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Solutions for a nonlinear fractional boundary value problem with sign-changing Green's function

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Abstract

This paper considers the existence, uniqueness and non-existence of solution for a quasi-linear fractional boundary value problems with sign-changing Green's function. Under certain growth conditions on the nonlinear term, we employ the Leray-Schauder alternative fixed point theorem to obtain an existence result of nontrivial solution and use the Banach contraction mapping principle to obtain a uniqueness result. Moreover, the existence result of positive solutions is obtained when the nonlinear term is also allowed to change sign.

Keywords: Fractional boundary value problem, fixed point theorem, sign-changing Green's function, positive solution, existence.
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1. Introduction

Fractional differential equations have received widely attention in recent decade since fractional order models are successfully used in cybernetics, viscoelastic mechanics, image processing, biological tissues etc. For example, [12] fractional order circuit elements have provided a useful model for the transient and the sinusoidal steady state frequency response of dielectrics, biological tissues and bioelectrodes. A basic cardiac tissue electrode impedance can be represented by a series combination of a resistor and two fractional lumped circuit elements. The overall transient voltage $V(t)$ for this model can be represented by the following fractional differential equation:

$$C_\alpha \frac{\mathrm{d}^\alpha V(t)}{\mathrm{d}t^\alpha} = RC_\alpha \frac{\mathrm{d}^\alpha I(t)}{\mathrm{d}t^\alpha} + I(t) + \frac{C_\alpha}{C_\beta} \frac{\mathrm{d}^{\alpha-\beta} I(t)}{\mathrm{d}t^{\alpha-\beta}}.$$

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For more details of fractional applications, for instance, see [1, 8, 9, 12, 13]. At the same time, we also notice that the existence research of (positive) solutions for boundary value problems of fractional differential equations have received considerable attention owing to their importance in the mathematic theory and application. Fractional differential equations of α order ($0 < \alpha \le 1, 1 < \alpha \le 2$ and until to $n-1 < \alpha \le n$) subject to various kinds of boundary value conditions(two points, multiple points, integral boundary value, periodic and anti-periodic etc.), even with impulsive effect or p-Laplacian operator etc., are investigated by many authors. Different techniques and methods have been employed to deal with the solvability of such boundary value problems, for example the use of fixed point index theory, the classic cone-compressions and expansions fixed point theorems, the method of upper and lower solutions, and Leggett-Williams theorem and its extensions. We refer reader to [2, 3, 4, 10, 14, 15, 16, 17, 18, 19, 20, 21, 23, 25, 26] and reference therein.

The referred above papers mostly considered the differential equations with the form $D^\alpha u = f(t,u)$ or $D^\alpha u = f(t,u,D^\beta u)$ and their Green's kernels are nonnegative. Some authors also considered "quasi-linear" differential equations. Graef *et al.* [4] investigated existence and uniqueness of solutions for the following Riemann-Liouville fractional boundary value problems with Dirichlet boundary condition

$$\begin{cases} -D^\alpha_{0^+} u + a(t)u = w(t)f(t,u), t \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. A special Green's function was constructed and used.

Fourth-order differential equations have extensively applications in material mechanics and the existence of solutions for many boundary value problems of the equations are investigated. Li [11] studied the existence of positive solutions for the following boundary value problem for fourth-order differential equation

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t,u(t)),\ t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.2}$$

Sufficient conditions for existence of at least one positive solutions are established by the fixed point index theory. Wei *et al.* [20] gave a necessary and sufficient condition for the existence of positive solutions for the following integral boundary value problems of differential equations with Captuo fractional derivative

$$\begin{cases} {}^cD^\alpha_{0^+} u(t) + f(t,u(t)) = 0,\ 0 < t < 1, \\ au(0) - bu'(0) = 0, cu(1) + du'(1) = 0, \\ u''(0) + u'''(0) = \int_0^1 u''(s)dp(s), \\ u''(1) + u'''(1) + \int_0^1 u''(s)dq(s) = 0, \end{cases} \tag{1.3}$$

where $3 < \alpha < 4$, under the growth condition on the nonlinear term f and f may be singular on $t = 0$ and $u = 0$. The main tools are the cone-compressions and expansions fixed point theorem with norm form for a compact operator and the method of upper and lower solutions.

