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Lower and upper solutions for a discrete first-order nonlocal problems at resonance

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Abstract

We discuss the existence of solutions for the discrete first-order nonlocal problem

$$\begin{cases} \Delta u(t-1) = f(t, u(t)), \quad t \in \{1, 2, \cdots, T\}, \\ u(0) + \sum\limits_{i=1}^{m} \alpha_i u(\xi_i) = 0, \end{cases}$$

where $f : \{1, \cdots, T\} \times \mathbb{R} \to \mathbb{R}$ is continuous, $T > 1$ is a fixed natural number, $\alpha_i \in (-\infty, 0]$, $\xi_i \in \{1, \cdots, T\}(i = 1, \cdots, m,\ 1 \le m \le T,\ m \in \mathbb{N})$ are given constants such that $\sum\limits_{i=1}^{m} \alpha_i + 1 = 0$. We develop the methods of lower and upper solutions by the connectivity properties of the solution set of parameterized families of compact vector fields.

Keywords: Coincidence point, first-order discrete nonlocal problem, contraction, lower and upper solutions, connected sets.
2010 MSC: 47H10, 54H25.

1. Introduction

Let $T \in \mathbb{N}$ be an integer with $T > 1$, $\mathbb{T} := \{1, \cdots, T\}$, $\hat{\mathbb{T}} := \{0, 1, \cdots, T\}$. we are concerned with the following first-order discrete nonlocal problem

$$\Delta u(t-1) = f(t, u(t)), \quad t \in \mathbb{T}, \tag{1.1}$$

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$$u(0)+\sum_{i=1}^{m}\alpha_i u(\xi_i)=0, \tag{1.2}$$

where $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous, $\alpha_i\in(-\infty,0], \xi_i\in\{1,\cdots,T\}(i=1,\cdots,m,\ 1\le m\le T,\ m\in\mathbb{N})$ are given constants such that $\sum\limits_{i=1}^{m}\alpha_i+1=0$. If we take $m=1$, $\xi_1=T$ and $\alpha_1=-1$, one can see problem (1.1), (1.2) is the first-order discrete periodic boundary value problem.

Problem (1.1), (1.2) happens to be at resonance in the sense that the associated linear homogeneous problem

$$\Delta u(t-1)=0,\quad t\in\mathbb{T}, \tag{1.3}$$

$$u(0)+\sum_{i=1}^{m}\alpha_i u(\xi_i)=0 \tag{1.4}$$

has $u(t)=c, c\in\mathbb{R}$, as nontrivial solutions.

In recent years, since the nonlocal problems of difference equations play an important role in many fields such as computer science, economics, neural network, ecology, cybernetics, more and more people pay attention to it, see references[1-9,12-18] and the references therein. However, there are few papers dealt with the nonlocal problems of first-order difference equations.

Our ideas arise from [10,11]. In 2002, Ma [10] considered the first-order three-point boundary value problems for differential equations

$$u'=f(t,u),\qquad t\in(a,c), \tag{1.5}$$

$$Mu(a)+Nu(b)+Ru(c)=\alpha, \tag{1.6}$$

where $b\in(a,c)$, $f:[a,c]\times\mathbb{R}^n\to\mathbb{R}^n$ is a Carathédory function, $M,\ N,\ R\in M_{n\times n}$ and $\alpha\in\mathbb{R}$ are given. He established the existence and uniqueness results for boundary value problem (1.5), (1.6) at nonresonance. Then, in 2003, Ma [11] investigated the following second-order m-point boundary value problems at resonance,

$$u''=f(t,u(t)),\ t\in(0,1), \tag{1.7}$$

$$u'(0)=0,\ u(1)=\sum_{i=1}^{m-1}a_i u(\xi_i), \tag{1.8}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, $a_i\in(0,\infty)$ and $\xi_i\in(0,1)$ are given constants such that $\sum_{i=1}^{m-1}a_i=1$. he obtained the existence results and multiplicity results for (1.7), (1.8) by using the connectivity properties of the solution set of parameterized families of compact vector fields.

To our knowledge, the existence of solutions for the problem (1.1), (1.2) at resonance has not been studied. So, in this paper, we will develop the methods of lower and upper solution for boundary value problem (1.1), (1.2) by using the connectivity properties of the solution set of parameterized families of compact vector fields, and obtain the existence results under the case of well order upper and lower solution and the case of upper and lower solutions with opposite order.

