty)$). Here, by a positive solution of (1.1) we mean a function $u \in C[-\tau, 1]$ such that $u(t) > 0$ for $t \in (0, 1]$ and u solves (1.1).

Differential equations with delay arise from a variety of areas in applied mathematics, physics and mathematical ecology. Clearly, comparing to the equations without delay, such equations, to a certain extent,

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reflect even more exactly the physical reality. Therefore, in recent years, there have been increasing interests in the study of this kind of problems and have received a lot of attention, see [1, 7, 12] and the references therein. Due to their wide applications, the existence and multiplicity of positive solutions for delay boundary value problems has also attracted increasing attention over the last decades, see [2, 3, 4, 5, 11, 14] and the references therein.

Recently, some authors also pay their attention to the existence and multiplicity of positive solutions for delay p -Laplacian boundary value problems, see [6, 9, 10, 13].

In [6], by using Guo-Krasnosel'skii fixed point theorem and generalization of the Leggett-Williams fixed point theorem due to Avery and Peterson, Du et al. considered the following multi-point boundary value problem with delay and one dimensional p -Laplacian

$$\begin{cases} (\phi_p(x'(t)))' + \lambda p(t)f(t, x(t-\tau)) = 0, t \in (0, 1), \\ x(t) = 0, -\tau \leq t \leq 0, x(1) = x(\eta), \end{cases} \quad (1.2)$$

where τ, η are given constants with $\tau > 0$, $0 < \eta < 1$, λ is a positive parameter. They obtained (1.2) has at least one positive solution or three positive solutions.

In [10], by virtue of Guo-Krasnosel'skii fixed point theorem, Jiang et al. established the existence of single and multiple nonnegative solutions to the problem

$$\begin{cases} (\phi_p(x'(t)))' + q(t)f(t, x(t-\tau)) = 0, t \in (0, 1) \setminus \{\tau\}, \\ x(t) = \xi(t), -\tau \leq t \leq 0, x(1) = 0. \end{cases} \quad (1.3)$$

In [13], Wang et al. studied the following delay differential equation with one-dimensional p -Laplacian $(\phi_p(x'(t)))' + q(t)f(t, x(t), x(t-1), x'(t)) = 0, t \in (0, 1)$, subject to one of the following two pairs of boundary conditions $x(t) = \xi(t), -1 \leq t \leq 0, x(1) = 0$, and $x(t) = \xi(t), -1 \leq t \leq 0, x'(1) = 0$. By using Avery-Peterson fixed point theorem, they obtained some results for the existence three positive solutions of the above two problems, respectively.

Motivated by the above works, we investigate the existence and multiplicity of positive solutions for (1.1). We first convert the problem to an equivalent integral equation. Then we introduce an appropriate linear operator and get its first eigenvalue and eigenfunction. Under some conditions concerning the first eigenvalue, by virtue of fixed point index theory, we establish several new existence theorems for (1.1).

2. Preliminaries

We first offer several basic facts used throughout this paper.

Lemma 2.1. *The problem (1.1) is equivalent to the following integral equation*

$$u(t) = \begin{cases} \varphi(t), & -\tau \leq t \leq 0, \\ \int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds, & 0 \leq t \leq 1. \end{cases} \quad (2.1)$$

This proof is very simple, so we omit it here. From (2.1) we should turn our aim to $t \in [0, 1]$, if there exists $u(t) \in C[0, 1]$, $u(t) > 0, \forall t \in (0, 1]$ such that the second equation of (2.1) holds true, then u defined by (2.1) is a positive solution for (1.1).

Let $E := C[0, 1]$, $\|u\| := \max_{t \in [0, 1]} |u(t)|$, $P := \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space and P is a cone on E . We let $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

Define an operator $A : P \rightarrow P$ by

$$(Au)(t) := \int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds, \quad 0 \leq t \leq 1. \quad (2.2)$$

The Arzela–Ascoli theorem, together with the continuity of f , implies that A is a completely continuous operator. Moreover, the existence of positive solutions for (1.1) is equivalent to the existence of positive fixed points of A .