Inspired by the above works, in this paper, we consider the following boundary value problem

$$\begin{cases} {}^cD^\alpha_{0^+} u(t) - m^2\ {}^cD^{\alpha-2}_{0^+} u(t) = \mu g(t)f(u(t)),\ t \in (0,1), \\ u'(0) = u'(1) = 0, \\ u''(0) - m^2u(0) = 0, u''(1) - m^2u(1) = 0, \end{cases} \tag{1.4}$$

where constants $\mu > 0, m > 0, 3 < \alpha \le 4$, ${}^cD^\alpha$ is the Caputo fractional derivative of order α. $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g(t) : [0,1] \to \mathbb{R}$ is continuous. A function $u(t) \not\equiv 0$ is called a positive solution of (1.4) if it satisfies the problem and is nonnegative.

To the best of our knowledge, no paper has considered the problem (1.4). Our technique and method are derived from [14, 20]. We first translate the problem (1.4) into two boundary value problems of "lower" order differential equations by means of an appropriate variable substitution, so that we can easily obtain their solutions without complicated calculation. Next, we consider some properties of the Green's function, which is sign-changing, of the boundary value problems. Finally, under certain growth conditions on the nonlinear term, several sufficient conditions for the existence and uniqueness of nontrivial solution are obtained by the Leray-Schauder alternative fixed point theorem and the Banach contraction mapping principle, respectively. Finally, the existence and non-existence results of positive solutions are also obtained under the condition of sign-changing both Green's kernel and nonlinear term.

This paper is organized as follows. Sect. 2 contains some preliminary results and considers some properties of Green's functions. Sect. 3 is devoted to the existence, uniqueness and non-existence of solution for problem (1.4). Sec. 4 provides an example to illustrate Theorem 3.5.

2. Preliminaries and Lemmas

For the reader's convenience, we present firstly some definitions of fractional calculus theory and lemmas. Throughout the paper, $C[0,1]$ denotes a Banach space of all continuous functions mapping $[0,1]$ to real number set $\mathbb{R}$ equipped with maximum norm $\|.\|$.

Definition 2.1 ([8, 20])**.** The Riemann-Liouville fractional integral I^{α}_{0+} and D^{α}_{0+} are defined by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

and

$$D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}f(s)ds = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{0+}f)(t)$$

where $n-1 < \alpha \le n,\ n \in \mathbb{N}$, respectively, provided the integrals exist.

Definition 2.2. A function $f : [0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as

$${}^cD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s) = (I^{n-\alpha}_{0+}f^{(n)})(t)ds,$$

provided the integral exists.

It is known (see [8]) that.

Lemma 2.3. *Let $n-1 < \alpha \le n,\ n \in \mathbb{N}$. If $u(t) \in C^n[a,b]$, then*

$$I^{\alpha}_{0+}({}^cD^{\alpha}_{0+}u(t)) = u(t) - c_0 - c_1 t - c_2 t^2 - ... - c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R},\ i = 0,1,2,...,n-1$.

Lemma 2.4. *For any $h(t) \in C[0,1]$, $3 < \alpha \le 4$, the unique solution of the boundary value problems*

$$\begin{cases} {}^cD^{\alpha-2}_{0+}u(t) + h(t) = 0,\ t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha-2)}\begin{cases} t(1-s)^{\alpha-3} - (t-s)^{\alpha-3}, & 0 \le s \le t \le 1; \\ t(1-s)^{\alpha-3}, & 0 \le t \le s \le 1, \end{cases} \tag{2.3}$$

and $G(t,s)$ has the following properties:

(1) *$G(t, s)$ is continuous on* $[0, 1] \times [0, 1]$*;*

(2) $|G(t, s)| \leq 1/\Gamma(\alpha - 2)$ *for all* $t, s \in [0, 1]$*;*

(3) $0 \leq \int_0^1 G(t, s)dt = \frac{1}{\Gamma(\alpha-2)}(1 - s)^{\alpha-3}(\frac{1}{2} - \frac{1-s}{\alpha-2})$ *for all* $s \in [1 - (\alpha - 2)/2, 1]$ *and*

$\frac{1}{\Gamma(\alpha-2)}(s - \frac{1}{2})(1 - s)^{\alpha-3} < \int_0^1 G(t, s)dt \leq 0$ *for all* $s \in [0, 1 - (\alpha - 2)/2]$.

