Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 763–775
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

A model of the groundwater flowing within a leaky aquifer using the concept of local variable order derivative

Abdon Atangana[a], Emile Franc Doungmo Goufo[b,*]

[a] *Institute for groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, 9300 Bloemfontein South Africa.*

[b] *Department of Mathematical Sciences, University of South Africa, Florida Sciences Campus, 003 South Africa.*

Communicated by J. J. Nieto

Abstract

One of the big problems we encounter in groundwater modeling is to provide a correct model that can be used to describe the movement of water via a particular geological formation. In this work, in order to further enhance the model of groundwater flow in a leaky aquifer, we made use of a new derivative called the local variable order derivative. The derivative includes into mathematical formula the complexity of the leaky aquifer, which is for instance the variation of the aquifer, or the heterogeneity of the leaky aquifer. The modified equation was solved using the concept of iterative method. We presented in detail the stability and the uniqueness of the special solution.

Keywords: Leaky aquifer, variable order derivative, stability and uniqueness analysis, special solution.
2010 MSC: 26A33, 35B35, 34K07.

1. Introduction

For the first time, Pythagoras realized that mathematics tools can be used to describe the pattern of real physical problems. Later on, Diophantus of Alexandria realized that, these natural patterns can be described via mathematical equations [10]. The notion has been used intensively in the circle of mathematics; however the idea of motion was not already introduced by that time. In the year 18^{th}, Sir Isaac Newton and

*Corresponding authors
Email addresses: abdonatangana@yahoo.fr (Abdon Atangana), franckemile2006@yahoo.ca (Emile Franc Doungmo Goufo)

Received 2015-3-25

Gottfried W. Leibniz independently introduced the concept of motion leading to the concept of derivative [4, 11, 12]. Since then, this concept has been used in almost all the branches of sciences to model real world problems[3, 7, 8, 9]. It is perhaps important to note that, the big challenge in this process is to include into mathematical formula all the detail surrounding the physical problem under observation. It happens to appear that, the Newtonian concept of derivative cannot satisfy all the complexity of the natural occurrences. For instance, how, do we explain accurately the movement of water within the leaky aquifer? An attempt to answer this question, Hantush has proposed and equation based on the model proposed by Theis in [1, 2, 13, 14]. Although this model has being used by many hydro-geologists, it is worth noting that, the model does not take into account all the details surrounding the movement of water through a leaky geological formation. A first attempt to enhance model, was to introduce the concept of derivative with fractional order [3]. This model has improved the description of this physical problem at a certain extend. Nonetheless, to be accurate, when dealing complexes systems, even the concept of fractional order derivative has some limitations, for instance it is not possible to accurately model the trap of water under matrices rocks. We shall mention that, a mathematical model will be considered accurate if and only if the numerical representation of the mathematic solution is in good agreement with the observed facts. If not there are two questions that need to be answered: the first one is to know if the experimental data were accurately measured. The second one will be to know if the mathematical equation is accurately implemented. If the second question appears to be negative, then, the model needs to be revised. In the case of leaky aquifer model with non-integer and integer order derivatives have failed to do the job. The aim of our paper is to revise this model by introducing the concept of variable order derivative, which so far appears to be the best concept for complexes systems.

2. Groundwater water flow equation using the local variable order derivative

The initial proposed groundwater equation within the leaky aquifer that was proposed by Hantush is given by:

$$\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}=\frac{S}{T}\frac{\partial S(r,t)}{\partial t} \tag{2.1}$$

The above equation then modified by Atangana [3] as follows

$$\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}=\frac{S}{T}\frac{\partial^\alpha S(r,t)}{\partial t^\alpha}, \tag{2.2}$$

$$\frac{\partial^\alpha S(r,t)}{\partial r^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-x)^{1-\alpha}\frac{\partial S(r,x)}{\partial r}dx,\quad 0<\alpha\leq 1.$$

