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Common fixed point theorems for weakly C-contractive mappings in ordered partial metric spaces

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Abstract

In this paper, we investigate some common fixed point theorems for weakly C-contractive mappings in ordered partial metric spaces. Presented theorems generalize the results of Karapınar and Shatanawi [E. Karapınar, W. Shatanawi, Abstr. Appl. Anal., **2012** (2012), 17 pages]. An example is also given to support our main result.

Keywords: Fixed point, common fixed point, partial metric space, weakly C-contraction.
2010 MSC: 47H10, 54H25.

1. Introduction and Preliminaries

In [23], Matthews introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. In partial metric spaces, the self distance $d(x,x)$ is no longer necessarily zero. In the same reference, some fixed point theorems were investigated. Recently, many authors paid much attention to partial metric spaces, and generalized the fixed point theorems in metric spaces into theorems in partial metric spaces (see, e.g., [1, 2, 4, 5, 6, 10, 11, 12, 14, 16, 18, 19, 20, 26, 28] and references therein).

In [13], Choudhury introduced the concept of weakly C-contractive mapping as follows.

Definition 1.1 ([13])**.** Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be weakly C-contractive (or a weakly C-contraction) if for all $x, y \in X$, the following inequality holds:

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$$d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(Tx, y)) - \phi(d(x, Ty), d(Tx, y)),$$

where $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function such that $\phi(x, y) = 0$ if and only if $x = y = 0$.

After that, some results have appeared related to weakly C-contractive mappings (see, for example, [8, 9, 15, 27]). Karapınar and Shatanawi [20] introduced the weakly (C, ψ, ϕ)-contractive mappings and investigated fixed point theorems for such mappings in ordered partial metric spaces.

Definition 1.2 ([20]). Let $(X, \preceq)$ be a partially ordered set and p be a partial metric on X. Then the mappings $S, T : X \times X \to X$ are said to be weakly (C, ψ, ϕ)-contractive mappings if T and S are weakly increasing with respect to $\preceq$ and for any comparable x and y, we have

$$\psi(p(Tx, Sy)) \leq \psi\left(\frac{p(Tx, y) + p(x, Sy)}{2}\right) - \phi(p(Tx, y), p(x, Sy)),$$

where

(a) $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function;
(b) $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function with $\phi(t, s) = 0$ if and only if $t = s = 0$.

Some concepts in the Definition 1.2 will be introduced in the next section.

For weakly (C, ψ, ϕ)-contractive mappings, Karapınar and Shatanawi obtained some results in [20].

Theorem 1.3 ([20]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \times X \to X$ are weakly (C, ψ, ϕ)-contractive mappings. If T and S are continuous, then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.*

Definition 1.4 ([20]). X is called to satisfy property (P) if $\{x_n\}$ is a nondecreasing sequence in X such that $\lim_{n\to+\infty} p(x_n, x) = p(x, x)$, then $x_n \leq x$ for all $n \in N$.

Theorem 1.5 ([20]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Suppose that $T, S : X \times X \to X$ are weakly (C, ψ, ϕ)-contractive mappings. If X satisfies property (P), then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.*

In this paper, we investigate some common fixed point theorems for weakly C-contractive mappings in ordered partial metric spaces. Our results generalize and extend the results in [20]. Throughout this paper, the letters N, N^+ and R^+ will denote the set of all nonnegative integer numbers, the set of all positive integer numbers and the set of all nonnegative real numbers, respectively.

Let us recall some definitions and some properties of partial metric spaces which will be needed in the sequel.

Definition 1.6 ([23]). A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(P_1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$,
(P_2) $p(x, x) \leq p(x, y)$,
(P_3) $p(x, y) = p(y, x)$,
(P_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P_1) and (P_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

For a partial metric p on X, the function $d_p : X \times X \to [0, +\infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is an usual metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

In a partial metric space, the concepts of convergence, completeness and continuity are defined as follows.

Definition 1.7 ([22, 23])**.** Let (X, p) be a partial metric space. Then:

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_m, x_n)$.

(iii) A partial metric space is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_m, x_n)$.

(iv) A mapping $T : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x_0, \delta)) \subset B_p(Tx_0, \varepsilon)$.

The following lemmas play an important role in the proof of our main results.