(4) $0 \leq \int_0^1 G(t, s)ds = \frac{1}{\Gamma(\alpha-1)}(t - t^{\alpha-2}) \leq \frac{\alpha-3}{\Gamma(\alpha-1)} \left(\frac{1}{\alpha-2}\right)^{\frac{\alpha-2}{\alpha-3}}$ *for all* $t \in [0, 1]$.

Proof. By Lemma 2.3, for some constants $c_0,\ c_1 \in \mathbb{R}$, we have

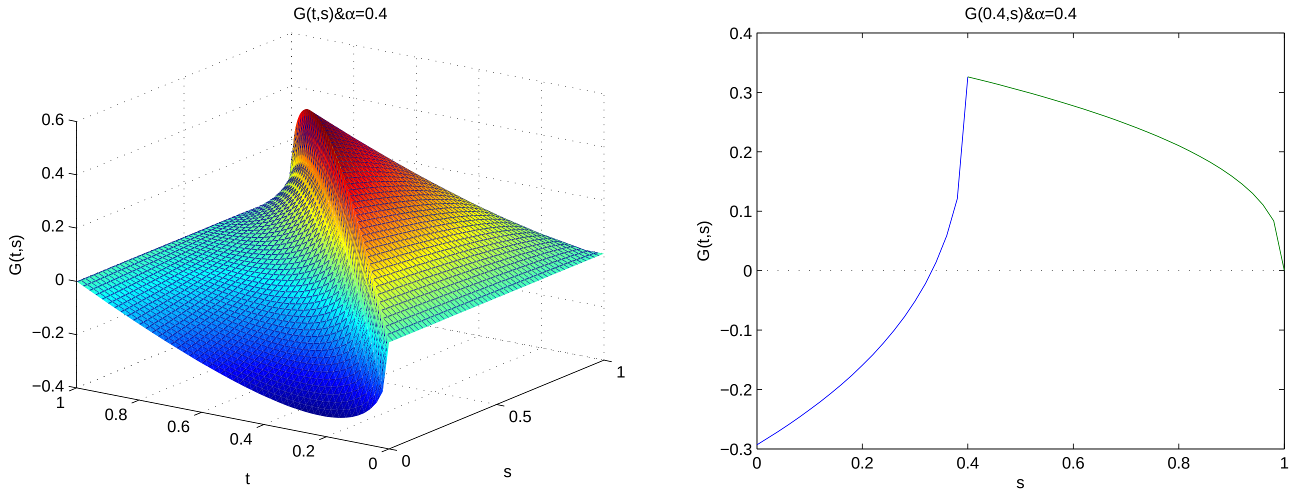
$$u(t) = c_0 + c_1 t - I_{0+}^{\alpha-2}(h(t)) = c_0 + c_1 t - \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha-3} h(s)ds. \tag{2.4}$$

By the boundary conditions $u(0) = u(1) = 0$, we have $c_0 = 0,\ c_1 = \frac{1}{\Gamma(\alpha-2)} \int_0^1 (1 - s)^{\alpha-3} h(s)ds$. Substituting the values of $c_0,\ c_1$ into (2.4), we have

$$\begin{aligned} u(t) &= \frac{t}{\Gamma(\alpha - 2)} \int_0^1 (1 - s)^{\alpha-3} y(s)ds - \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha-3} h(s)ds \\ &= \frac{t}{\Gamma(\alpha - 2)} \left(\int_0^t (1 - s)^{\alpha-3} h(s)ds + \int_t^1 (1 - s)^{\alpha-3} h(s)ds \right) - \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha-3} h(s)ds \\ &= \int_0^1 G(t, s)h(s)ds. \end{aligned} \tag{2.5}$$

The others are simple calculations and therefore omitted. This completes the proof. □

Remark 2.5. Let $3 < \alpha \leq 4$. From (3) in Lemma 2.4, one can know that the $G(t, s)$ defined by (2.3) changes its sign, Indeed, as $\alpha = 3.5, G(0.5, 0.25) = \frac{1}{2\sqrt{\pi}}[\sqrt{\frac{3}{4}} - 1] < 0$. Furthermore, it has infinitely many sign-changing points. We can see the following images of the $G(t, s)$ and $G(0.4, s)$ when order $\alpha = 0.4$. Obviously, this makes it difficult to investigate the existence of positive solution for the problem (2.1) by some techniques of cone theory.



