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The upper bound estimation for the spectral norm of r-circulant and symmetric r-circulant matrices with the Padovan sequence

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Abstract

In this paper, we gives an upper bound estimation of the spectral norm for matrices A and B such that the entries in the first row of $n \times n$ r-circulant matrix $A = \text{Circ}_r(a_1, a_2, \ldots, a_n)$ and $n \times n$ symmetric r-circulant matrix $B = \text{SCirc}_r(a_1, a_2, \ldots, a_n)$ are $a_i = P_i$ or $a_i = P_i^2$ or $a_i = P_{i-1}$ or $a_i = P_{i-1}^2$, where $\{P_i\}_{i=0}^{\infty}$ is Padovan sequence. At the last section, some illustrative numerical example is furnished which demonstrate the validity of the hypotheses and degree of utility of our results.

Keywords: r-circulant matrices, symmetric r-circulant matrices, Padovan sequence, Hadamard product.
2010 MSC: 15A15, 15A18, 15A24, 15A48.

1. Introduction and preliminaries

The r-circulant matrices plays an important role in many branches of applied mathematics such as signal processing, coding theory, image processing and linear forecast.

Definition 1.1. An $n \times n$ matrix A is called an r-circulant matrix if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ ra_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ra_3 & ra_4 & \cdots & a_1 & a_2 \\ ra_2 & ra_3 & \cdots & ra_n & a_1 \end{pmatrix},$$

where $r, a_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$.

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The elements of the r-circulant matrix are determined by its first row elements $a_1, a_2, \ldots, a_n$ and the parameter r, thus we denote $A = \text{Circ}_r(a_1, a_2, \ldots, a_n)$. Especially when $r = 1$, we write $\text{Circ}(a_1, a_2, \ldots, a_n)$ instead of $\text{Circ}_1(a_1, a_2, \ldots, a_n)$, that is,

$$\text{Circ}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \cdots & a_1 & a_2 \\ a_2 & a_3 & \cdots & a_n & a_1 \end{pmatrix}$$

and it is called a circulant matrix.

Definition 1.2. An $n \times n$ matrix is called a symmetric r-circulant matrix if it is of the form

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & ra_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & ra_{n-3} & ra_{n-2} \\ a_n & ra_1 & \cdots & ra_{n-2} & ra_{n-1} \end{pmatrix},$$

where $r, a_i \in \mathbb{C}$ for all $i = 1, 2, \ldots, n$.

The elements of the symmetric r-circulant matrix are determined by its first row elements $a_1, a_2, \ldots, a_n$ and the parameter r, thus we denote $A = \text{SCirc}_r(a_1, a_2, \ldots, a_n)$. Especially when $r = 1$, we write $\text{SCirc}(a_1, a_2, \ldots, a_n)$ instead of $\text{SCirc}_1(a_1, a_2, \ldots, a_n)$, that is,

$$\text{SCirc}(a_1, a_2, \ldots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_n & \cdots & a_{n-3} & a_{n-2} \\ a_n & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$$

and it is called a symmetric circulant matrix.

Example 1.3. Let

$$A = \begin{pmatrix} -3 & 1 & 4 & 0 & 2 \\ 2 & -3 & 1 & 4 & 0 \\ 0 & 2 & -3 & 1 & 4 \\ 4 & 0 & 2 & -3 & 1 \\ 1 & 4 & 0 & 2 & -3 \end{pmatrix}, B = \begin{pmatrix} -3 & 1 & 4 & 0 & 2 \\ 6 & -3 & 1 & 4 & 0 \\ 0 & 6 & -3 & 1 & 4 \\ 12 & 0 & 6 & -3 & 1 \\ 3 & 12 & 0 & 6 & -3 \end{pmatrix},$$

$$C = \begin{pmatrix} -3 & 1 & 4 & 0 & 2 \\ 1 & 4 & 0 & 2 & -3 \\ 4 & 0 & 2 & -3 & 1 \\ 0 & 2 & -3 & 1 & 4 \\ 2 & -3 & 1 & 4 & 0 \end{pmatrix}, D = \begin{pmatrix} -3 & 1 & 4 & 0 & 2 \\ 1 & 4 & 0 & 2 & -9 \\ 4 & 0 & 2 & -9 & 3 \\ 0 & 2 & -9 & 3 & 12 \\ 2 & -9 & 3 & 12 & 0 \end{pmatrix}.$$

Then $A = \text{Circ}(-3, 1, 4, 0, 2)$, $B = \text{Circ}_3(-3, 1, 4, 0, 2)$, $C = \text{SCirc}(-3, 1, 4, 0, 2)$ and $B = \text{SCirc}_3(-3, 1, 4, 0, 2)$.