Lemma 1.8 ([3])**.** *Let (X, p) be a partial metric space.*

(A) *A sequence $\{x_n\}$ is a Cauchy sequence in a partial metric space (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in a metric space (X, d_p).*

(B) *A partial metric space (X,p) is complete if and only if the metric space (X, d_p) is complete. Moreover,*

$$\lim_{n\to+\infty} d_p(x_n, x) = 0 \Leftrightarrow P(x, x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \tag{1.1}$$

Lemma 1.9 ([3, 17])**.** *Assume that $x_n \to z$ as $n \to +\infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to+\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 1.10 ([24])**.** *Let (X, p) be a partial metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to+\infty} p(x_{n+1}, x_n) = 0.$$

If $\{x_{2n}\}$ is not a Cauchy sequence in (X, p), then there exists $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ε when $k \to +\infty$:

$$p(x_{2m(k)}, x_{2n(k)}),\quad p(x_{2m(k)}, x_{2n(k)+1}),\quad p(x_{2m(k)-1}, x_{2n(k)}),\quad p(x_{2m(k)-1}, x_{2n(k)+1}). \tag{1.2}$$

Lemma 1.11 ([20, 24])**.** *Let (X, p) be a partial metric space, $T : X \to X$ be a given mapping. Suppose that T is continuous at $x_0 \in X$. Then, for each sequence $\{x_n\}$ in X, $x_n \to x_0$ in $\tau_p \Rightarrow Tx_n \to Tx_0$ in τ_p holds.*

2. Main results

In this section, we begin with the following definitions(see, e.g., [7, 21, 25] and references therein).

Definition 2.1 ([21])**.** The function $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is called an altering distance function, if the following properties are satisfied:
(1) φ is continuous and nondecreasing;
(2) $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.2 ([7])**.** Let $(X, \preceq)$ be a partially ordered set. Two mappings $T, S : X \to X$ are said to be weakly increasing if $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in X$.

Definition 2.3 ([25])**.** Let $(X, \preceq)$ be a partially ordered set and let $T, S : X \to X$ be two mappings. The mapping S is said to be T-weakly isotone increasing if for all $x \in X$, we have $Sx \preceq TSx \preceq STSx$.

Remark 2.4 ([25]). If $T, S : X \to X$ are weakly increasing, then S is T-weakly isotone increasing.

Now we give our first result.

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Let $T, S : X \times X \to X$ be mappings. Suppose that T and S are continuous. If S is T-weakly isotone increasing with respect to $\preceq$ and for any comparable x and y in X, we have*

$$\psi\left(p(Tx, Sy)\right) \leq \varphi\left(\frac{p(Tx, y) + p(x, Sy)}{2}\right) - \phi(p(Tx, y), p(x, Sy)), \tag{2.1}$$

where
(1) $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are altering distance functions with

$$\psi(t) - \varphi(t) \geq 0 \tag{2.2}$$

for all $t \geq 0$.
(2) $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a continuous function with $\phi(t, s) = 0$ if and only if $t = s = 0$.
Then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.

Proof. Let x_0 be an arbitrary point in X. Set $Sx_0 = x_1$ and $Tx_1 = x_2$. Continuing this process, we can construct sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n \in N. \tag{2.3}$$

Since S is T-weakly isotone increasing with respect to $\preceq$, we have

$$\begin{aligned} x_1 = Sx_0 \preceq TSx_0 = Tx_1 = x_2 \preceq STSx_0 = Sx_2 = x_3, \\ x_3 = Sx_2 \preceq TSx_2 = Tx_3 = x_4 \preceq STSx_2 = Sx_4 = x_5, \end{aligned}$$

and continuing this process, we get

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots . \tag{2.4}$$

Now, we discuss the following two cases.
Case 1. If $p(x_n, x_{n+1}) = 0$ for some $n = n_0 \in N$, then S and T have at least one common fixed point. In fact, since $p(x_{n_0}, x_{n_0+1}) = 0$, we get $x_{n_0} = x_{n_0+1}$. If $n_0 = 2k (k \in N)$, then $x_{2k} = x_{2k+1}$. Using (2.1) and (P_2), we have

$$\begin{aligned} \psi(p(x_{2k+1}, x_{2k+2})) &= \psi(p(Tx_{2k+1}, Sx_{2k},)) \\ &\leq \varphi\left(\frac{p(Tx_{2k+1}, x_{2k}) + p(x_{2k+1}, Sx_{2k})}{2}\right) - \phi(p(Tx_{2k+1}, x_{2k}), p(x_{2k+1}, Sx_{2k})) \end{aligned}$$