Lemma 2.6 ([6])**.** *Let* $y \in C[0, 1]$*, the second-order linear boundary value problem*

$$\begin{cases} -u''(t) + m^2 u(t) = y(t),\ t \in (0, 1), \\ u'(0) = u'(1) = 0 \end{cases} \tag{2.6}$$

has a unique solution

$$u(t) = \int_0^1 H(t,s)y(s)ds, \tag{2.7}$$

where $H(t,s) = \dfrac{1}{m\sinh m}\begin{cases} \cosh(m(1-t))\cdot\cosh ms, & 0 \le s \le t \le 1, \\ \cosh(m(1-s))\cdot\cosh mt, & 0 \le t \le s \le 1. \end{cases}$ *satisfies*

$$0 < \frac{1}{m\sinh m} \le H(t,s) \le \frac{\cosh^2 m}{m\sinh m} =: \mathcal{H} \tag{2.8}$$

for $t, s \in [0,1]$ *and* $\int_0^1 H(t,s)ds = \int_0^1 H(t,s)dt = 1/m^2$.

Lemma 2.7. *Let* $3 < \alpha < 4$. *Then the boundary value problem* (1.4) *is equivalent to the following integral equation*

$$u(t) = \int_0^1 \mu\mathcal{G}(t,\tau)g(\tau)f(u(\tau))d\tau, \tag{2.9}$$

where $\mathcal{G}(t,\tau) = \int_0^1 H(t,s)G(s,\tau)ds$ *and* $G(t,s)$ *is given by* (2.3), $H(t,s)$ *is given by* (2.7). *Furthermore,* $\mathcal{G}(t,\tau)$ *is continuous and sign-changing on* $[0,1]\times[0,1]$.

Proof. Let $y(t) = -u''(t) + m^2u(t)$. Then $y(t)$ satisfies that

$$\begin{cases} {}^cD_{0^+}^{\alpha-2}y(t) + \mu g(t)f(u(t)) = 0, \ t \in (0,1), \\ y(0) = y(1) = 0 \end{cases} \tag{2.10}$$

and

$$\begin{cases} -u''(t) + m^2u(t) = y(t), \ t \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{2.11}$$

By the Lemma 2.6 and the Lemma 2.7, exchanging integral order we obtain that

$$\begin{aligned} u(t) &= \int_0^1 H(t,s)\int_0^1 \mu G(s,\tau)g(\tau)f(u(\tau))d\tau ds \\ &= \int_0^1 \mu g(\tau)f(u(\tau))\int_0^1 H(t,s)G(s,\tau)dsd\tau \\ &= \int_0^1 \mu\mathcal{G}(t,\tau)g(\tau)f(u(\tau))d\tau. \end{aligned} \tag{2.12}$$

In order to prove that $\mathcal{G}(t,\tau)$ is sign-changing, we consider that

$$\begin{aligned} \int_0^1 \mathcal{G}(t,\tau)dt &= \int_0^1\int_0^1 H(t,s)G(s,\tau)dsdt \\ &= \int_0^1 G(s,\tau)\int_0^1 H(t,s)dtds \\ &= \frac{1}{m^2}\int_0^1 G(s,\tau)ds \end{aligned} \tag{2.13}$$

by Lemma 2.6. Note that (3) of Lemma 2.4, we know that $\int_0^1 \mathcal{G}(t,\tau)dt$ is sign-changing, which implies that $\mathcal{G}(t,\tau)$ is sing-changing too. This completes the proof. □

Define an integral operator $\mathcal{A} : C[0,1] \to C[0,1]$ by

$$\mathcal{A}u(t) := \int_0^1 \mu\mathcal{G}(t,\tau)g(\tau)f(u(\tau))d\tau. \tag{2.14}$$

It is obvious that the operator $\mathcal{A}$ is completely continuous by the continuities of $\mathcal{G}$ and f. Clearly, the problem (1.4) has a solution if and only if the operator $\mathcal{A}$ has a fixed point in $C[0,1]$. So we only need to seek a fixed point of the operator $\mathcal{A}$ in $C[0,1]$.