Lemma 2.2. *If g is well defined and non-negative, non-increasing on $[0, 1]$. Then for any $t \in [0, 1]$,*

$$\int_0^t g(s)ds \geq t \int_0^1 g(s)ds.$$

Lemma 2.3. *Let $P_0 = \{u \in P : u(t) \geq t\|u\|, \forall t \in [0, 1]\}$. Then $A(P) \subset P_0$.*

Proof. For any $u \in P$, we have by (2.2), $(Au)(t) \leq \int_0^1 \left(\int_s^1 f(x, u(x-\tau))dx \right)^{\frac{1}{p-1}} ds$. On the other hand, let $g(s) := \left(\int_s^1 f(x, u(x-\tau))dx \right)^{\frac{1}{p-1}}$ and then g is non-negative and non-increasing on $[0, 1]$. From Lemma 2.2, we find $\int_0^t g(s)ds \geq t \int_0^1 g(s)ds$ and thus

$$\int_0^t \left(\int_s^1 f(x, u(x-\tau))dx \right)^{\frac{1}{p-1}} ds \geq t \int_0^1 \left(\int_s^1 f(x, u(x-\tau))dx \right)^{\frac{1}{p-1}} ds.$$

Therefore, $(Au)(t) \geq t\|Au\|, t \in [0, 1]$, as claimed. This completes the proof. \square

Define the linear operator $(Lu)(t) := \int_0^1 G(t, s)u(s)ds, u \in E$, where $G(t, s) := \min\{t, s\}$. Then $L : E \rightarrow E$ is a completely continuous and positive operator.

Lemma 2.4. ([8]) *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If there exists $v_0 \in P \setminus \{0\}$ such that $v - Av \neq \lambda v_0$ for all $v \in \partial\Omega \cap P$ and $\lambda \geq 0$, then $i(A, \Omega \cap P, P) = 0$.*

Lemma 2.5. ([8]) *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \rightarrow P$ is a completely continuous operator. If $v \neq \lambda Av$ for all $v \in \partial\Omega \cap P$ and $0 \leq \lambda \leq 1$, then $i(A, \Omega \cap P, P) = 1$.*

Lemma 2.6. *Let $\psi(t) := \sin \frac{\pi}{2}t$. Then $\psi \in P \setminus \{0\}$ and*

$$\int_0^1 G(t, s)\psi(s)ds = \frac{4}{\pi^2}\psi(t). \quad (2.3)$$

Lemma 2.7. (see [15, Lemma 2.6]) *Let $\theta > 0$ and $\varphi \in C([0, 1], \mathbb{R}^+)$. Then*

$$\left(\int_0^1 \varphi(t)dt \right)^\theta \leq \int_0^1 (\varphi(t))^\theta dt \quad \text{if } \theta \geq 1, \quad \left(\int_0^1 \varphi(t)dt \right)^\theta \geq \int_0^1 (\varphi(t))^\theta dt \quad \text{if } 0 < \theta \leq 1.$$

3. Main results

Let $p_* := \min\{1, p-1\}, p^* := \max\{1, p-1\}, N_\tau := \int_0^1 t^{p^*} \psi(t)dt, M_\tau := \int_0^{1-\tau} t^{p^*} \psi(t+\tau)dt,$

$$\alpha := \frac{N_\tau \pi^2}{4(1-\tau)}, \quad \beta := \frac{\pi^2}{4} \left[\frac{\tau(2 + \pi(1-\tau))}{2M_\tau} + 1 \right].$$

We now list our hypotheses on f .

(H1) $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$.

(H2) There exist $a_1 > \beta^{\frac{p-1}{p^*}}$ and $c > 0$ such that $f(t, x) \geq a_1 x^{p-1} - c$ for all $x \in \mathbb{R}^+$ and $t \in [0, 1]$.

(H3) There exist $b_1 \in (0, \alpha^{\frac{p-1}{p^*}})$ and $r > 0$ such that $f(t, x) \leq b_1 x^{p-1}$ for all $x \in [0, r]$ and $t \in [0, 1]$.