As we said before, the above model was also unable to describe accurately the complexity of the geological formation. Therefore in order to further include into mathematical formula the complexity of the aquifer through which the flow take place, we shall proposed the following version

$$\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}=\frac{S}{T}{}_0^AD_\alpha^t\left(S(r,t)\right), \tag{2.3}$$

$${}_0^AD_\alpha^t\left(S(r,t)\right)=\lim_{\varepsilon\to 0}\frac{S\left(r,t+\varepsilon\left(t+\frac{1}{\Gamma(1-l(r,t))}\right)^{1-l(r,t)}\right)-S\left(r,t\right)}{\varepsilon}.$$

Here the function $l\left(r,t\right)$ accounts for the complexity associated with the leaky aquifer, $S(r,t)$ is the change of level of water, S is the storativity, T is the transmissivity and B is the factor that account for the leakage. We shall show some useful properties of the local variable order derivative.

Theorem 2.1. *Assume that $f(x,y)$ is function which $\partial_x^{l(x)}\left[\partial_y^{o(y)}\left(f(x,y)\right)\right]$ and $\partial_y^{o(y)}\left[\partial_x^{l(x)}\left(f(x,y)\right)\right]$ exist and is continous over the domain $D\subset\mathbb{R}_2$ then [5]*

$$\partial_x^{l(x)}\left[\partial_y^{o(y)}\left(f(x,y)\right)\right]=\partial_y^{o(y)}\left[\partial_x^{l(x)}\left(f(x,y)\right)\right]. \tag{2.4}$$

Theorem 2.2. *Assuming that, a given function says $f\ :[a,\ 8)\to\mathbb{R}$ is $o(x)-$differentiable at a given point, say $x_0\geq a,$ then, f is also continuous at x_0.*

Proof. Assuming that f is $o(x)-$differentiable then

$$ {}_0^AD_x^{o(x)}\left(f(x_0)\right)=\lim_{\varepsilon\to 0}\frac{f\left(x_0+\varepsilon\left(x_0+\frac{1}{\Gamma(1-o(x))}\right)^{1-o(x)}\right)-f(x_0)}{\varepsilon}. \tag{2.5}$$

□

Theorem 2.3. *Assuming that f is $o(x)-$differentiable on an open interval $(a,\ b)$ then*

1. *If ${}_0^AD_x^{o(x)}\left(f(x)\right)<0$ for all $x\in(a,\ b)$ then f is decreasing there,*
2. *If ${}_0^AD_x^{o(x)}\left(f(x)\right)>0$ for all $x\in(a,\ b)$ then f is increasing there,*
3. *If ${}_0^AD_x^{o(x)}\left(f(x)\right)=0$ for all $x\in(a,\ b)$ then f is constant there.*

Definition 2.4. Let $f:[a,\infty)\to\mathbb{R}$ is given function, then we propose that the anti-variable derivative of f is

$$ {}_a^AI_x^{o(x,t)}\left(f(x)\right)=\int_a^x\left(t+\frac{1}{\Gamma\left(1-o\left(x,\ t\right)\right)}\right)^{o(x,\ t)-1}f\left(t\right)dt. \tag{2.6}$$

The above operator is the inverse operator of the proposed fractional derivative. We shall present to underpin this statement by the following theorem.

Theorem 2.5. *Fundamental theorem of local variable calculus: ${}_0^AD_x^{o(x,t)}\left[{}_0^AI_x^{o(x,t)}f(x)\right]=f\left(x\right)$ for all $x\ \geq\ a$ with f a given continuous and differentiable function.*