Next, we give the concepts of the spectral norm, the maximum column length norm and the maximum row length norm of arbitrary matrix.

Definition 1.4. Let $A = (a_{ij})_{m \times n}$ be a matrix, where $a_{ij} \in \mathbb{C}$ for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

1. The spectral norm of the matrix A is defined by
$$\|A\|_S := \sqrt{\max_{1\le i\le n} \lambda_i(A^H A)},$$
where $\lambda_i(A^H A)$ is the eigenvalue of $A^H A$ and A^H is the conjugate transpose of matrix A.
2. The maximum column length norm of the matrix A is defined by
$$c_1(A) := \max_{1\le j\le n} \sqrt{\sum_{1\le i\le m} |a_{ij}|^2}.$$
3. The maximum row length norm of the matrix A is defined by
$$r_1(A) := \max_{1\le i\le m} \sqrt{\sum_{1\le j\le n} |a_{ij}|^2}.$$

Here we give the following important lemma about the spectral norm, the maximum column length norm and the maximum row length norm which is proved by Mathias [2] in 1990.

Lemma 1.5 ([2])**.** *Let A, B and C be $m \times n$ matrices. If $A = B \circ C$, where $B \circ C$ is the Hadamard product of B and C, then*
$$\|A\|_S \le r_1(B)c_1(C) \tag{1.1}$$
and
$$\|A\|_S \le \|B\|_S\|C\|_S. \tag{1.2}$$

In recent years, several mathematicians were concerned with r-circulant matrices associated with a number sequence. For example, Solak [4, 5] and Shen and Cen [3] calculated and estimated the Frobenius norm and the spectral norm of a circulant matrix where the elements of the r-circulant matrix are Fibonacci numbers and Lucas numbers. More recently, He *et al.* [1] approximated upper bound of the spectral norm of a r-circulant and symmetric r-circulant matrices where the elements of these matrices are Fibonacci numbers and Lucas numbers.

Inspired by the recent work, we gives an upper bound estimation of the spectral norm for r-circulant and symmetric r-circulant matrices with Padovan sequences. Some illustrative numerical example is furnished which demonstrate the validity of the hypotheses and degree of utility of our results.

2. Main results

Firstly, we give the concept of Padovan sequence $\{P_n\}_{n=0}^{\infty}$ which is defined by
$$P_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, 2, \\ P_{n-3} + P_{n-2}, & n = 3, 4, 5, \dots . \end{cases}$$

If we start by zero, then the Padovan sequence is given by

n	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
P_n	0	1	1	1	2	2	3	4	5	7	9	12	16	$\cdots$

(2.1)

Remark 2.1. The Padovan sequence $\{P_n\}_{n=0}^{\infty}$ satisfies the following properties:

1. $\sum_{s=0}^{n} P_s = \sum_{s=1}^{n} P_s = P_{n+5} - 2$ for each fixed $n \in \mathbb{N}$;

2. $\sum_{s=0}^{n} P_s^2 = \sum_{s=1}^{n} P_s^2 = P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2$ for each fixed $n \in \mathbb{N}$ with $n \geq 3$.

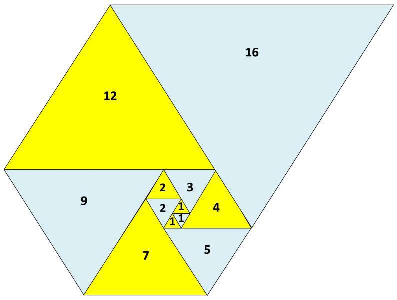

Figure 1: Spiral of equilateral triangles with side lengths which follow the Padovan sequence

Now, we give an upper bound estimation of the spectral norm for r-circulant and symmetric r-circulant matrices with Padovan sequences.