(H4) There exist $a_2 > \beta^{\frac{p-1}{p^*}}$ and $r > 0$ such that $f(t, x) \geq a_2 x^{p-1}$ for all $x \in [0, r]$ and $t \in [0, 1]$.

(H5) There exist $b_2 \in (0, \alpha^{\frac{p-1}{p^*}})$ and $c > 0$ such that $f(t, x) \leq b_2 x^{p-1} + c$ for all $x \in \mathbb{R}^+$ and $t \in [0, 1]$.

(H6) There are $\omega > 0$ and $\zeta \in (0, (\frac{p}{p-1})^{p-1})$ such that

$$f(t, x) \leq \zeta \omega^{p-1} \quad \text{for all } x \in [0, \omega] \text{ and } t \in [0, 1].$$

Theorem 3.1. Suppose that (H1)–(H3) are satisfied. Then (1.1) has at least one positive solution.

Proof. Let $\mathcal{M}_1 := \{u \in P : u = Au + \lambda\psi \text{ for some } \lambda \geq 0\}$, where ψ is determined by Lemma 2.6. We claim \mathcal{M}_1 is bounded. Indeed, $u \in \mathcal{M}_1$ implies $u(t) \geq (Au)(t)$ and thus

$$u(t) \geq \int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds.$$

Note $p_*, \frac{p_*}{p-1} \in [0, 1]$. For all $u \in \mathcal{M}_1$, Lemma 2.7 and (H2) imply

$$\begin{aligned} u^{p_*}(t) &\geq \left(\int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds \right)^{p_*} \\ &\geq \int_0^t \int_s^1 f^{\frac{p_*}{p-1}}(x, u(x-\tau)) dx ds \\ &= \int_0^1 G(t, s) f^{\frac{p_*}{p-1}}(s, u(s-\tau)) ds \\ &\geq \int_0^1 G(t, s) (a_1 u^{p-1}(s-\tau) - c)^{\frac{p_*}{p-1}} ds \\ &\geq \int_0^1 G(t, s) \left(a_1^{\frac{p_*}{p-1}} u^{p_*}(s-\tau) - c^{\frac{p_*}{p-1}} \right) ds \\ &\geq a_1^{\frac{p_*}{p-1}} \int_0^1 G(t, s) u^{p_*}(s-\tau) ds - \frac{c^{\frac{p_*}{p-1}}}{2}. \end{aligned} \quad (3.1)$$

Multiplying $\psi(t)$ on both sides of the above and integrating over $[0, 1]$, we find by (2.3)

$$\int_0^1 u^{p_*}(t) \psi(t) dt \geq \frac{4a_1^{\frac{p_*}{p-1}}}{\pi^2} \int_0^1 u^{p_*}(t-\tau) \psi(t) dt - \frac{c^{\frac{p_*}{p-1}}}{\pi} = \frac{4a_1^{\frac{p_*}{p-1}}}{\pi^2} \int_0^{1-\tau} u^{p_*}(t) \psi(t+\tau) dt - \frac{c^{\frac{p_*}{p-1}}}{\pi}$$

and thus

$$\begin{aligned} \frac{4a_1^{\frac{p_*}{p-1}}}{\pi^2} \int_0^{1-\tau} u^{p_*}(t) \psi(t+\tau) dt &\leq \int_0^1 u^{p_*}(t) \psi(t) dt - \int_0^{1-\tau} u^{p_*}(t) \psi(t+\tau) dt + \frac{c^{\frac{p_*}{p-1}}}{\pi} \\ &= \int_0^{1-\tau} u^{p_*}(t) (\psi(t) - \psi(t+\tau)) dt + \int_{1-\tau}^1 u^{p_*}(t) \psi(t) dt + \frac{c^{\frac{p_*}{p-1}}}{\pi} \\ &\leq \frac{\tau(2 + \pi(1-\tau))}{2} \|u\|^{p_*} + \frac{c^{\frac{p_*}{p-1}}}{\pi}. \end{aligned} \quad (3.2)$$

