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# On Opial-Rozanova type inequalities

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**Abstract**

In the present paper we establish some inverses of Rozanova's type integral inequalities. The results in special cases yield reverse Rozanova's, Godunova's and Pölya's inequalities. 

*Keywords:* Opial's inequality, Jensen's inequality, Rozanova's inequality.
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## 1. Introduction

The well-known inequality due to Opial can be stated as follows (see [12]).

**Theorem 1.1.** *Suppose $f \in C^1[0,h]$ satisfies $f(0)=f(h)=0$ and $f(x)>0$ for all $x \in (0,h)$. Then*

$$\int_0^h \left|f(x)f'(x)\right| dx \le \frac{h}{4}\int_0^h (f'(x))^2 dx. \tag{1.1}$$

The first Opial's type inequality was established by Willett [16] as follows:

**Theorem 1.2.** *If $x(t)$ be absolutely continuous in $[0,a]$, and $x(0)=0$, then*

$$\int_0^a |x(t)x'(t)|dt \le \frac{a}{2}\int_0^a |x'(t)|^2 dt. \tag{1.2}$$

A non-trivial generalization of Theorem 1.2 was established by Hua [10] as follows:

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**Theorem 1.3.** *Let $x(t)$ be absolutely continuous in $[0, a]$ and $x(0) = 0$. If $l$ be a positive integer, then*

$$\int_0^a |x(t)x'(t)|dt \leq \frac{a^l}{l+1}\int_0^a |x'(t)|^{l+1}dt. \tag{1.3}$$

A sharper inequality was established by Godunova [9] as follows:

**Theorem 1.4.** *Let $f(t)$ be convex and increasing function on $[0,\infty)$ with $f(0) = 0$. If $x(t)$ is absolutely continuous on $[0, \tau]$, and $x(\alpha) = 0$, then*

$$\int_\alpha^\tau f'(|x(t)|)|x'(t)|dt \leq f\left(\int_\alpha^\tau |x'(t)|dt\right). \tag{1.4}$$

Rozanova [14] proved an extension of Inequality (1.4) which is embodied in the following:

**Theorem 1.5.** *Let $f(t)$ and $g(t)$ be convex and increasing functions on $[0,\infty)$ with $f(0) = 0$ and let $p(t) \geq 0$, $p'(t) > 0, t \in [\alpha, a]$ with $p(\alpha) = 0$. If $x(t)$ is absolutely continuous on $[\alpha, a)$ and $x(\alpha) = 0$, then*

$$f\left(\int_\alpha^a p'(t)\cdot g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \geq \int_\alpha^a p'(t)\cdot g\left(\frac{|x'(t)|}{p'(t)}\right)\cdot\left[f'\left(p(t)\cdot g\left(\frac{|x(t)|}{p(t)}\right)\right)\right]dt. \tag{1.5}$$

The Inequality (1.5) will be called as Rozanova's inequality in the paper.

Opial's inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1, 4, 5, 6, 7, 8, 11] and [17]. For Opial type integral inequalities involving high-order partial derivatives see [3] and [18]. For an extensive survey on these inequalities, see [2].

The aim of the present paper is to establish some inverses of the Rozanova's Inequality (1.5) as follows.

**Theorem 1.6.** *Let $f(t)$ and $g(t)$ be convex and decreasing functions on $[0,\infty)$ with $f(0) = 0$ and let $p(t) \geq 0$, $p'(t) > 0$, $t \in [\alpha, \tau]$ with $p(\alpha) = 0$. If $x(t)$ is absolutely continuous on $[\alpha, \tau)$ and $x(\alpha) = 0$, then there exists $\lambda$ $(0 \leq \lambda \leq 1)$, following inequality holds*

$$f\left(\int_\alpha^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \leq \int_\alpha^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)f'\left((C_{g,\lambda}(\alpha,t))\cdot p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)dt. \tag{1.6}$$

*where*

$$C_{g,\lambda}(\alpha,t) = \frac{\lambda g(\alpha) + (1-\lambda)g(t)}{g(\lambda\alpha + (1-\lambda)t)}.$$

*Remark* 1.7. The reverse inequality in Theorem 1.6 is achieved. Moreover, in Theorem 1.5 we deal with convex and increasing functions $f$ and $g$, while the reverse inequality in Theorem 1.6 is achieved for convex and decreasing functions $f$ and $g$.