Theorem 2.2. *Let $A = Circ_r(P_1, P_2, \ldots, P_n)$ be an r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \leq \begin{cases} n, & n = 1, 2, \\ \sqrt{n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots; \end{cases}$
2. *If $|r| \geq 1$, then* $\|A\|_S \leq \begin{cases} \sqrt{n(n-1)|r|^2 + n}, & n = 1, 2, \\ \sqrt{[(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots. \end{cases}$

Proof. Since $A = \mathrm{Circ}_r(P_1, P_2, \ldots, P_n)$ is a r-circulant matrix, it is of the form

$$\begin{pmatrix} P_1 & P_2 & \cdots & P_{n-1} & P_n \\ rP_n & P_1 & \cdots & P_{n-2} & P_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rP_3 & rP_4 & \cdots & P_1 & P_2 \\ rP_2 & rP_3 & \cdots & rP_n & P_1 \end{pmatrix}.$$

Setting the matrices B and C are

$$B = \begin{pmatrix} P_1 & 1 & \cdots & 1 & 1 \\ r & P_1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & P_1 & 1 \\ r & r & \cdots & r & P_1 \end{pmatrix}, \quad C = \begin{pmatrix} P_1 & P_2 & \cdots & P_{n-1} & P_n \\ P_n & P_1 & \cdots & P_{n-2} & P_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_3 & P_4 & \cdots & P_1 & P_2 \\ P_2 & P_3 & \cdots & P_n & P_1 \end{pmatrix}.$$

Now we obtain that $A = B \circ C$, where $B \circ C$ is the Hadamard product of B and C. It is easy to see that

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{1 \leq j \leq n} |b_{ij}|^2} = \begin{cases} \sqrt{n}, & |r| < 1, \\ \sqrt{(n-1)|r|^2 + 1}, & |r| \geq 1 \end{cases} \tag{2.2}$$

and

$$c_1(C) = \max_{1 \leq j \leq n} \sqrt{\sum_{1 \leq i \leq n} |c_{ij}|^2} = \sqrt{\sum_{s=1}^{n} P_s^2} = \begin{cases} \sqrt{n}, & n = 1, 2, \\ \sqrt{P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2}, & n = 3, 4, \ldots. \end{cases} \tag{2.3}$$

By using inequality (1.1) in Lemma 1.5, from (2.2) and (2.3), we obtain the following results.

- If $|r| < 1$, then we get $\|A\|_S \leq \begin{cases} n, & n = 1, 2, \\ \sqrt{n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots; \end{cases}$
- If $|r| \geq 1$, then we get $\|A\|_S \leq \begin{cases} \sqrt{n(n-1)|r|^2 + n}, & n = 1, 2, \\ \sqrt{[(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots. \end{cases}$

□

Corollary 2.3. *Let $A = Circ_r(P_1^2, P_2^2, \ldots, P_n^2)$ be an r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \leq \begin{cases} n^2, & n = 1, 2, \\ n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2], & n = 3, 4, \ldots; \end{cases}$
2. *If $|r| \geq 1$, then* $\|A\|_S \leq \begin{cases} n(n-1)|r|^2 + n, & n = 1, 2, \\ [(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2], & n = 3, 4, \ldots. \end{cases}$

Proof. It is easy to see that $A = \mathrm{Circ}_r(P_1^2, P_2^2, \ldots, P_n^2) = B \circ B$, where $B = \mathrm{Circ}_r(P_1, P_2, \ldots, P_n)$ and $B \circ B$ is the Hadamard product of B and B. From inequality (1.2) in Lemma 1.5 and Theorem 2.2, we get this result. □

By similar the proof in Theorem 2.2, we get the following results for symmetric r-circulant matrix.

Theorem 2.4. *Let $A = SCirc_r(P_1, P_2, \ldots, P_n)$ be a symmetric r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \leq \begin{cases} n, & n = 1, 2, \\ \sqrt{n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots; \end{cases}$
2. *If $|r| \geq 1$, then* $\|A\|_S \leq \begin{cases} \sqrt{n(n-1)|r|^2 + n}, & n = 1, 2, \\ \sqrt{[(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots. \end{cases}$

Proof. Since $A = \mathrm{SCirc}_r(P_1, P_2, \ldots, P_n)$ is a symmetric r-circulant matrix, it is of the form

$$\begin{pmatrix} P_1 & P_2 & \cdots & P_{n-1} & P_n \\ P_2 & P_3 & \cdots & P_n & rP_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-1} & P_n & \cdots & rP_{n-3} & rP_{n-2} \\ P_n & rP_1 & \cdots & rP_{n-2} & rP_{n-1} \end{pmatrix}.$$