The rest of the paper is organized as follows. In section 2, we give some preliminaries. In section 3, we consider the case of well order lower and upper solutions. In section 4, we deal with the case of upper and lower solutions with opposite order.

2. Preliminaries

Let $X:=\{u|u:\hat{\mathbb{T}}\to\mathbb{R}\}$, $Y:=\{u|u:\mathbb{T}\to\mathbb{R}\}$ be equipped with the norm

$$\|u\|_X=\max_{k\in\hat{\mathbb{T}}}|u(k)|,\qquad\qquad \|u\|_Y=\max_{k\in\mathbb{T}}|u(k)|,$$

respectively. It is easy to see that $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ are Banach spaces.

The proofs of the methods of lower and upper solution are based on the connectivity properties of the solution sets of parameterized families of compact vector fields; they are a direct consequence of Mawhin[6,Lemma 2.3].

Theorem 2.1. *Let E be a Banach space and let $C \subset E$ be a nonempty, bounded, closed convex subset. Suppose that $T : [a, b] \times C \to C$ is completely continuous. Then the set*

$$S = \{(\lambda, x)|T(\lambda, x) = x, \quad \lambda \in [a, b]\}$$

contains a closed connected subset Σ which connects $\{a\} \times C$ to $\{b\} \times C$.

Definition 2.2. We say that the function $x \in X$ is an upper solution of problem (1.1), (1.2) if

$$\Delta x(t-1) \geq f(t, x(t)), \qquad t \in \mathbb{T}, \tag{2.1}$$

$$x(0) + \sum_{i=1}^{m} \alpha_i x(\xi_i) \geq 0, \tag{2.2}$$

and $y \in X$ is a lower solution of problem (1.1), (1.2) if

$$\Delta y(t-1) \leq f(t, y(t)), \qquad t \in \mathbb{T}, \tag{2.3}$$

$$y(0) + \sum_{i=1}^{m} \alpha_i y(\xi_i) \leq 0. \tag{2.4}$$

If the inequalities in (2.1) and (2.3) are strict, then x and y are called strict upper and lower solutions.

Define a linear operator $L : D(L) \subset X \to Y$ by setting

$$D(L) = \{u \in X|u(0) + \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0\}$$

and for $u \in D(L)$

$$Lu(t) = \Delta u(t-1), \qquad t \in \mathbb{T}. \tag{2.5}$$

Lemma 2.3. *Let L be defined as (2.5). Then*

$$Ker(L) = \{c|c \in \mathbb{R}\} \tag{2.6}$$

and

$$Im(L) = \Big\{y \in Y|\sum_{i=1}^{m} \alpha_i \sum_{k=1}^{\xi_i} y(k) = 0\Big\}. \tag{2.7}$$

Proof. For $u(t) = c \in X$,

$$\sum_{i=1}^{m} \alpha_i u(\xi_i) = c\sum_{i=1}^{m} \alpha_i = -u(0)$$

implies that (2.6) holds.

If $y \in Im(L)$, then there exists a function $u \in D(L)$ such that $y(t) = \Delta u(t-1)$. Thus, we obtain

$$u(t) = u(0) + \sum_{l=1}^{t} y(l)$$

and

$$\sum_{i=1}^{m}\alpha_i u(\xi_i)=\sum_{i=1}^{m}\alpha_i u(0)+\sum_{i=1}^{m}\alpha_i\sum_{l=1}^{\xi_i}y(l).$$

Then, Combine with (1.2),

$$\sum_{i=1}^{m}\alpha_i\sum_{k=1}^{\xi_i}y(k)=0.$$

On the other hand, we suppose $y\in Y$ and it satisfies $\sum\limits_{i=1}^{m}\alpha_i\sum\limits_{k=1}^{\xi_i}y(k)=0$. It's not difficult to prove there exists $y\in Im(L)$. □

For $y\in Y$, we define

$$Qy=\Gamma_0\sum_{i=1}^{m}\alpha_i\sum_{k=1}^{\xi_i}y(k), \tag{2.8}$$

where

$$\Gamma_0=\frac{1}{\sum_{i=1}^{m}\alpha_i\xi_i}<0.$$

So Y =Im$(L)+\mathbb{R}$. Also Im$(L)\cap\mathbb{R}=0$. Hence Y =Im$(L)\oplus\mathbb{R}$. Let $P:X\to$Ker(L) be such that $(Pu)(t)=u(0)$. Then X =Ker$(P)\oplus$Ker(L).