**Theorem 1.8.** *Assume that*

(I) *$f(t)$, $g(t)$ and $x(t)$ are as in Theorem 1.6,*

(II) *$p(t)$ is increasing on $[0, \tau]$ with $p(0) = 0$,*

(III) *$h(t)$ is concave and increasing on $[0,\infty)$,*

(IV) *$\phi(t)$ is increasing on $[0, a]$ with $\phi(0) = 0$,*

(V) *For $y(t) = \int_0^t p'(s)g\left(\frac{|x'(s)|}{p'(s)}\right)ds$,*

$$f'(y(t))\,y'(t)\cdot\phi\left(\frac{1}{y'(t)}\right) \geq \frac{f(y(\tau))}{y(\tau)}\cdot\phi'\left(\frac{t}{y(\tau)}\right). \tag{1.7}$$

*Then there exists $\lambda$ and $\mu$ $(0 \le \lambda,\ \mu \le 1)$, following inequality holds*

$$\omega\left(\int_0^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \le E_{h,\mu}(0,\tau)\int_0^\tau f'\left(E_{g,\lambda}^{-1}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt, \quad (1.8)$$

*where*

$$E_{g,\lambda}(0,t) = \frac{g((1-\lambda)t)}{\lambda g(0) + (1-\lambda)g(t)},$$

$$E_{h,\mu}(0,\tau) = \frac{h((1-\mu)\tau)}{\mu h(0) + (1-\mu)h(\tau)},$$

$$v(z) = zh\left(\phi\left(\frac{1}{z}\right)\right), \quad (1.9)$$

*and*

$$w(z) = f(z)h\left(\phi\left(\frac{\tau}{z}\right)\right). \quad (1.10)$$

*Remark* 1.9. Inequality (1.8) just is an inverse of the following inequality established by Rozanova [15].

$$\omega\left(\int_0^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right) \ge \int_0^\tau f'\left(p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt.$$

On the other hand, for $x(t) = x_1(t)$, $x_1'(t) > 0$, $x_1'(0) = 0$, $x(\tau) = b$, $g(t) = t$, $f(t) = \phi(t) = t^2$ and $h(t) = \sqrt{1+t}$, the inequality (1.8) reduces to an inverse of the following inequality established by Pölya [13].

$$2\int_0^\tau x_1(t)\left(1 + (x_1'(t))^2\right)^{1/2} dt \le b(\tau^2 + b^2)^{1/2}.$$

## 2. Proof of main results

**Lemma 2.1.** *Let $p$ be a positive continuous function and $\phi$ be continuous function on $[a,b]$. Let $f$ be a positive, convex and continuous function on an interval containing both $[a,b]$ and $\phi[a,b]$ as subsets. Then there exist $\lambda$ $(0 \le \lambda \le 1)$ such that*

$$f\left(\frac{\int_a^b p(x)\phi(x)dx}{\int_a^b p(x)dx}\right) \ge E_{f,\lambda}(a,b)\frac{\int_a^b p(x)f(\phi(x))dx}{\int_a^b p(x)dx}, \quad (2.1)$$

*where*

$$E_{f,\lambda}(a,b) = \frac{f(\lambda a + (1-\lambda)b)}{\lambda f(a) + (1-\lambda)f(b)}. \quad (2.2)$$

*Proof.* For any finite sequence of real numbers $\{u_i\}$ in a fixed closed interval $[a,b]$ and any sequence of positive numbers $\{q_i\}$, since $a \le u_i \le b$, there is a sequence $t_i \in [0,1]$ such that $u_i = t_i a + (1-t_i)b$. Therefore

$$\frac{\dfrac{\sum_{i=1}^n q_i f(u_i)}{\sum_{i=1}^n q_i}}{f\left(\dfrac{\sum_{i=1}^n q_i u_i}{\sum_{i=1}^n q_i}\right)} = \frac{\dfrac{\sum_{i=1}^n q_i f(t_i a + (1-t_i)b)}{\sum_{i=1}^n q_i}}{f\left(\dfrac{\sum_{i=1}^n q_i(t_i a + (1-t_i)b)}{\sum_{i=1}^n q_i}\right)}$$

$$\le \frac{\dfrac{\sum_{i=1}^n q_i(t_i f(a) + (1-t_i)f(b))}{\sum_{i=1}^n q_i}}{f\left(\dfrac{\sum_{i=1}^n q_i(t_i a + (1-t_i)b)}{\sum_{i=1}^n q_i}\right)}$$

$$= \frac{\dfrac{f(a)\sum_{i=1}^{n} q_i t_i + f(b)\sum_{i=1}^{n} q_i(1-t_i))}{\sum_{i=1}^{n} q_i}}{f\left(\dfrac{a\sum_{i=1}^{n} q_i t_i + b\sum_{i=1}^{n} q_i(1-t_i))}{\sum_{i=1}^{n} q_i}\right)}.$$