Setting the matrices B and C are

$$B = \begin{pmatrix} P_1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & rP_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & r & r \\ 1 & rP_1 & \cdots & r & r \end{pmatrix}, \quad C = \begin{pmatrix} P_1 & P_2 & \cdots & P_{n-1} & P_n \\ P_2 & P_3 & \cdots & P_n & P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-1} & P_n & \cdots & P_{n-3} & P_{n-2} \\ P_n & P_1 & \cdots & P_{n-2} & P_{n-1} \end{pmatrix}.$$

Now we obtain that $A = B \circ C$, where $B \circ C$ is the Hadamard product of B and C. It is easy to see that

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{1 \leq j \leq n} |b_{ij}|^2} = \begin{cases} \sqrt{n}, & |r| < 1, \\ \sqrt{(n-1)|r|^2 + 1}, & |r| \geq 1 \end{cases} \tag{2.4}$$

and

$$c_1(C) = \max_{1 \leq j \leq n} \sqrt{\sum_{1 \leq i \leq n} |c_{ij}|^2} = \sqrt{\sum_{s=1}^{n} P_s^2} = \begin{cases} \sqrt{n}, & n = 1, 2, \\ \sqrt{P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2}, & n = 3, 4, \ldots. \end{cases} \tag{2.5}$$

By using inequality (1.1) in Lemma 1.5, from (2.4) and (2.5), we obtain the following results.

- If $|r| < 1$, then we get $\|A\|_S \le \begin{cases} n, & n = 1, 2, \\ \sqrt{n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots; \end{cases}$
- If $|r| \ge 1$, then we get $\|A\|_S \le \begin{cases} \sqrt{n(n-1)|r|^2 + n}, & n = 1, 2, \\ \sqrt{[(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2]}, & n = 3, 4, \ldots. \end{cases}$

□

Corollary 2.5. *Let $A = SCirc_r(P_1^2, P_2^2, \ldots, P_n^2)$ be a symmetric r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \le \begin{cases} n^2, & n = 1, 2, \\ n[P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2], & n = 3, 4, \ldots; \end{cases}$
2. *If $|r| \ge 1$, then* $\|A\|_S \le \begin{cases} n(n-1)|r|^2 + n, & n = 1, 2, \\ [(n-1)|r|^2 + 1][P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2], & n = 3, 4, \ldots. \end{cases}$

Next, we give second main result in this work.

Theorem 2.6. *Let $A = Circ_r(P_0, P_1, \ldots, P_{n-1})$ be an r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \le \begin{cases} n-1, & n = 1, 2, 3, \\ \sqrt{[n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4, 5, \ldots; \end{cases}$
2. *If $|r| \ge 1$, then* $\|A\|_S \le \begin{cases} (n-1)|r|, & n = 1, 2, 3, \\ \sqrt{[(n-1)|r|^2][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4, 5, \ldots. \end{cases}$

Proof. Since $A = \mathrm{Circ}_r(P_0, P_1, \ldots, P_{n-1})$ is a r-circulant matrix, it is of the form

$$\begin{pmatrix} P_0 & P_1 & \cdots & P_{n-2} & P_{n-1} \\ rP_{n-1} & P_0 & \cdots & P_{n-3} & P_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rP_2 & rP_3 & \cdots & P_0 & P_1 \\ rP_1 & rP_2 & \cdots & rP_{n-1} & P_0 \end{pmatrix}.$$

Setting the matrices B and C are

$$B = \begin{pmatrix} P_0 & 1 & \cdots & 1 & 1 \\ r & P_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \cdots & P_0 & 1 \\ r & r & \cdots & r & P_0 \end{pmatrix}, \quad C = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & \cdots & P_{n-3} & P_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & \cdots & P_0 & P_1 \\ P_1 & P_2 & \cdots & P_{n-1} & P_0 \end{pmatrix}.$$

Now we obtain that $A = B \circ C$, where $B \circ C$ is the Hadamard product of B and C. It is easy to see that

$$r_1(B) = \max_{1 \le i \le n} \sqrt{\sum_{1 \le j \le n} |b_{ij}|^2} = \begin{cases} \sqrt{n-1}, & |r| < 1, \\ \sqrt{(n-1)|r|^2}, & |r| \ge 1 \end{cases} \tag{2.6}$$

and

$$c_1(C) = \max_{1 \le j \le n} \sqrt{\sum_{1 \le i \le n} |c_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} P_s^2} = \begin{cases} \sqrt{n-1}, & n = 1, 2, 3, \\ \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}, & n = 4, 5, \ldots. \end{cases} \tag{2.7}$$

By using inequality (1.1) in Lemma 1.5, from (2.6) and (2.7), we obtain the following results.