Let $\tilde{X}$:=Ker$(P)=\{u\in X|u(0)=0\}$. For every $u\in X$, we have the unique decomposition $u(t)=\rho+w(t)$, where $\rho\in\mathbb{R}$ and $w\in Y$. Let $L_P=L|_{D(L)\cap\tilde{X}}$; then L_P is a one to one operator from $D(L)\cap\tilde{X}$ to Im(L).

Define

$$K_P=L_P^{-1}$$

and

$$K_{PQ}=K_P(I-Q).$$

Let $N:X\to Y$ be the nonlinear operator defined by

$$(Nu)(t)=f(t,u(t)),\qquad t\in\mathbb{T}.$$

3. Well Order Lower and Upper Solutions

In this section, we assume that x is a strict upper solution and y a strict lower solution for (1.1), (1.2) satisfying $x(t)>y(t)$ on $\hat{\mathbb{T}}$.

Set

$$D=\{(t,u)|y(t)\le u(t)\le x(t),t\in\hat{\mathbb{T}}\}.$$

Define an auxiliary function

$$f^*(t,u(t))=\begin{cases} f(t,x(t)), & u(t)>x(t),\ t\in\mathbb{T},\\ f(t,u), & y(t)\le u(t)\le x(t),\ t\in\mathbb{T},\\ f(t,y(t)), & u(t)<y(t),\ t\in\mathbb{T}\end{cases}$$

and consider the problem

$$\Delta u(t-1)=f^*(t,u(t)),\qquad t\in\mathbb{T}, \tag{3.1}$$

$$u(0)+\sum_{i=1}^{m}\alpha_i u(\xi_i)=0. \tag{3.2}$$

Let $N^*:X\to Y$ be the nonlinear operator defined by

$$(N^*u)(t)=f^*(t,u(t)),\qquad t\in\mathbb{T}. \tag{3.3}$$

Since X is finite dimensional, it's easy to see $K_{PQ}N^*:X\to X$ is completely continuous.

Lemma 3.1. *If there is a solution u of (3.1), (3.2), then*

$$y(t) \le u(t) \le x(t), \qquad t \in \hat{\mathbb{T}}. \tag{3.4}$$

In other words, u is a solution of (1.1), (1.2).

Proof. We first prove that $u(t) \le x(t)$ for all $t \in \hat{\mathbb{T}}$. Set $m(t) = u(t) - x(t)$. Suppose on the contrary that

$$m(t_0) = \max\{u(t) - x(t) | t \in \hat{\mathbb{T}}\} > 0$$

for some $t_0 \in \hat{\mathbb{T}}$. We divide the following proof into two steps.

Step 1. If $t_0 \in \mathbb{T}$, then $\Delta m(t_0 - 1) \ge 0$. On the other hand,

$$\begin{aligned} \Delta m(t_0 - 1) &= \Delta u(t_0 - 1) - \Delta x(t_0 - 1) \\ &< f^*(t_0, u(t_0)) - f(t_0, x(t_0)) \\ &= 0. \end{aligned}$$

A contradiction!

Step 2. If $t_0 = 0$, then $u(0) - x(0) > 0$. On the other hand, by (1.2) and (2.2), we have

$$u(0) - x(0) + \sum_{i=1}^{m} \alpha_i (u - x)(\xi_i) \le 0.$$

Thus, there exists a $\xi_{i_0} \in \mathbb{T}$ such that $m(\xi_{i_0}) = (u - x)(\xi_{i_0}) > 0$. Now, similar to Step 1, we also can get a contradiction.