The following lemmas are used to prove our main results.

Lemma 2.8 ([5]). *Let X be a real Banach space and Ω be a open subset of $X, 0 \in \Omega$, $T : \overline{\Omega} \to X$ be a completely continuous operator. Then either exist $x \in \partial\Omega, \lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

Lemma 2.9 ([22]). *Let $\Omega \subset \mathbb{R}^n$ be bounded domain, $X := C(\overline{\Omega})$ be a Banach space. $T : X \to X$ is a compact operator with the form $(Tu)(t) = \int_\Omega K(t,s)g(s)f(u(s))ds, \forall u \in X$, where g is a continuous function, $K : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ is integral kernel. If*

(C1) $f : [0,\infty) \to \mathbb{R}$ is continuous and $f(0) > 0$.

(C2) There exists a number $k > 1$ such that

$$\int_\Omega (K(t,s)g(s))^+ ds \geq k \int_\Omega (K(t,s)g(s))^- ds, \qquad t \in \overline{\Omega}$$

where $(K(t,s)g(s))^+, (K(t,s)g(s))^-$ denote the positive and negative part of the function $K(t,s)g(s)$ for fixed $t \in \overline{\Omega}$, respectively.

Then there exists a positive number λ^* such that nonlinear operator equation $u = \lambda Tu$ has a positive solution for $\lambda < \lambda^*$.

Lemma 2.10. *Assume that f is non-decreasing on [0,1], then*

$$\int_0^x f(t)dt \leq x \int_0^1 f(t)dt, \quad x \in (0,1).$$

3. Main Results

For convenience, we introduce the following notations. Let $k_1 = \int_0^1 k(t)dt,\ h_1 = \int_0^1 h(t)dt, c_1 = \int_0^1 c(t)dt$ (for detail, see Theorem 3.1, 3.2). Two assumptions conditions are listed as following.

(H1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g(t)f(0) \not\equiv 0$, $g(t) : [0,1] \to \mathbb{R}$ is continuous.

(H2) $f : [0,+\infty) \to \mathbb{R}$ is continuous and $f(0) > 0$, $g(t) : [0,1] \to [0,+\infty)$ is continuous and $g(t)(1-t)^{\alpha-3}$ is nondecreasing.

Theorem 3.1. *Suppose that (H1) holds and there exist nonnegative functions $k, h \in L^1[0,1]$ such that*

$$|\mu g(t)f(u(t))| \leq k(t)|u(t)| + h(t),\ a.e.\ (t,u) \in [0,1] \times \mathbb{R}, \tag{3.1}$$

and $k_1\mathcal{H} < \Gamma(\alpha-2), \mathcal{H}$ is defined of (2.8). Then the problem (1.4) has at least one nontrivial solutions $u \in C[0,1]$.

Proof. By condition (3.1), we know $h(t) \geq g(t)f(0)\ a.e.t \in [0,1]$. Since $g(t)f(0) \not\equiv 0$, one has $h_1 > 0$. Let

$$r = \mathcal{H}h_1/(\Gamma(\alpha-2))^2(\Gamma(\alpha-2) - k_1\mathcal{H}), \quad \Omega = \{u \in C[0,1] : \|u\| < r\}.$$

Since the operator $\mathcal{A}$ is a completely continuous, $\mathcal{A} : \overline{\Omega} \to C[0,1]$ is also completely continuous.