- If $|r| < 1$, then we get $\|A\|_S \leq \begin{cases} n-1, & n = 1,2,3, \\ \sqrt{[n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4,5,\ldots; \end{cases}$

- If $|r| \geq 1$, then we get $\|A\|_S \leq \begin{cases} (n-1)|r|, & n = 1,2,3, \\ \sqrt{[(n-1)|r|^2][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4,5,\ldots. \end{cases}$

□

Corollary 2.7. *Let $A = Circ_r(P_0^2, P_1^2, \ldots, P_{n-1}^2)$ be an r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \leq \begin{cases} (n-1)^2, & n = 1,2,3, \\ [n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2], & n = 4,5,\ldots; \end{cases}$
2. *If $|r| \geq 1$, then* $\|A\|_S \leq \begin{cases} (n-1)^2|r|^2, & n = 1,2,3, \\ [(n-1)|r|^2][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2], & n = 4,5,\ldots. \end{cases}$

Proof. It is easy to see that $A = \text{Circ}_r(P_0^2, P_1^2, \ldots, P_{n-1}^2) = B \circ B$, where $B = \text{Circ}_r(P_0, P_1, \ldots, P_{n-1})$ and $B \circ B$ is the Hadamard product of B and B. From inequality (1.2) in Lemma 1.5 and Theorem 2.6, we get this result. □

By similar the proof in Theorem 2.6, we get the following results for symmetric r-circulant matrix.

Theorem 2.8. *Let $A = SCirc_r(P_0, P_1, \ldots, P_{n-1})$ be a symmetric r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \leq \begin{cases} n-1, & n = 1,2,3, \\ \sqrt{[n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4,5,\ldots; \end{cases}$
2. *If $|r| \geq 1$, then* $\|A\|_S \leq \begin{cases} 0, & n = 1, \\ \sqrt{(n-1)[(n-2)|r|^2+1]}, & n = 2,3, \\ \sqrt{[(n-2)|r|^2+1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n = 4,5,\ldots. \end{cases}$

Proof. Since $A = \text{SCirc}_r(P_0, P_1, \ldots, P_{n-1})$ is a symmetric r-circulant matrix, it is of the form

$$\begin{pmatrix} P_0 & P_1 & \cdots & P_{n-2} & P_{n-1} \\ P_1 & P_2 & \cdots & P_{n-1} & rP_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-2} & P_{n-1} & \cdots & rP_{n-4} & rP_{n-3} \\ P_{n-1} & rP_0 & \cdots & rP_{n-3} & rP_{n-2} \end{pmatrix}.$$

Setting the matrices B and C are

$$B = \begin{pmatrix} P_0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & rP_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & r & r \\ 1 & rP_0 & \cdots & r & r \end{pmatrix}, \quad C = \begin{pmatrix} P_0 & P_1 & \cdots & P_{n-2} & P_{n-1} \\ P_1 & P_2 & \cdots & P_{n-1} & P_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{n-2} & P_{n-1} & \cdots & P_{n-4} & P_{n-3} \\ P_{n-1} & P_0 & \cdots & P_{n-3} & P_{n-2} \end{pmatrix}.$$

Now we obtain that $A = B \circ C$, where $B \circ C$ is the Hadamard product of B and C. It is easy to see that

$$r_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{1 \leq j \leq n} |b_{ij}|^2} = \begin{cases} \sqrt{n-1}, & |r| < 1, \\ 0, & |r| \geq 1 \text{ and } n = 1, \\ \sqrt{(n-2)|r|^2+1}, & |r| \geq 1 \text{ and } n = 2,3,\ldots \end{cases} \tag{2.8}$$

and

$$c_1(C) = \max_{1\le j\le n}\sqrt{\sum_{1\le i\le n}|c_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1}P_s^2} = \begin{cases} \sqrt{n-1}, & n=1,2,3, \\ \sqrt{P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2}, & n=4,5,\ldots. \end{cases} \tag{2.9}$$

By using (1.1) in Lemma 1.5, from (2.8) and (2.9), we obtain the following results.