Similarly we can show that $u(t) \ge y(t)$ for $t \in \hat{\mathbb{T}}$. □

Theorem 3.2. *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that x and y are a strict upper solution and a strict lower solution for (1.1), (1.2), respectively, satisfying $x(t) > y(t)$ on $\hat{\mathbb{T}}$. Then (1.1), (1.2) have a solution $u \in D$.*

Proof. From Lemma 3.1, we only need to show that

$$\Delta u(t - 1) = f^*(t, u(t)), \qquad t \in \mathbb{T}, \tag{3.5}$$

$$u(0) + \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0 \tag{3.6}$$

has a solution. It is easy to see that $K_{PQ}N^* : X \to X$ is completely continuous, and (3.5), (3.6) are equivalent to the system

$$w(t) = K_{PQ}N^*(\rho + w(t)), \tag{3.7}$$

$$QN^*(\rho + w(t)) = 0. \tag{3.8}$$

Since f^* is bounded, we know from (3.7) and the Schauder fixed point theorem that for every $\rho \in \mathbb{R}$, the set $W(\rho) := \{w \in \tilde{Y} | (\rho, w) \text{ satisfies } (3.7)\} \ne \emptyset$. Moreover, by Theorem 2.1, the set

$$S := \{(\rho, w) \in \mathbb{R} \times \tilde{Y} | (\rho, w) \text{ satisfies } (3.7)\} \tag{3.9}$$

contains a connected subset Σ which joins $\{a\} \times W(a)$ and $\{b\} \times W(b)$ for every $a, b \in \mathbb{R}$ with $a < b$. Put

$$W := \{w \in \tilde{Y} | (\rho, w) \in S\}.$$

Then by (3.7), there exists a constant $M > 0$, independent of ρ, such that

$$\|w\|_\infty \le M, \qquad \text{for all } w \in W.$$

Hence if we choose $\rho \in \mathbb{R}$ so large that for all $w \in W$

$$\rho_1 + w(t) > x(t), \qquad \text{for } t \in \hat{\mathbb{T}},$$

this implies that $f^*(t, \rho_1 + w(t)) \equiv f(t, x(t))$ and $W(\rho_1)$ reduces to the single-point set $\{K_{PQ} f(t, x(t))\}$. Moreover, for every $w \in W(\rho_1)$, we have

$$\begin{aligned} QN^*(\rho_1 + w(t)) &= \Gamma_0 \sum_{i=1}^{m} \alpha_i \sum_{k=1}^{\xi_i} f^*(k, \rho_1 + w(k)) \\ &= \Gamma_0 \sum_{i=1}^{m} \alpha_i \sum_{k=1}^{\xi_i} f(k, x(k)) \\ &< \Gamma_0 \sum_{i=1}^{m} \alpha_i \sum_{k=1}^{\xi_i} \Delta x(k-1) \\ &= \Gamma_0 \sum_{i=1}^{m} \alpha_i (x(\xi_i) - x(0)) \\ &\le 0. \end{aligned}$$

Similarly, we can choose ρ_2 with $\rho_2 < \rho_1$ such that for every $w \in W(\rho_2)$, $\rho_2 + w(t) < y(t)$, for $t \in \hat{\mathbb{T}}$. This implies $f^*(t, \rho_2 + w(t)) \equiv f(t, y(t))$ and $W(\rho_2)$ reduces to the single-point set $\{K_{PQ} f(t, y(t))\}$. Moreover, for every $w \in W(\rho_2)$, we have $QN^*(\rho_2 + w(t)) > 0$. Therefore, by the connectivity of Σ, there must exist some $\rho_0 \in (\rho_2, \rho_1)$ and $w(\rho_0) \in W(\rho_0)$ such that $(\rho_0, w(\rho_0)) \in \Sigma$ and (3.8) holds. Thus $\rho_0 + w(\rho_0)$ is a solution of (3.5), (3.6). □

Example 3.1 Consider the problem

$$\begin{cases} \Delta u(t-1) = 3 + (u - \sin \frac{\pi t}{T+1})(u+2)(u-5), & t \in \mathbb{T}, \\ u(0) - u(\eta) = 0, \end{cases} \tag{3.10}$$

where $\eta \in \mathbb{T}$. It's not difficult to see that $x(t) = 3$ and $y(t) = \sin \frac{\pi t}{T+1}$ are the strict upper solution and the strict lower solution of (3.10), respectively. So by Theorem 3.2, (3.10) has at least one solution. □