$$
\begin{aligned}
&= \varphi\left(\frac{p(x_{2k+2},x_{2k})+p(x_{2k+1},x_{2k+1})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&\leq \varphi\left(\frac{p(x_{2k},x_{2k+1})+p(x_{2k+1},x_{2k+2})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&\leq \varphi\left(max\{p(x_{2k},x_{2k+1}),p(x_{2k+1},x_{2k+2})\}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&= \varphi\left(max\{p(x_{2k+1},x_{2k+1}),p(x_{2k+1},x_{2k+2})\}\right)\\
&\quad - \phi(p(x_{2k+2},x_{2k+1}),p(x_{2k+1},x_{2k+1}))\\
&= \varphi\left(p(x_{2k+1},x_{2k+2})\right) - \phi(p(x_{2k+2},x_{2k+1}),p(x_{2k+1},x_{2k+1})).
\end{aligned}
\tag{2.5}
$$

From (2.2) and (2.5), we have $\phi(p(x_{2k+2},x_{2k+1}),p(x_{2k+1},x_{2k+1}))=0$, by the property of ϕ, we get $p(x_{2k+1},x_{2k+2})=0$, that is, $x_{2k+1}=x_{2k+2}$. By similar arguments, we obtain $x_{2k+2}=x_{2k+3}$ and so on. Thus $\{x_n\}$ becomes a constant from $n=2k$, that is

$$x_{2k}=x_{2k+1}=x_{2k+2}=\cdots. \tag{2.6}$$

(2.3) and (2.6) yield

$$x_{2k}=Sx_{2k}=Tx_{2k}, \tag{2.7}$$

which implies that x_{2k} is the common fixed point of S and T.

Similarly, one can show that if $n_0=2k+1(k\in N)$, then S and T have at least one common fixed point. Therefore, we have proved that if $p(x_n,x_{n+1})=0$ for some $n=n_0\in N$, then S and T have at least one common fixed point. **Case 2.** If $p(x_n,x_{n+2})=0$ for some $n=n_0\in N$, then S and T have at least one

common fixed point. Indeed, if $n_0=2k(k\in N)$, then $p(x_{2k},x_{2k+2})=0$, hence $x_{2k}=x_{2k+2}$, due to (2.1), we have

$$
\begin{aligned}
\psi(p(x_{2k+1},x_{2k+2})) &= \psi(Tx_{2k+1},Sx_{2k})\\
&\leq \varphi\left(\frac{p(Tx_{2k+1},x_{2k})+p(x_{2k+1},Sx_{2k})}{2}\right) - \phi(p(Tx_{2k+1},x_{2k}),p(x_{2k+1},Sx_{2k}))\\
&= \varphi\left(\frac{p(x_{2k+2},x_{2k})+p(x_{2k+1},x_{2k+1})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&\leq \varphi\left(\frac{p(x_{2k},x_{2k+1})+p(x_{2k+1},x_{2k+2})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&= \varphi\left(\frac{p(x_{2k+2},x_{2k+1})+p(x_{2k+1},x_{2k+2})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&= \varphi(p(x_{2k+2},x_{2k+1})) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1})).
\end{aligned}
\tag{2.8}
$$

Applying (2.2) and (2.8), we obtain $\phi(p(x_{2k},x_{2k+2}),p(x_{2k+1},x_{2k+1}))=0$, using the property of ϕ, we have

$$p(x_{2k+1},x_{2k+1}))=0. \tag{2.9}$$

Since

$$
\begin{aligned}
\psi(p(x_{2k+1},x_{2k+2})) &= \psi(p(Tx_{2k+1},Sx_{2k}))\\
&\leq \varphi\left(\frac{p(Tx_{2k+1},x_{2k})+p(x_{2k+1},Sx_{2k})}{2}\right) - \phi(p(Tx_{2k+1},x_{2k}),p(x_{2k+1},Sx_{2k}))\\
&= \varphi\left(\frac{p(x_{2k+2},x_{2k})+p(x_{2k+1},x_{2k+1})}{2}\right) - \phi(p(x_{2k+2},x_{2k}),p(x_{2k+1},x_{2k+1}))\\
&= \varphi(0)-\phi(0,0)
\end{aligned}
\tag{2.10}
$$

Applying (2.10) and the property of φ and ϕ, we obtain that $\psi(p(x_{2k+1},x_{2k+2}))=0$, and thus $p(x_{2k+1},x_{2k+2})=0$, which means that S and T have at least one common fixed point from Case 1. Similarly, it is easy to show that if $n_0 = 2k+1$ ($k \in N$), then S and T have at least one fixed point, this completes the proof of case 2.