Suppose that $u \in \partial\Omega, \lambda > 1$ such that $\mathcal{A}u = \lambda u$, then

$$\begin{aligned}
\lambda r = \lambda\|u\| = \|\mathcal{A}u\| &= \max_{0\leq t\leq 1} |\mathcal{A}u(t)| \\
&\leq \int_0^1 H(t,s) \int_0^1 \mu|G(s,\tau)g(\tau)f(u(\tau))|d\tau ds \\
&\leq \mathcal{H} \int_0^1 \int_0^1 \mu|G(s,\tau)||g(\tau)f(u(\tau))|d\tau ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\mathcal{H}}{\Gamma(\alpha-2)}\int_0^1 |\mu g(\tau)f(u(\tau))|d\tau ds \\
&\le \frac{\mathcal{H}}{\Gamma(\alpha-2)}\left(\|u\|\int_0^1 |k(\tau)|d\tau + \int_0^1 |h(\tau)|d\tau\right) \\
&= \frac{\mathcal{H}}{\Gamma(\alpha-2)}(rk_1+h_1),
\end{aligned}
$$

by (2) of Lemma 2.4. Therefore $\lambda \le 1$, which contradicts $\lambda > 1$. By Lemma 2.8, A has a fixed point in $\overline{\Omega}$.

Since $g(t)f(0) \not\equiv 0$ for $t \in [0,1]$, the problem (1.4) has a nontrivial solution in $\overline{\Omega}$. This completes the proof. □

Corollary 3.2. *Suppose that (H1) holds and there exists nonnegative functions $h \in L^1[0,1]$ such that*

$$|\mu g(t)f(u(t))| \le h(t), \ \text{a.e. } (t,u) \in [0,1]\times\mathbb{R}, \tag{3.2}$$

Then the problem (1.4) has at least one nontrivial solutions $u \in C[0,1]$.

The uniqueness result will be considered in the following.

Theorem 3.3. *Suppose that (H1) holds and there exists nonnegative function $c(t) \in L^1[0,1]$ such that*

$$|\mu g(t)f(u_1) - \mu g(t)f(u_2)| \le c(t)|u_1-u_2|$$

a.e.$(t,u_i) \in [0,1]\times\mathbb{R}(i=1,2)$ and $0 < c_1\mathcal{H} < \Gamma(\alpha-2)$, $\mathcal{H}$ defined by (2.8). Then the problem (1.4) has a unique nontrivial solution.

Proof. For any $u_1, u_2 \in C[0,1]$, it follows that

$$
\begin{aligned}
\|\mathcal{A}u_1 - \mathcal{A}u_2\| &= \max_{0\le t\le 1} |\mathcal{A}u_1(t) - \mathcal{A}u_2(t)| \\
&\le \int_0^1 |H(t,s)|\int_0^1 \mu|G(s,\tau)||g(\tau)f(u_1(\tau)) - g(\tau)f(u_2(\tau))|d\tau ds \\
&\le \frac{\mathcal{H}}{\Gamma(\alpha-2)}\int_0^1\int_0^1 \mu|g(\tau)f(u_1(\tau)) - g(\tau)f(u_1(\tau))|d\tau ds \\
&\le \frac{\mathcal{H}}{\Gamma(\alpha-2)}\int_0^1 c(t)|u_1(\tau)-u_2(\tau)|d\tau \\
&\le \frac{\mathcal{H}}{\Gamma(\alpha-2)}\|u_1-u_2\|\int_0^1 c(t)dt \\
&= \frac{\mathcal{H}c_1}{\Gamma(\alpha-2)}\|u_1-u_2\|.
\end{aligned}
\tag{3.3}
$$

Thus $\mathcal{A}$ is a contraction map. Banach contraction Theorem yields that $\mathcal{A}$ has a unique fixed point in $C[0,1]$. Since $g(t)f(0) \not\equiv 0$ for $t \in [0,1]$, the problem (1.4) has a unique nontrivial solution. The proof is completed. □

Corollary 3.4. *Suppose that (H1) holds and*

$$|\mu g(t)f(u_1) - \mu g(t)f(u_2)| \le c|u_1-u_2|$$

a.e.$(t,u_i) \in [0,1]\times\mathbb{R}(i=1,2)$ and $0 < c\mathcal{H} < \Gamma(\alpha-2), c > 0$. Then the problem (1.4) has a unique nontrivial solution.