Hence

$$f\left(\frac{\sum_{i=1}^{n} q_i u_i}{\sum_{i=1}^{n} q_i}\right) \geq \frac{f\left(a\frac{\sum_{i=1}^{n} q_i t_i}{\sum_{i=1}^{n} q_i} + b\left(1-\frac{\sum_{i=1}^{n} q_i t_i}{\sum_{i=1}^{n} q_i}\right)\right)}{f(a)\frac{\sum_{i=1}^{n} q_i t_i}{\sum_{i=1}^{n} q_i} + f(b)\left(1-\frac{\sum_{i=1}^{n} q_i t_i}{\sum_{i=1}^{n} q_i}\right)} \cdot \frac{\sum_{i=1}^{n} q_i f(u_i)}{\sum_{i=1}^{n} q_i}.$$

On the other hand, letting $x_i = a + \left(\frac{b-a}{n}\right) i,\ \ i = 0, 1, \ldots, n$, we have

$$\triangle x_i = x_i - x_{i-1} = \frac{b-a}{n}, \qquad i = 1, \ldots, n\ .$$

Let $u_i := \phi(x_i)$ and $q_i := p(x_i) \triangle x_i$, $i = \ldots, n$, we obtain

$$f\left(\frac{\sum_{i=1}^{n} p(x_i)\phi(x_i) \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i}\right) \geq E_f'(a,b)\frac{\sum_{i=1}^{n} p(x_i) f(\phi(x_i)) \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i}\ ,$$

where

$$E_f'(a,b) = \frac{f\left(a\frac{\sum_{i=1}^{n} p(x_i)t(x_i)\triangle x_i}{\sum_{i=1}^{n} p(x_i)\triangle x_i} + b\left(1-\frac{\sum_{i=1}^{n} p(x_i)t(x_i)\triangle x_i}{\sum_{i=1}^{n} p(x_i)\triangle x_i}\right)\right)}{f(a)\frac{\sum_{i=1}^{n} p(x_i)t(x_i)\triangle x_i}{\sum_{i=1}^{n} p(x_i)\triangle x_i} + f(b)\left(1-\frac{\sum_{i=1}^{n} p(x_i)t(x_i)\triangle x_i}{\sum_{i=1}^{n} p(x_i)\triangle x_i}\right)}.$$

By taking limits as $n \to \infty$, we get

$$f\left(\frac{\int_a^b p(x)\phi(x)dx}{\int_a^b p(x)dx}\right) \geq E_f(a,b)\frac{\int_a^b p(x)f(\phi(x))dx}{\int_a^b p(x)dx},$$

where

$$E_f(a,b) = \frac{f(ma+nb)}{mf(a)+nf(b)}$$

for some $0 \leq m, n \leq 1$ with $m + n = 1$.

If $m = \lambda$ and $n = 1 - \lambda$, (2.1) easily follows. □

Lemma 2.1 was also proved in [19] by the author, but there's a little mistake in that proof. A complete and correct proof has shown here.

**Proof of Theorem 1.6**

*Proof.* Let $y(t) = \int_\alpha^t |x'(s)|ds$, $t \in [\alpha, \tau]$ so that $y'(t) = |x'(t)|$ and in view of

$$|x(t)| \leq \int_\alpha^t |x'(s)|ds,$$

we have

$$y(t) \geq |x(t)|\,.$$

From the hypotheses and in view of the reverse Jensen's inequality in Lemma 2.1, we obtain for $0 \leq \lambda \leq 1$

$$\begin{aligned} g\left(\frac{|x(t)|}{p(t)}\right) &\geq g\left(\frac{y(t)}{p(t)}\right) \\ &= g\left(\frac{\int_\alpha^t p'(s)\frac{|x'(s)|}{p'(s)}ds}{\int_\alpha^t p'(s)ds}\right) \\ &\geq \left(\frac{g(\lambda\alpha+(1-\lambda)t)}{\lambda g(\alpha)+(1-\lambda)g(t)}\right)\frac{1}{p(t)}\int_\alpha^t p'(s)g\left(\frac{|x'(s)|}{p'(x)}\right)ds. \end{aligned} \tag{2.3}$$