Taking $p(x_n, x_{n+1}) > 0$ and $p(x_n, x_{n+2}) > 0$ for $n \in N$, we shall show that for every $n \in N$,

$$p(x_{2n+2}, x_{2n+1}) \leq p(x_{2n+1}, x_{2n}). \tag{2.11}$$

Suppose, to the contrary, that $p(x_{2n+2}, x_{2n+1}) > p(x_{2n+1}, x_{2n})$ for some $n = n_0$, that is,

$$p(x_{2n_0+2}, x_{2n_0+1}) > p(x_{2n_0+1}, x_{2n_0}).$$

Using (2.1), we obtain

$$\begin{aligned}
&\psi\left(p(x_{2n_0+1}, x_{2n_0+2})\right)\\
&\quad= \psi\left(p(Tx_{2n_0+1}, Sx_{2n_0}\right)\\
&\quad\leq \varphi\left(\frac{p(Tx_{2n_0+1}, x_{2n_0}) + p(x_{2n_0+1}, Sx_{2n_0})}{2}\right) - \phi(p(Tx_{2n_0+1}, x_{2n_0}), p(x_{2n_0+1}, Sx_{2n_0}))\\
&\quad= \varphi\left(\frac{p(x_{2n_0+2}, x_{2n_0}) + p(x_{2n_0+1}, x_{2n_0+1})}{2}\right) - \phi(p(x_{2n_0+2}, x_{2n_0}), p(x_{2n_0+1}, x_{2n_0+1}))\\
&\quad\leq \varphi\left(\frac{p(x_{2n_0}, x_{2n_0+1}) + p(x_{2n_0+1}, x_{2n_0+2})}{2}\right) - \phi(p(x_{2n_0+2}, x_{2n_0}), p(x_{2n_0+1}, x_{2n_0+1}))\\
&\quad\leq \varphi\left(max\{p(x_{2n_0}, x_{2n_0+1}), p(x_{2n_0+1}, x_{2n_0+2})\}\right) - \phi(p(x_{2n_0+2}, x_{2n_0}), p(x_{2n_0+1}, x_{2n_0+1}))\\
&\quad= \varphi\left(p(x_{2n_0+1}, x_{2n_0+2})\right) - \phi(p(x_{2n_0+2}, x_{2n_0}), p(x_{2n_0+1}, x_{2n_0+1})).
\end{aligned} \tag{2.12}$$

Equation (2.12) together with (2.2) gives $\phi(p(x_{2n_0+2}, x_{2n_0}), p(x_{2n_0+1}, x_{2n_0+1})) = 0$. Applying the property of ϕ, we get $p(x_{2n_0}, x_{2n_0+2}) = 0$, which contradicts with $p(x_n, x_{n+2}) > 0$ for $n \in N$, hence (2.11) holds. Similarly, one can show that for every $n \in N^+$, the following inequality holds.

$$p(x_{2n+1}, x_{2n}) \leq p(x_{2n}, x_{2n-1}). \tag{2.13}$$

Equations (2.11) and (2.13) imply that the sequence $\{p(x_n, x_{n+1})\}$ is nonincreasing. Consequently there exists some $r \geq 0$ such that

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = r. \tag{2.14}$$

By (2.14) and the following inequalities

$$p(x_{2n}, x_{2n+2}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) - p(x_{2n+1}, x_{2n+1}) \leq p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}),$$

we get that $\{p(x_{2n}, x_{2n+2})\}$ is bounded. Hence there exists some subsequence $\{p(x_{2n(k)}, x_{2n(k)+2})\}$ of $\{p(x_{2n}, x_{2n+2})\}$ converging to some r_0, that is