Theorem 3.5. *Assume that (H2) holds. Then there exists a positive number μ^* such that problem (1.4) has a positive solution for $\mu < \mu^*$.*

Proof. Let $Tu(t) := \int_0^1 \mathcal{G}(t,\tau)g(\tau)f(u(\tau))d\tau$. Then $T : C[0,1] \to C[0,1]$ is completely continuous. Now we need only check (C2) in Lemma 2.9. By (2.3), note that $\alpha > 3$, for $0 \le s \le 1$, we compute

$$\begin{aligned}\int_0^1 G(s,\tau)g(\tau)d\tau &= \int_0^s G(s,\tau)g(\tau)d\tau + \int_s^1 G(s,\tau)g(\tau)d\tau\\ &= \frac{1}{\Gamma(\alpha-2)}\left(\int_0^1 sg(\tau)(1-\tau)^{\alpha-3}d\tau - \int_0^s g(\tau)(s-\tau)^{\alpha-3}d\tau\right)\\ &\ge \frac{1}{\Gamma(\alpha-2)}\left(s\int_0^1 g(\tau)(1-\tau)^{\alpha-3}d\tau - \int_0^s g(\tau)(1-\tau)^{\alpha-3}d\tau\right)\\ &\ge 0\end{aligned} \tag{3.4}$$

by Lemma 2.10. Then, from (3.4), we obtain

$$\begin{aligned}\int_0^1 \mathcal{G}(t,\tau)g(\tau)d\tau &= \int_0^1\int_0^1 H(t,s)G(s,\tau)g(\tau)dsd\tau\\ &= \int_0^1 H(t,s)\int_0^1 G(s,\tau)g(\tau)d\tau ds\\ &> \frac{1}{m\sinh m}\int_0^1 G(s,\tau)g(\tau)d\tau ds > 0\end{aligned}$$

by (2.8). Which implies that there exists a number $k > 1$ such that

$$\int_0^1 \mathcal{G}^+(t,\tau)g(\tau)d\tau \ge k\int_0^1 \mathcal{G}^-(t,\tau)g(\tau)d\tau, \quad t\in[0,1], \tag{3.5}$$

where $\mathcal{G}^+(t,\tau)$ and $\mathcal{G}^-(t,\tau)$ denote positive and negative parts of the function $\mathcal{G}(t,\cdot)$ for fixed $t \in [0,1]$, respectively. That is, the condition (C2) in Lemma 2.9 holds. Therefore there exists $\mu^* > 0$ such that the operator equation $u = \mu Tu = \mathcal{A}u$ has a positive solution for $\mu < \mu^*$. Equivalently, problem (1.4) has a positive solution for $\mu < \mu^*$. □

Finally, A result on non-existence of solution for the problem (1.4) is given.

Theorem 3.6. *Assume that (H2) holds and*

$$\mu g(t)f(u) < M|u| := |u|\left(\sup_{t\in[0,1]}\int_0^1 \mathcal{G}^+(t,s)ds\right)^{-1}$$

for all $t \in [0,1]$ and $u \in \mathbb{R}\setminus\{0\}$, where $\mathcal{G}^+(t,s)$ is positive part of a function $\mathcal{G}(t,\cdot)$ for a fixed t. Then the problem (1.4) has no non-trival solution.

Proof. Assume, on the contrary, that there exists $u \not\equiv 0$ such that $u = \mathcal{A}u$ and let $t_0 \in [0,1]$ such that $\|u\| = |u(t_0)|$. Then we have, by (3.5),

$$\begin{aligned}\|u\| = |u(t_0)| &= |\int_0^1 \mu\mathcal{G}(t_0,s)g(s)f(u(s))ds|\\ &\le \max\{\int_0^1 \mu\mathcal{G}^+(t_0,s)g(s)f(u(s))ds, \int_0^1 \mu\mathcal{G}^-(t_0,s)g(s)f(u(s))ds\}\\ &< \max\{\int_0^1 \mathcal{G}^+(t_0,s)M|u|ds, \int_0^1 \mathcal{G}^-(t_0,s)M|u|ds\}\\ &\le \max\{\int_0^1 \mathcal{G}^+(t_0,s)ds, \int_0^1 \mathcal{G}^-(t_0,s)ds\}M\|u\|\\ &= M\|u\|\int_0^1 \mathcal{G}^+(t_0,s)ds = \|u\|,\end{aligned}$$

a contradiction. Therefore, problem (1.4) has no solution. □

4. Example

Example 4.1. The following example is used to illustrate theorem 3.5.