- If $|r| < 1$, then we get $\|A\|_S \le \begin{cases} n-1, & n=1,2,3, \\ \sqrt{[n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n=4,5,\ldots; \end{cases}$

- If $|r| \ge 1$, then we get $\|A\|_S \le \begin{cases} 0, & n=1, \\ \sqrt{(n-1)[(n-2)|r|^2+1]}, & n=2,3, \\ \sqrt{[(n-2)|r|^2+1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2]}, & n=4,5,\ldots. \end{cases}$

□

Corollary 2.9. *Let $A = SCirc_r(P_0^2, P_1^2, \ldots, P_{n-1}^2)$ be a symmetric r-circulant matrix such that $\{P_n\}_{n=0}^{\infty}$ is a Padovan sequence. Then the following assertions hold:*

1. *If $|r| < 1$, then* $\|A\|_S \le \begin{cases} (n-1)^2, & n=1,2,3, \\ [n-1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2], & n=4,5,\ldots; \end{cases}$
2. *If $|r| \ge 1$, then* $\|A\|_S \le \begin{cases} 0, & n=1, \\ (n-1)[(n-2)|r|^2+1], & n=2,3, \\ [(n-2)|r|^2+1][P_{n+1}^2 - P_{n-2}^2 - P_{n-4}^2], & n=4,5,\ldots. \end{cases}$

3. Examples

Example 3.1. Let $\widetilde{A} = \mathrm{Circ}_r(P_1, P_2, \ldots, P_n)$ be an r-circulant matrix and $\widehat{A} = \mathrm{SCirc}_r(P_1, P_2, \ldots, P_n)$ be a symmetric r-circulant matrix, in which $\{P_n\}_{n=0}^{\infty}$ denotes the Padovan sequence. It is easy to find that the upper bounds for the spectral norm of $\widetilde{A}$ and $\widehat{A}$ from Theorems 2.2 and 2.4 (see in Tables 1, 2 and 3).

Table 1 Numerical results for $|r| < 1$

n	upper bound for the spectral norm from Theorems 2.2 and 2.4
2	2
3	$\sqrt{9} = 3$
4	$\sqrt{28} \approx 5.29150$
5	$\sqrt{55} \approx 7.41620$
6	$\sqrt{120} \approx 10.95445$
7	$\sqrt{252} \approx 15.87451$

Table 2 Numerical results for $r = -2, 2$

n	upper bound for the spectral norm from Theorems 2.2 and 2.4
2	$\sqrt{10} \approx 3.16228$
3	$\sqrt{27} \approx 5.19615$
4	$\sqrt{91} \approx 9.53939$
5	$\sqrt{187} \approx 13.67479$
6	$\sqrt{420} \approx 20.49390$
7	$\sqrt{900} = 30$

Table 3 Numerical results for $r = -3, 3$

n	upper bound for the spectral norm from Theorems 2.2 and 2.4
2	$\sqrt{20} \approx 4.47214$
3	$\sqrt{57} \approx 7.54983$
4	$\sqrt{196} = 14$
5	$\sqrt{407} \approx 20.17424$
6	$\sqrt{920} \approx 30.33150$
7	$\sqrt{1980} \approx 44.49719$

Example 3.2. Let $A = \text{Circ}_r(P_0, P_1, \ldots, P_{n-1})$ be an r-circulant matrix, in which $\{P_n\}_{n=0}^{\infty}$ denotes the Padovan sequence. It is easy to find that the upper bounds for the spectral norm from Theorem 2.6 (see in Table 4,5,6).

Table 4 Numerical results for $|r| < 1$

n	upper bound for the spectral norm from Theorem 2.6
2	1
3	2
4	$\sqrt{9} = 3$
5	$\sqrt{28} \approx 5.29150$
6	$\sqrt{55} \approx 7.41620$
7	$\sqrt{120} \approx 10.95445$

Table 5 Numerical results for $r = -2, 2$

n	upper bound for the spectral norm from Theorem 2.6
2	2
3	4
4	$\sqrt{36} = 6$
5	$\sqrt{112} \approx 10.58301$
6	$\sqrt{220} \approx 14.83240$
7	$\sqrt{480} \approx 21.90890$

Table 6 Numerical results for $r = -3, 3$

n	upper bound for the spectral norm from Theorem 2.6
2	3
3	6
4	$\sqrt{81} = 9$
5	$\sqrt{252} \approx 15.87451$
6	$\sqrt{495} \approx 22.24860$
7	$\sqrt{1,080} \approx 32.86335$

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