$$\lim_{k\to+\infty} p(x_{2n(k)}, x_{2n(k)+2}) = r_0. \tag{2.15}$$

Due to (P_2), we have

$$p(x_{2n(k)+1}, x_{2n(k)+1}) \leq p(x_{2n(k)+1}, x_{2n(k)}),$$

using (2.14), we get that $\{p(x_{2n(k)+1}, x_{2n(k)+1})\}$ is bounded, thus there exists a subsequence $\{p(x_{2n(k_i)+1}, x_{2n(k_i)+1})\}$ of $\{p(x_{2n(k)+1}, x_{2n(k)+1})\}$ such that $\{p(x_{2n(k_i)+1}, x_{2n(k_i)+1})\}$ converges to some r_1, that is

$$\lim_{k\to+\infty} p(x_{2n(k_i)+1}, x_{2n(k_i)+1}) = r_1. \tag{2.16}$$

By (2.1), we have

$$\psi\left(p(x_{2n(k_i)+1}, x_{2n(k_i)+2})\right) = \psi\left(p(Tx_{2n(k_i)+1}, Sx_{2n(k_i)}\right)$$

$$
\begin{aligned}
&\leq \varphi\left(\frac{p(Tx_{2n(k_i)+1}, x_{2n(k_i)}) + p(x_{2n(k_i)+1}, Sx_{2n(k_i)})}{2}\right) \\
&\quad -\phi(p(Tx_{2n(k_i)+1}, x_{2n(k_i)}), p(x_{2n(k_i)+1}, Sx_{2n(k_i)})) \\
&= \varphi\left(\frac{p(x_{2n(k_i)+2}, x_{2n(k_i)}) + p(x_{2n(k_i)+1}, x_{2n(k_i)+1})}{2}\right) \\
&\quad -\phi(p(x_{2n(k_i)+2}, x_{2n(k_i)}), p(x_{2n(k_i)+1}, x_{2n(k_i)+1})) \\
&\leq \varphi\left(\frac{p(x_{2n(k_i)}, x_{2n(k_i)+1}) + p(x_{2n(k_i)+1}, x_{2n(k_i)+2})}{2}\right) \\
&\quad -\phi(p(x_{2n(k_i)+2}, x_{2n(k_i)}), p(x_{2n(k_i)+1}, x_{2n(k_i)+1})). \qquad (2.17)
\end{aligned}
$$

Letting $i \to +\infty$ in the above inequalities, and using (2.14), (2.15) and (2.16), we obtain

$$\psi(r) \leq \varphi(r) - \phi(r_0, r_1), \tag{2.18}$$

which means that $\phi(r_0, r_1) = 0$, hence $r_0 = 0$ and $r_1 = 0$.

Since

$$
\begin{aligned}
\psi\left(p(x_{2n(k_i)+1}, x_{2n(k_i)+2})\right) &= \psi\left(p(Tx_{2n(k_i)+1}, Sx_{2n(k_i)})\right) \\
&\leq \varphi\left(\frac{p(Tx_{2n(k_i)+1}, x_{2n(k_i)}) + p(x_{2n(k_i)+1}, Sx_{2n(k_i)})}{2}\right) \\
&\quad -\phi(p(Tx_{2n(k_i)+1}, x_{2n(k_i)}), p(x_{2n(k_i)+1}, Sx_{2n(k_i)})) \\
&= \varphi\left(\frac{p(x_{2n(k_i)+2}, x_{2n(k_i)}) + p(x_{2n(k_i)+1}, x_{2n(k_i)+1})}{2}\right) \\
&\quad -\phi(p(x_{2n(k_i)+2}, x_{2n(k_i)}), p(x_{2n(k_i)+1}, x_{2n(k_i)+1})),
\end{aligned}
$$

taking the limit as $i \to +\infty$, by the property of φ and ϕ, we have $\psi(r) \leq \varphi(0) - \phi(0, 0) = 0$, which implies that $r = 0$, that is

$$\lim_{n \to +\infty} p(x_n, x_{n+1}) = 0. \tag{2.19}$$

From (P_2) and (2.19) , we have

$$\lim_{n \to +\infty} p(x_n, x_n) = 0. \tag{2.20}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p)(and $\{x_n\}$ is a Cauchy sequence in the space (X, p) by Lemma 1.8). For this, it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence in (X, d_p). Suppose that this is not the case, then using Lemma 1.8 we have that $\{x_{2n}\}$ is not a Cauchy sequence in (X, p). By Lemma 1.10, we have that there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and sequences in (1.2) tend to ε when $k \to +\infty$.