4. Upper and Lower Solutions with Opposite Order

Let x, y be strict upper solution and lower solution of (1.1), (1.2) satisfying $x(t) < y(t),\ t \in \hat{\mathbb{T}}$. Then there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$, $x(t) - \frac{1}{n}$ and $y(t) + \frac{1}{n}$ are also strict upper solution and strict lower solution for (1.1), (1.2). For each $n \ge n_0$, we define an auxiliary operator $\tilde{f} : \mathbb{T} \times Y \to Y$ by

$$\tilde{f}_n(t, u(t)) = \begin{cases} f(t, y(t) + \frac{1}{n}),\ u(t) \ge y(t) + \frac{1}{n},\ t \in \mathbb{T}, \\ f(t, u(t)) + n\gamma_u[f(t, y(t) + \frac{1}{n}) - f(t, u(t))], \\ \qquad \text{if } y(t) \le u(t)\ \forall\ t \in \mathbb{T} \text{ and } \exists\ t_u,\ \text{s. t.} u(t_u) < y(t_u) + \frac{1}{n}, \\ f(t, u(t)),\ \exists\ t_u \in \mathbb{T},\ \text{s. t. } x(t_u) < u(t_u) < y(t_u), \\ f(t, u(t)) + n\sigma_u[f(t, u(t) - f(t, x(t) - \frac{1}{n}))], \\ \qquad \text{if } x(t) \ge u(t)\ \forall\ t \in \mathbb{T} \text{ and } \exists\ t_u \text{ s. t. } u(t_u) > x(t_u) - \frac{1}{n}, \\ f(t, x(t) - \frac{1}{n}),\ u(t) \le x(t) - \frac{1}{n},\ t \in \mathbb{T}, \end{cases} \tag{4.1}$$

where

$$\gamma_u = \min_{t \in \mathbb{T}} |u(t) - y(t)|, \qquad \sigma_u = \min_{t \in \mathbb{T}} |x(t) - u(t)|.$$

Clearly, if $y(t) \leq u(t)$, $\forall\ t \in \hat{\mathbb{T}}$ and there exists t_u satisfying $u(t_u) < y(t_u) + \frac{1}{n}$, then $\gamma_u \in [0, \frac{1}{n})$; if $x(t) \geq u(t)\ \forall\ t \in \hat{\mathbb{T}}$ and there exists t_u satisfying $u(t_u) > y(t_u) - \frac{1}{n}$, then $\sigma_u \in [0, \frac{1}{n})$.

Moreover the operator $\tilde{f}_n : \mathbb{T} \times Y \to Y$ is continuous. Let us consider the problem

$$\begin{aligned} \Delta u(t-1) &= \tilde{f}_n(t, u(t)), \qquad t \in \mathbb{T}, \\ u(0) + \sum_{i=1}^{m} \alpha_i u(\xi_i) &= 0. \end{aligned} \tag{4.2}$$

Let $\tilde{N}_n : X \to Y$ be the nonlinear operator defined by

$$\tilde{N}_n u(t) = \tilde{f}_n(t, u(t)), \qquad t \in \mathbb{T}. \tag{4.3}$$

Then $K_p(I-Q)\tilde{N}_n : X \to X$ is completely continuous.

Lemma 4.1. *If there is a solution u of (4.2), then*

$$x(t_u) - \frac{1}{n} < u(t_u) < y(t_u) + \frac{1}{n} \qquad \textit{for a } t_u \in \hat{\mathbb{T}}.$$

In other words, u is a solution of (1.1), (1.2).

Proof. Assume on the contrary that there is no $t_u \in \hat{\mathbb{T}}$, such that $x(t_u) - \frac{1}{n} < u(t_u) < y(t_u) + \frac{1}{n}$. Then either

$$u(t) \leq x(t) - \frac{1}{n}, \qquad t \in \hat{\mathbb{T}}, \tag{4.4}$$

or

$$u(t) \geq y(t) + \frac{1}{n}, \qquad t \in \hat{\mathbb{T}}. \tag{4.5}$$

If (4.4) holds, then from (4.1) we know that $\tilde{f}_n(t, u(t)) = f(t, x(t) - \frac{1}{n})$, $t \in \mathbb{T}$. Set $z(t) = u(t) - (x(t) - \frac{1}{n})$. Then we have from (4.1) and the fact that $x(t) - \frac{1}{n}$ is a strict upper solution of (1.1), (1.2) that

$$\begin{aligned} \Delta z(t-1) &= \Delta u(t-1) - \Delta x(t-1) \\ &= \tilde{f}_n(t, u(t)) - \Delta x(t-1) \\ &= f(t, x(t) - \frac{1}{n}) - \Delta x(t-1) \\ &< 0, \end{aligned} \tag{4.6}$$

and

$$z(0) + \sum_{i=1}^{m} \alpha_i z(\xi_i) = \sum_{i=1}^{m} \alpha_i (z(\xi_i) - z(0)) > 0. \tag{4.7}$$

But on the other hand,

$$z(0) + \sum_{i=1}^{m} \alpha_i z(\xi_i) = -x(0) - \sum_{i=1}^{m} \alpha_i u(\xi_i) \leq 0. \tag{4.8}$$

A contradiction.