On the other hand, we have by Lemma 2.3,

$$\int_0^{1-\tau} u^{p_*}(t) \psi(t+\tau) dt \geq \int_0^{1-\tau} (t\|u\|)^{p_*} \psi(t+\tau) dt = \|u\|^{p_*} \int_0^{1-\tau} t^{p_*} \psi(t+\tau) dt. \quad (3.3)$$

Combining this and (3.2), we find

$$\frac{4a_1^{\frac{p_*}{p-1}}}{\pi^2} \int_0^{1-\tau} t^{p_*} \psi(t+\tau) dt \|u\|^{p_*} \leq \frac{\tau(2 + \pi(1-\tau))}{2} \|u\|^{p_*} + \frac{c^{\frac{p_*}{p-1}}}{\pi}.$$

Consequently,

$$\|u\|^{p_*} \leq \frac{\frac{c^{\frac{p_*}{p-1}}}{\pi}}{\frac{4a_1^{\frac{p_*}{p-1}}}{\pi^2} M_\tau - \frac{\tau(2 + \pi(1-\tau))}{2}}$$

for all $u \in \mathcal{M}_1$, which implies the boundedness of \mathcal{M}_1 , as claimed. Taking $R > \sup\{\|u\| : u \in \mathcal{M}_1\}$, we have $u - Au \neq \lambda\psi$, $\forall u \in \partial B_R \cap P$, $\lambda \geq 0$. Now by virtue of Lemma 2.4, we obtain

$$i(A, B_R \cap P, P) = 0. \quad (3.4)$$

Let $\mathcal{M}_2 := \{u \in \overline{B}_r \cap P : u = \lambda Au \text{ for some } \lambda \in [0, 1]\}$. We shall prove $\mathcal{M}_2 = \{0\}$. Indeed, if $u \in \mathcal{M}_2$, we have

$$u(t) \leq (Au)(t) = \int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds, \forall u \in \overline{B}_r \cap P.$$

Notice $p^*, \frac{p^*}{p-1} \geq 1$. Now by Lemma 2.7 and (H3), we obtain

$$\begin{aligned} u^{p^*}(t) &\leq \left(\int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds \right)^{p^*} \\ &\leq \int_0^t \int_s^1 f^{\frac{p^*}{p-1}}(x, u(x-\tau)) dx ds \\ &= \int_0^1 G(t, s) f^{\frac{p^*}{p-1}}(s, u(s-\tau)) ds \\ &\leq b_1^{\frac{p^*}{p-1}} \int_0^1 G(t, s) u^{p^*}(s-\tau) ds, \quad \forall u \in \mathcal{M}_2. \end{aligned} \quad (3.5)$$

Multiplying $\psi(t)$ on both sides of the above and integrating over $[0, 1]$, we find by (2.3)

$$\int_0^1 u^{p^*}(t) \psi(t) dt \leq \frac{4b_1^{\frac{p^*}{p-1}}}{\pi^2} \int_0^1 u^{p^*}(t-\tau) \psi(t) dt = \frac{4b_1^{\frac{p^*}{p-1}}}{\pi^2} \int_0^{1-\tau} u^{p^*}(t) \psi(t+\tau) dt. \quad (3.6)$$

Consequently, $\|u\|^{p^*} \int_0^1 t^{p^*} \psi(t) dt \leq \frac{4(1-\tau)b_1^{\frac{p^*}{p-1}}}{\pi^2} \|u\|^{p^*}$, $N_\tau > \frac{4(1-\tau)b_1^{\frac{p^*}{p-1}}}{\pi^2}$ implies $\mathcal{M}_2 = \{0\}$ and thus $u \neq \lambda Au, \forall u \in \partial B_r \cap P, \lambda \in [0, 1]$. Now Lemma 2.5 yields

$$i(A, B_r \cap P, P) = 1. \quad (3.7)$$

Combining this with (3.4) gives $i(A, (B_R \setminus \overline{B}_r) \cap P, P) = 0 - 1 = -1$. Hence the operator A has at least one fixed point in $(B_R \setminus \overline{B}_r) \cap P$ and therefore (1.1) has at least one positive solution. This completes the proof. \square

Theorem 3.2. Suppose that (H1), (H4) and (H5) are satisfied. Then (1.1) has at least one positive solution.