From (2.1), we obtain

$$
\begin{aligned}
\psi\left(p(x_{2n(k)+1}, x_{2m(k)})\right) &= \psi\left(Tx_{2m(k)-1}, Sx_{2n(k)}\right) \\
&\leq \varphi\left(\frac{p(Tx_{2m(k)-1}, x_{2n(k)}) + p(x_{2m(k)-1}, Sx_{2n(k)})}{2}\right) \\
&\quad -\phi\left(p(Tx_{2m(k)-1}, x_{2n(k)}), p(x_{2m(k)-1}, Sx_{2n(k)})\right) \\
&= \varphi\left(\frac{p(x_{2m(k)}, x_{2n(k)}) + p(x_{2m(k)-1}, x_{2n(k)+1})}{2}\right) \\
&\quad -\phi\left(p(x_{2m(k)}, x_{2n(k)}), p(x_{2m(k)-1}, x_{2n(k)+1})\right).
\end{aligned}
$$

Letting $k \to +\infty$ and using the continuity of ψ, φ and ϕ, we get

$$\psi(\varepsilon) \leq \varphi(\varepsilon) - \phi(\varepsilon, \varepsilon),$$

which implies that $\phi(\varepsilon,\varepsilon)=0$. Hence $\varepsilon=0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in (X,d_p), and $\{x_n\}$ is also a Cauchy sequence in (X,p). Since (X,p) is complete then the sequence $\{x_n\}$ converges to some $z\in X$, that is

$$p(z,z)=\lim_{n\to+\infty}p(x_n,z)=\lim_{n,m\to+\infty}p(x_n,x_m).$$

Moreover, the sequence $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to $z\in X$, that is

$$p(z,z)=\lim_{n\to+\infty}p(x_{2n},z)=\lim_{n,m\to+\infty}p(x_{2n},x_{2m}).$$

and

$$p(z,z)=\lim_{n\to+\infty}p(x_{2n+1},z)=\lim_{n,m\to+\infty}p(x_{2n+1},x_{2m+1}).$$

Since $\{x_n\}$ is a Cauchy sequence in (X,d_p) , we have $\lim\limits_{n,m\to+\infty}d_p(x_n,x_m)=0$. By $d_p(x_n,x_m)=2p(x_n,x_m)-p(x_n,x_n)-p(x_m,x_m)$, we obtain $\lim\limits_{n,m\to+\infty}p(x_n,x_m)=0$, then we have

$$p(z,z)=\lim_{n\to+\infty}p(x_n,z)=\lim_{n\to+\infty}p(x_{2n},z)=\lim_{n\to+\infty}p(x_{2n+1},z)=0. \tag{2.21}$$

By Lemma 1.9, we conclude that

$$\lim_{n\to+\infty}p(x_n,Tz)=p(z,Tz), \tag{2.22}$$

and

$$\lim_{n\to+\infty}p(x_n,Sz)=p(z,Sz). \tag{2.23}$$

Since T is continuous, from (2.21), (2.22) and by Lemma 1.11, we get

$$p(z,Tz)=\lim_{n\to+\infty}p(x_n,Tz)=\lim_{n\to+\infty}p(x_{2n},Tz)=\lim_{n\to+\infty}p(Tx_{2n-1},Tz)=p(Tz,Tz). \tag{2.24}$$

Similarly, we can prove

$$p(z,Sz)=p(Sz,Sz). \tag{2.25}$$

Applying (2.1), (2.24) and (2.25) we have

$$\begin{aligned}
\psi\left(\frac{p(z,Tz)+p(z,Sz)}{2}\right) &= \psi\left(\frac{p(Tz,Tz)+p(Sz,Sz)}{2}\right)\\
&\le \psi\left(\frac{p(Tz,Sz)+p(Sz,Tz)}{2}\right)\\
&= \psi\left(p(Tz,Sz)\right)\\
&\le \psi\left(\frac{p(Tz,z)+p(z,Sz)}{2}\right)-\phi(p(Tz,z),p(z,Sz)),
\end{aligned}$$

which implies $\phi(p(Tz,z),p(z,Sz))=0$, from the property of ϕ, we get $p(Tz,z)=0,\ p(z,Sz)=0$, hence, $z=Tz$, $z=Sz$. Therefore $z=Sz=Tz$, that is, z is the common fixed point of T and S. □

Replacing the condition that S is T-weakly isotone increasing by condition that T,S are weakly increasing in Theorem 2.5, we get the following corollary.