If (4.5) holds, using the same argument we can get a desired contradiction again. □

From now on, we need the following assumptions:

$(H1)$ $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t, u)| \leq p(t)|u(t)| + r(t), \qquad t \in \mathbb{T},$$

where $r,\ p : \mathbb{T} \to \mathbb{R}$ and $\sum\limits_{s=1}^{T} |p(s)| < \frac{1}{2}$.

$(H2)$ There exist a strict lower solution α and a strict upper solution β such that

$$\alpha(t) < x(t) < y(t) < \beta(t) \qquad \text{for } t \in \hat{\mathbb{T}}.$$

Lemma 4.2. *Let x and y be the strict upper solution and strict lower solution of (1.1), (1.2) and satisfy $x(t) < y(t)$ for all $t \in \hat{\mathbb{T}}$. Assume that f satisfies $(H1)$, Then there exists constant $M^* \in (0, \infty)$, independent of $n \geq n_0$, such that*

(i) for every solution u of the problems (4.2), the implication

$$\exists t_u \in \hat{\mathbb{T}} : x(t_u) - \frac{1}{n} < u(t_u) < y(t_u) + \frac{1}{n} \Rightarrow \|u\|_X < M^*$$

is valid.

(ii) for every solution u of the problem (4.2), we have

$$\|u\|_X < M^*$$

Proof. Let u be a solution of (4.2) with $x(t_u) - \frac{1}{n} < u(t_u) < y(t_u) + \frac{1}{n}$ for some $t_u \in \hat{\mathbb{T}}$. Let us put $\gamma := \{\|x\|_X, \|y\|_X\} + \frac{1}{n}$. It's easy to see that

$$|u(t_u)| \leq \gamma. \tag{4.9}$$

The condition $(H1)$ and the definition of $\tilde{f}_n$ imply that

$$\begin{aligned} &- 2p(t)|u(t)| - p(t)\max\{|y(t)| + \frac{1}{n}, |x(t)| + \frac{1}{n}\} - 3r(t) \\ \leq &\Delta u(t-1) \\ \leq &2p(t)|u(t)| + p(t)\max\{|y(t)| + \frac{1}{n}, |x(t)| + \frac{1}{n}\} + 3r(t). \end{aligned} \tag{4.10}$$

Summing (4.10) from $t_u + 1$ to t if $t_u < t$ or from t to t_u if $t \leq t_u$, we can get

$$|u(t)| \leq 2\|u\|_X \sum_{s=1}^{T} |p(s)| + 3\sum_{s=1}^{T} |r(s)| + \gamma.$$

Then

$$\|u\|_X \leq (1 - 2\sum_{s=1}^{T} |p(s)|)^{-1}(3\sum_{s=1}^{T} |r(s)| + \gamma) =: M^*$$

(ii) It is immediate consequence of (i) and Lemma 4.1.

Lemma 4.3. *Let $(H2)$ hold. Then for each $n \geq n_0$ with $\alpha(t) < x(t) - \frac{1}{n} < y(t) + \frac{1}{n} < \beta(t)$ and every solution u of (4.2), the implication*

$$\exists t_u \in \hat{\mathbb{T}} : x(t_u) - \frac{1}{n} < u(t_u) < y(t_u) + \frac{1}{n} \Rightarrow \alpha(t) < u(t) < \beta(t), \quad t \in \hat{\mathbb{T}}$$

is valid.