Proof. Let $\mathcal{M}_3 := \{u \in \overline{B}_r \cap P : u = Au + \lambda\psi \text{ for some } \lambda \geq 0\}$. We claim $\mathcal{M}_3 \subset \{0\}$. Indeed, if $u \in \mathcal{M}_3$, then we have $u \geq Au$ by definition

$$u(t) \geq \int_0^t \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds.$$

Note $p_*, \frac{p_*}{p-1} \in [0, 1]$. Now Lemma 2.7 and (H4) imply

$$u^{p_*}(t) \geq \int_0^1 G(t, s) f^{\frac{p_*}{p-1}}(s, u(s-\tau)) ds \geq a_2^{\frac{p_*}{p-1}} \int_0^1 G(t, s) u^{p_*}(s-\tau) ds, \quad \forall u \in \mathcal{M}_3.$$

Multiplying $\psi(t)$ on both sides of the above and integrating over $[0, 1]$, we find by (2.3)

$$\int_0^1 u^{p_*}(t) \psi(t) dt \geq \frac{4a_2^{\frac{p_*}{p-1}}}{\pi^2} \int_0^1 u^{p_*}(t-\tau) \psi(t) dt = \frac{4a_2^{\frac{p_*}{p-1}}}{\pi^2} \int_0^{1-\tau} u^{p_*}(t) \psi(t+\tau) dt.$$

Consequently,

$$\int_0^1 u^{p^*}(t)\psi(t)dt - \int_0^{1-\tau} u^{p^*}(t)\psi(t+\tau)dt \geq \frac{4a_2^{\frac{p^*}{p-1}} - \pi^2}{\pi^2} \int_0^{1-\tau} u^{p^*}(t)\psi(t+\tau)dt$$

and thus

$$\frac{\tau(2+\pi(1-\tau))}{2} \|u\|^{p^*} \geq \frac{4a_2^{\frac{p^*}{p-1}} - \pi^2}{\pi^2} \|u\|^{p^*} \int_0^{1-\tau} t^{p^*}\psi(t+\tau)dt, \quad \forall u \in \mathcal{M}_3.$$

$\frac{\tau(2+\pi(1-\tau))}{2} < \frac{4a_2^{\frac{p^*}{p-1}} - \pi^2}{\pi^2} M_\tau$ implies $\mathcal{M}_3 \subset \{0\}$, as claimed. As a result, we have $u - Au \neq \lambda\psi, \forall u \in \partial B_r \cap P, \lambda \geq 0$. Now Lemma 2.4 gives

$$i(A, B_r \cap P, P) = 0. \quad (3.8)$$

Let $\mathcal{M}_4 := \{u \in P : u = \lambda Au \text{ for some } \lambda \in [0, 1]\}$. We assert \mathcal{M}_4 is bounded. Indeed, if $u \in \mathcal{M}_4$, then u is concave and $u \leq Au$, i.e.,

$$u(t) \leq \int_0^t \left(\int_s^1 f(x, u(x-\tau))dx \right)^{\frac{1}{p-1}} ds.$$