On the other hand, from the hypotheses and by using Inequality (2.3), we have

$$\begin{aligned}
&\int_{\alpha}^{\tau} p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right) f'\left(\frac{\lambda g(\alpha)+(1-\lambda)g(t)}{g(\lambda\alpha+(1-\lambda)t)}\cdot p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)dt \\
&\geq \int_{\alpha}^{\tau} p'(t)g\left(\frac{y'(t)}{p'(t)}\right) f'\left(\int_{\alpha}^{t} p'(s)g\left(\frac{y'(s)}{p'(s)}\right)ds\right)dt \\
&= \int_{\alpha}^{\tau} \frac{d}{dt}\left[f\left(\int_{\alpha}^{t} p'(s)g\left(\frac{y'(s)}{p'(s)}\right)ds\right)\right]dt \\
&= f\left(\int_{\alpha}^{\tau} p'(t)g\left(\frac{y'(t)}{p'(t)}\right)dt\right) \\
&= f\left(\int_{\alpha}^{\tau} p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right).
\end{aligned}$$

This completes the proof. □

**Proof of Theorem 1.8**

*Proof.* From the reverse Jensen's inequality, we obtain

$$p(t)g\left(\frac{|x(t)|}{p(t)}\right) \geq E_{g,\lambda}(0,t)y(t),$$

where $E_{g,\lambda}(0,t)$ is as in (2.2). Because $g$ and $h$ are convex and concave functions, respectively, so there exists $0 \leq \lambda, \mu \leq 1$, so that

$$E^{-1}_{g,\lambda}(0,t) = \frac{\lambda g(0)+(1-\lambda)g(t)}{g((1-\lambda)t)} \geq 1,$$

and

$$E_{h,\mu}(0,\tau) = \frac{h((1-\mu)\tau)}{\mu h(0)+(1-\mu)h(\tau)} \geq 1.$$

Hence

$$E_{h,\mu}(0,\tau)\int_0^{\tau} f'\left(E^{-1}_{g,\lambda}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \geq E_{h,\mu}(0,\tau)\int_0^{\tau} f'\left(y(t)\right)\cdot v\left(y'(t)\right)dt. \tag{2.4}$$

From (1.9) and (2.4), we have

$$\begin{aligned}
&E_{h,\mu}(0,\tau)\int_0^{\tau} f'\left(E^{-1}_{g,\lambda}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \\
&\qquad \geq E_{h,\mu}(0,\tau)\int_0^{\tau} f'\left(y(t)\right)y'(t)h\left(\phi\left(\frac{1}{y'(t)}\right)\right)dt.
\end{aligned}\tag{2.5}$$

From (2.1), (2.5) and in view of $h$ is concave function, we obtain

$$\begin{aligned}
&E_{h,\mu}(0,\tau)\int_0^{\tau} f'\left(E^{-1}_{g,\lambda}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \\
&\geq E_{h,\mu}(0,\tau)\frac{\int_0^{\tau} f'\left(y(t)\right)y'(t)\cdot h\left(\phi\left(\frac{1}{y'(t)}\right)\right)dt}{\int_0^{\tau} f'\left(y(t)\right)y'(t)dt}\int_0^{\tau} f'\left(y(t)\right)y'(t)dt \\
&\geq h\left(\frac{\int_0^{\tau} f'\left(y(t)\right)y'(t)\cdot \phi\left(\frac{1}{y'(t)}\right)dt}{\int_0^{\tau} f'\left(y(t)\right)y'(t)dt}\right)f(y(\tau)).
\end{aligned}\tag{2.6}$$

From (1.7), (1.10), (2.6) and in view of $h$ is increasing function, we obtain

$$\begin{aligned}
&E_{h,\mu}(0,\tau)\int_0^\tau f'\left(E_{g,\lambda}^{-1}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right)\cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt\\
&\geq h\left(\frac{\int_0^\tau \frac{f(y(\tau))}{y(\tau)}\cdot\phi'\left(\frac{t}{y(\tau)}\right)dt}{\int_0^\tau f'\left(y(t)\right)y'(t)dt}\right)f(y(\tau))\\
&= h\left(\frac{\frac{f(y(\tau))}{y(\tau)}\cdot\int_0^\tau \phi'\left(\frac{t}{y(\tau)}\right)dt}{\int_0^\tau (f\left(y(t)\right))'dt}\right)f(y(\tau))\\
&= h\left(\phi\left(\frac{\tau}{y(\tau)}\right)\right)f(y(\tau))\\
&= \omega(y(\tau)) = \omega\left(\int_0^\tau p'(t)\left(\frac{|x'(t)|}{p'(t)}\right)dt\right).
\end{aligned}$$

This completes the proof. □

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