Corollary 2.6. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X,p) is complete. Let $T,S:X\times X\to X$ be mappings. Suppose that T and S are continuous. If $T,S:X\to X$ are weakly increasing and for any comparable x and y in X, we have*

$$\psi\left(p(Tx,Sy)\right)\le\varphi\left(\frac{p(Tx,y)+p(x,Sy)}{2}\right)-\phi(p(Tx,y),p(x,Sy)),$$

where

1. $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ *are altering distance functions with* $\psi(t) - \varphi(t) \geq 0$ *for all* $t \geq 0$.

2. $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is a continuous function with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$.

Then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.

Remark 2.7. Taking $\psi = \varphi$ in Corollary 2.6, we can get Theorem 1.3 .

Theorem 2.5 is still valid if S and T are not necessarily continuous.

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X, p) is complete. Let $T, S : X \times X \to X$ be mappings. Suppose that X satisfies property (P). If S is T-weakly isotone increasing with respect to $\preceq$ and for any comparable x and y in X, we have*

$$\psi\left(p(Tx, Sy)\right) \leq \varphi\left(\frac{p(Tx, y) + p(x, Sy)}{2}\right) - \phi(p(Tx, y), p(x, Sy)), \tag{2.26}$$

where

1. $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ *are altering distance functions with*
$$\psi(t) - \varphi(t) \geq 0 \tag{2.27}$$
for all $t \geq 0$.

2. $\phi : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is a continuous function with* $\phi(t, s) = 0$ *if and only if* $t = s = 0$.

Then T and S have a common fixed point; that is, there exists $u \in X$ such that $u = Tu = Su$.

Proof. As the proof of Theorem 2.5, we can construct a nondecreasing Cauchy sequence $\{x_n\}$ in X, such that $\lim_{n\to+\infty} p(x_n, z) = p(z, z) = 0$. Moreover,

$$\begin{aligned} \lim_{n\to+\infty} p(x_{2n}, z) = p(z, z) = 0, \\ \lim_{n\to+\infty} p(x_{2n+1}, z) = p(z, z) = 0. \end{aligned} \tag{2.28}$$

Since $\{x_n\}$ is a nondecreasing sequence, we have $x_n \preceq z$. By (2.1), we get

$$\begin{aligned} \psi(p(x_{2n+1}, Tz)) &= \psi(p(Tz, Sx_{2n})) \\ &\leq \varphi\left(\frac{p(Tz, x_{2n}) + p(z, Sx_{2n})}{2}\right) - \phi(p(Tz, x_{2n}), p(z, Sx_{2n})) \\ &= \varphi\left(\frac{p(Tz, x_{2n}) + p(z, x_{2n+1})}{2}\right) - \phi(p(Tz, x_{2n}), p(z, x_{2n+1})), \end{aligned}$$

letting $n \to +\infty$ in the above inequalities, and using (2.28) and Lemma 1.9, we have

$$\begin{aligned} \psi(p(z, Tz)) &\leq \varphi\left(\frac{p(Tz, z) + p(z, z)}{2}\right) - \phi(p(Tz, z), p(z, z)) \\ &\leq \varphi(p(Tz, z)) - \phi(p(Tz, z), 0). \end{aligned} \tag{2.29}$$

Eq. (2.2) and (2.29) yield $\phi(p(Tz, z), 0) = 0$, which means $p(Tz, z) = 0$, hence $Tz = z$. Similarly, we can show that $Sz = z$. Thus z is the common fixed point of S and T. □

Replacing the condition that S is T-weakly isotone increasing by condition that T, S are weakly increasing in Theorem 2.8, we get the following corollary.

Corollary 2.9. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a partial metric p on X such that (X,p) is complete. Let $T,S : X\times X\to X$ be mappings. Suppose that X satisfies property (P). If $T,S : X\to X$ are weakly increasing and for any comparable x and y in X, we have*

$$\psi\left(p(Tx,Sy)\right)\leq\varphi\left(\frac{p(Tx,y)+p(x,Sy)}{2}\right)-\phi(p(Tx,y),p(x,Sy)),$$

where

1. *$\psi,\varphi : [0,+\infty)\to[0,+\infty)$ are altering distance functions with $\psi(t)-\varphi(t)\geq 0$ for all $t\geq 0$.*
2. *$\phi : [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is a continuous function with $\phi(t,s)=0$ if and only if $t=s=0$.*

Then T and S have a common fixed point; that is, there exists $u\in X$ such that $u=Tu=Su$.