Proof. Similar to the proof of Lemma 3.1. □

Theorem 4.4. *Suppose there exist strict upper and lower solution x and y of (1.1), (1.2) with $x(t) < y(t)$ for $t \in \hat{\mathbb{T}}$. Assume that either $(H1)$ or $(H2)$ be fulfilled. Then there is a solution u to (1.1), (1.2) such that*

$$y(t_u) \leq u(t_u) \leq x(t_u) \ \textit{for a} \ t_u \in \hat{\mathbb{T}}.$$

Proof. If $\{u_n\}$ is a sequence of solutions of (4.2) satisfying

$$x(t_{u_n}) - \frac{1}{n} < u_n(t_{u_n}) < y(t_{u_n}) + \frac{1}{n} \ \text{ for a } t_{u_n} \in \hat{\mathbb{T}}, \tag{4.11}$$

then by Lemma 4.2 or Lemma 4.3, there exists a positive constant C, independent of n, such that

$$\|u_n\|_X \le C.$$

Thus, by standard argument, we can show that there exist $\{u_{n_j}\} \subseteq \{u_n\}$, $\bar{u} \in X$ and $t_{\bar{u}} \in \hat{\mathbb{T}}$, such that

$$\|u_{n_j} - \bar{u}\|_X \to 0,\ t_{u_{n_j}} \to t_{\bar{u}},\ \text{ as } j \to \infty.$$

Such $\bar{u}$ is a solution of (1.1), (1.2) satisfies $y(t_{\bar{u}}) \ge u(\bar{u}) \ge x(\bar{u})$. So we only need to show that for each $n \ge n_0$, (4.2) has a solution u_n satisfying (4.11).

In the following, we only prove the existence of u_n under $(H1)$ since the case that $(H2)$ is true can be treated by the similar way. We divide the proof into two steps.

Step 1 : f is bounded. In this case, the operator that $\tilde{f}_n : \mathbb{T} \times Y \to Y$ is bounded uniformly. Using the same arguments to prove Theorem 3.2, we can get that (4.2) has a solution u_n satisfying (4.11).

Step 2 : f is unbounded on $\mathbb{T} \times \mathbb{R}$. In this case, $\tilde{f}_n : \mathbb{T} \times Y \to Y$ may be unbounded. So we need to introduce an auxiliary operator $F_n : \mathbb{T} \times Y \to Y$ by

$$F_n(t, u(t)) = \tilde{f}_n(t, \phi(u(t)))$$

with

$$\phi(z) = \begin{cases} M^*, & z \ge M^*, \\ z, & -M^* < z < M^*, \\ -M^*, & z \le -M^*, \end{cases}$$

where M^* is given by Lemma 4.2. Now, consider the problem

$$\Delta u(t-1) = F_n(t, u_n(t)), \qquad t \in \mathbb{T}, \tag{4.12}$$

$$u(0) + \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0. \tag{4.13}$$

It is easy to see that $y(t)$ and $x(t)$ are the strict lower solution and strict upper solution of (4.12), (4.13) with $x(t) < y(t)$ for $t \in \hat{\mathbb{T}}$, and $F_n : \mathbb{T} \times Y \to Y$ is bounded uniformly. Moreover, applying the same argument as in the proof of Lemma 4.2 (i), we can get that for every solution u of (4.12), (4.13) satisfying $\|u\|_X < M^*$. This mean that every solution of (4.12), (4.13) is a solution of (4.2).

Now by step 1, (4.12), (4.13) has a solution u_n satisfying

$$x(t_{u_n}) - \frac{1}{n} < u(t_{u_n}) < y(t_{u_n}) + \frac{1}{n} \ \text{ for a } t_{u_n} \in \hat{\mathbb{T}}.$$

Therefore we get a solution u_n of (4.2) which satisfies (4.11). □

Example 4.1 Consider the problem

$$\begin{cases} \Delta u(t-1) = -e^u(-u+t+1), \qquad t \in \mathbb{T}, \\ u(0) + \sum_{i=1}^{m} \alpha_i u(\xi_i) = 0, \end{cases} \tag{4.14}$$

where $\xi_i \in \mathbb{T}$, $\alpha_i < 0 (i = 1, 2, \cdots, m)$ satisfy $1 + \sum_{i=1}^{m} \alpha_i = 0$. It's not difficult to see that $x(t) = 0$ and $y(t) = Tt + 1$ are the strict upper solution and the strict lower solution of (4.14), respectively, and satisfy $x(t) < y(t)$, $t \in \hat{\mathbb{T}}$. So by Theorem 4.4, (4.14) has at least one solution u satisfying $0 \le u(t_0) \le Tt + 1$ for some $t_0 \in \hat{\mathbb{T}}$. □

Similar to above, one can obtain the multiplicity results of (1.1), (1.2) by using Theorem 3.2 and Theorem 4.2.

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