Notice $p^*, \frac{p^*}{p-1} \geq 1$. Now by Lemma 2.7 and (H5), we obtain

$$\begin{aligned} u^{p^*}(t) &\leq \int_0^1 G(t, s) f^{\frac{p^*}{p-1}}(s, u(s-\tau))ds \\ &\leq \int_0^1 G(t, s) (b_2 u^{p-1}(s-\tau) + c)^{\frac{p^*}{p-1}} ds \\ &\leq b_3^{\frac{p^*}{p-1}} \int_0^1 G(t, s) u^{p^*}(s-\tau)ds + \frac{c_1^{\frac{p^*}{p-1}}}{2} \end{aligned} \quad (3.9)$$

for all $u \in \mathcal{M}_4$, $b_3 \in (b_2, \alpha_p)$ and $c_1 > 0$ being chosen so that $(b_2 z + c)^{\frac{p^*}{p-1}} \leq b_3^{\frac{p^*}{p-1}} z^{\frac{p^*}{p-1}} + c_1^{\frac{p^*}{p-1}}, \forall z \geq 0$. Multiplying $\psi(t)$ on both sides of the above and integrating over $[0, 1]$, we find by (2.3)

$$\int_0^1 u^{p^*}(t)\psi(t)dt \leq \frac{4b_3^{\frac{p^*}{p-1}}}{\pi^2} \int_0^{1-\tau} u^{p^*}(t)\psi(t+\tau)dt + \frac{c_1^{\frac{p^*}{p-1}}}{\pi}. \quad (3.10)$$

Therefore,

$$\|u\|^{p^*} \int_0^1 t^{p^*}\psi(t)dt \leq \frac{4(1-\tau)b_3^{\frac{p^*}{p-1}}}{\pi^2} \|u\|^{p^*} + \frac{c_1^{\frac{p^*}{p-1}}}{\pi}$$

and thus

$$\|u\|^{p^*} \leq \frac{\frac{c_1^{\frac{p^*}{p-1}}}{\pi}}{N_\tau - \frac{4(1-\tau)b_3^{\frac{p^*}{p-1}}}{\pi^2}}.$$

Now the boundedness of \mathcal{M}_4 , as asserted. Taking $R > \sup\{\|u\| : u \in \mathcal{M}_4\}$, we have $u \neq \lambda Au, \forall u \in \partial B_R \cap P, \lambda \in [0, 1]$. Now Lemma 2.5 yields

$$i(A, B_R \cap P, P) = 1. \quad (3.11)$$

Combining this with (3.8) gives $i(A, (B_R \setminus \overline{B_r}) \cap P, P) = 1 - 0 = 1$. Hence the operator A has at least one fixed point in $(B_R \setminus \overline{B_r}) \cap P$ and therefore (1.1) has at least one positive solution. This completes the proof. \square

Theorem 3.3. *Suppose that (H1), (H2), (H4) and (H6) are satisfied. Then (1.1) has at least two positive solutions.*

Proof. By (H6), we have

$$\begin{aligned}\|Au\| &= (Au)(1) = \int_0^1 \left(\int_s^1 f(x, u(x-\tau)) dx \right)^{\frac{1}{p-1}} ds \\ &\leq \int_0^1 \left(\int_s^1 \zeta \omega^{p-1} dx \right)^{\frac{1}{p-1}} ds \\ &= \frac{\zeta^{\frac{1}{p-1}} (p-1)}{p} \omega \\ &< \omega\end{aligned}$$

and thus $\|Au\| < \|u\|$ for all $u \in B_\omega \cap P$, so that $u \neq \lambda Au$, $\forall u \in \partial B_\omega \cap P, \lambda \in [0, 1]$. Lemma 2.5 yields

$$i(A, B_\omega \cap P, P) = 1. \quad (3.12)$$

On the other hand, in view of (H2) and (H4), we may choose $R > \omega$ and $r \in (0, \omega)$ so that (3.4) and (3.8) hold (see the proofs of Theorem 3.1 and 3.2). Combining (3.4), (3.8) and (3.12), we obtain $i(A, (B_R \setminus \overline{B}_\omega) \cap P, P) = 0 - 1 = -1$, $i(A, (B_\omega \setminus \overline{B}_r) \cap P, P) = 1 - 0 = 1$. Hence A has at least two fixed points, one in $(B_R \setminus \overline{B}_\omega) \cap P$ and the other in $(B_\omega \setminus \overline{B}_r) \cap P$. This proves that (1.1) has at least two positive solutions. This completes the proof. \square

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