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t) - m^2\ {}^cD_{0^+}^{\alpha-2}u(t) = \mu t^{\delta} f(u(t)),\ t \in (0,1), \\ u'(0) = u'(1) = 0, \\ u''(0) - m^2u(0) = 0, u''(1) - m^2u(1) = 0, \end{cases} \tag{4.1}$$

where $g(t) = t^{\delta},\ \delta > 0,\ \mu > 0,\ 3 < \alpha \leq 4$ and $f : [0, +\infty] \to \mathbb{R}, f(0) > 0$.

We compute

$$\begin{aligned} \int_0^1 G(s,\tau)\tau^{\delta}d\tau &= \int_0^s G(s,\tau)\tau^{\delta}d\tau + \int_s^1 G(s,\tau)\tau^{\delta}d\tau \\ &= \frac{1}{\Gamma(\alpha-2)}\left(\int_0^1 s\tau^{\delta}(1-\tau)^{\alpha-3}d\tau - \int_0^s \tau^{\delta}(s-\tau)^{\alpha-3}d\tau\right) \\ &= \frac{1}{\Gamma(\alpha-2)}\left(s\int_0^1 \tau^{\delta}(1-\tau)^{\alpha-3}d\tau - \mathbf{B}(\delta+1,\alpha-2)s^{\alpha+\delta-2}\right) \\ &= \frac{\mathbf{B}(\delta+1,\alpha-2)}{\Gamma(\alpha-2)}(s - s^{\delta+\alpha-2}) \\ &= \frac{\Gamma(\delta+1)}{\Gamma(\alpha+\delta-1)}(s - s^{\delta+\alpha-2}) > 0, \end{aligned}$$

where $\mathbf{B}(x, y)$ is Beta function, and then

$$\begin{aligned} \int_0^1 \mathcal{G}(t,\tau)\tau^{\delta}d\tau &= \int_0^1\int_0^1 H(t,s)G(s,\tau)\tau^{\delta}dsd\tau \\ &= \int_0^1 H(t,s)\int_0^1 G(s,\tau)\tau^{\delta}d\tau ds \\ &\geq \frac{1}{m\sinh m}\int_0^1\int_0^1 G(s,\tau)\tau^{\delta}d\tau ds \\ &= \frac{1}{m\sinh m}\frac{\Gamma(\delta+1)}{\Gamma(\alpha+\delta-1)}\int_0^1 (s - s^{\delta+\alpha-2})ds \\ &= \frac{1}{m\sinh m}\frac{\Gamma(\delta+1)}{\Gamma(\alpha+\delta-1)}\left(\frac{1}{2} - \frac{1}{\alpha+\delta-1}\right) > 0. \end{aligned}$$

So inequality (3.5) holds. Thus there exists a positive number μ^* such that example 4.1 has a positive solution for $\mu < \mu^*$ by Lemma 2.9.

In the paper, we mainly considered the existence, uniqueness and non-existence of solutions for the problem (1.4). Our technique can be also used to deal with the boundary value problem for the same fractional differential equation in problem (1.4) subjects to others boundary value conditions [24], for example the following conditions

$$u(0) = u(1), u'(0) = u'(1); u''(0) = m^2u(0), u''(1) = m^2u(1).$$

Similarly, by [6] and [7], we can also prove the existence of solutions for fractional differential equation with Riemann-Liouville fractional derivative, that is,

$$\begin{cases} D_{0^+}^{\alpha}u(t) - m^2D_{0^+}^{\alpha-2}u(t) = \mu g(t)f(u(t)),\ t \in (0,1), \\ u'(0) = u'(1) = 0, \\ u''(0) - m^2u(0) = 0, u''(1) - m^2u(1) = 0, \end{cases} \tag{4.2}$$

where $m > 0, 3 < \alpha \leq 4$ is a real number, D^{α} is the Riemann-Liouville fractional derivative of order α.

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