Remark 2.10. Taking $\psi=\varphi$ in Corollary 2.9, we can get Theorem 1.5 .

Now, we introduce an example to illustrate the validity of our main result.

Example 2.11. Let $X=[0,1]$ be endowed with the usual partial metric $p : X\times X\to[0,+\infty)$ defined by $p(x,y)=max\{x,y\}$ and the relation $\preceq$ by $y\preceq x$ if and only if $x\leq y$. Also, define the mappings $S,T : X\to X$ by $Tx=\frac{x^3}{8}$ and $Sx=\frac{x^2}{4}$, respectively. Let us take $\psi,\varphi : [0,+\infty)\to[0,+\infty)$ such that $\psi(t)=t^2$ and $\varphi(t)=\frac{t^2}{2}$, respectively. Let $\phi : [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ such that $\phi(t,s)=\frac{(t+s)^2}{16}$. Then one has the following.

1. $(X,p,\preceq)$ is a complete ordered partial metric space.
2. S and T are continuous.
3. $T,S : X\to X$ are weakly increasing.
4. For any comparable x and y in X, we have

$$\psi\left(p(Tx,Sy)\right)\leq\varphi\left(\frac{p(Tx,y)+p(x,Sy)}{2}\right)-\phi(p(Tx,y),p(x,Sy)).$$

Proof. It is clear that (1) and (2) hold. Now, we show that (3) is true. Let $x\in X$, since $TSx=T(\frac{x^2}{4})=\frac{x^6}{512}\leq\frac{x^2}{4}=Sx$, we have $Sx\preceq TSx$. Similarly, we have $Tx\preceq STx$. Thus $T,S : X\to X$ are weakly increasing. To prove (4), for any two comparable elements x and y in X, if $x\preceq y$, then

$$p(Tx,Sy)=max\left\{\frac{x^3}{8},\frac{y^2}{4}\right\}\leq\frac{x^2}{4},$$

and

$$p(Tx,y)+p(x,Sy)=p\left(\frac{x^3}{8},y\right)+p\left(x,\frac{y^2}{4}\right)=p\left(\frac{x^3}{8},y\right)+max\left\{x,\frac{y^2}{4}\right\}=p\left(\frac{x^3}{8},y\right)+x.$$

So, we have

$$\begin{aligned}\psi\left(p(Tx,Sy)\right)&\leq\frac{x^4}{16}\leq\frac{x^2}{16}\\&\leq\frac{(x+p(\frac{x^3}{8},y))^2}{16}\\&=\frac{(x+p(\frac{x^3}{8},y))^2}{8}-\frac{(x+p(\frac{x^3}{8},y))^2}{16}\\&=\varphi\left(\frac{p(Tx,y)+p(x,Sy)}{2}\right)-\phi(p(Tx,y),p(x,Sy)).\end{aligned}$$

If $y \preceq x$, then

$$p(Tx, Sy) = max\left\{\frac{x^3}{8}, \frac{y^2}{4}\right\} = \frac{y^2}{4},$$

and

$$p(Tx, y) + p(x, Sy) = p\left(\frac{x^3}{8}, y\right) + p\left(x, \frac{y^2}{4}\right) = max\left\{\frac{x^3}{8}, y\right\} + p\left(x, \frac{y^2}{4}\right) = y + p\left(x, \frac{y^2}{4}\right).$$

So, we have

$$\begin{aligned}\psi\left(p(Tx, Sy)\right) &= \frac{y^4}{16} \le \frac{y^2}{16} \\ &\le \frac{(y + p(x, \frac{y^2}{4}))^2}{16} \\ &= \frac{(y + p(x, \frac{y^2}{4}))^2}{8} - \frac{(y + p(x, \frac{y^2}{4}))^2}{16} \\ &= \varphi\left(\frac{p(Tx, y) + p(x, Sy)}{2}\right) - \phi(p(Tx, y), p(x, Sy)).\end{aligned}$$

From the above arguments, we conclude that (4) holds, hence all the required hypotheses of Corollary 2.6 are satisfied, and thus, we deduce the existence of a common fixed point of S and T. Here, 0 is the common fixed point of S and T. □

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