Proof. Let f be a continous function, then by definition if we let ${}_0^AI_x^{\alpha}f\left(x\right)=F(x)$, we have

$$\begin{aligned}
{}_0^AD_x^{o(x)}\left[{}_0^AI_x^{o(x)}f(x)\right]&=\lim_{\varepsilon\to 0}\frac{F\left(x+\varepsilon\left(x+\frac{1}{\Gamma(1-o(x))}\right)^{1-0(x)}\right)-F(x)}{\varepsilon}\\
&=\left(x+\frac{1}{\Gamma\left(1-o(x)\right)}\right)^{1-o(x)}\frac{dF(x)}{dx}\\
&=\left(x+\frac{1}{\Gamma\left(1-o(x)\right)}\right)^{1-o(x)}\frac{d}{dx}\left\{\int_a^x\left(t+\frac{1}{\Gamma\left(1-o(t)\right)}\right)^{o(t)-1}f(t)dt\right\}\\
&=\left(x+\frac{1}{\Gamma\left(1-o(x)\right)}\right)^{1-o(x)}\left(x+\frac{1}{\Gamma\left(1-o(x)\right)}\right)^{o(x)-1}f\left(x\right)=f\left(x\right).
\end{aligned}$$

This completes the proof. □

3. Construction of a possible special solution

The aim of this section is to construct a possible solution of the novel groundwater flow within a leaky aquifer. To construct a solution to the new equation, we employ the $o\left(x,t\ \right)$-Laplace operator defined as

Definition 3.1. Let g be a function defined in $(0,\ \infty)$, then, we defined the $o(x,t)$-Laplace transform of f as

$$L_{o(x)}\left(f\left(x\right)\right)(s)=\int_0^\infty\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{o(x,t)-1}e^{-st}f(t)dt. \tag{3.1}$$

We shall give some properties of the above operator. The above operator satisfies the following properties, $F(s)$ is the Laplace transform of $f(t)$

$$L_{o(x,t)}\left({}_0^AD_x^{o(x,t)}\left(\frac{df(x)}{dx}\right)\right)(s)=s^2F\left(s\right)-sf\left(0\right)-f(0).$$

The proposed operator satisfies the following properties, $F(s)$ is the Laplace of $f(t)$

1. Linearity

$$L_{o(x)}\left(af\left(x\right)+bg\left(x\right)\right)(s)=aL_{o(x)}\left(f\left(x\right)\right)(s)+bL_{o(x)}\left(g\left(x\right)\right)(s),$$

2. Time delay

$$L_{o(x)}\left({}_0^AD_x^{o(x,t)}\left\{f\left(x-a\right).\delta(x-a)\right\}\right)(s)=se^{-sa}F(s),$$

3. First derivative

$$L_{o(x)}\left({}_0^AD_x^{o(x,t)}\left(\frac{df(x)}{dx}\right)\right)(s)=s^2F\left(s\right)-sf\left(0\right)-f(0),$$

4. N order derivative

$$L_{o(x)}\left({}_0^AD_x^{o(x,t)}\left(\frac{d^nf(x)}{dx^n}\right)\right)(s)=s^{n+1}F\left(s\right)-\sum_{j=0}^{n}s^jf^{(n-1)}(0),$$

5. Fractional derivative Caputo type

$$L_{o(x)}\left({}_0^AD_x^{o(x)}\left(\frac{d^\alpha f(x)}{dx^\alpha}\right)\right)(s)=L_{o(x)}\left({}_0^AD_x^{o(x)}\left(\frac{d^\alpha f(x)}{dx^\alpha}\right)\right)(s)=s^{\alpha+1}F\left(s\right)-\sum_{j=0}^{n}s^{\alpha-k}f^{(n-1)}\left(0\right),$$

   $n-1<\alpha\leq n.$

6. Integral

$$L_{o(x)}\left({}_0^AD_x^{o(x)}\left(\int_0^x f(t)dt\right)\right)(s)=F(s),$$

7. Convolution

$$L_{o(x)}\left({}_0^AD_x^{o(x)}\left(f*g(x)\right)\right)(s)=sF(s)G(s),$$

8. Multiplication by distance

$$L_{o(x)}\left({}_0^AD_x^{o(x)}\left\{xf(x)\right\}\right)(s)=-sF'(s),$$

9. Complex shift

$$L_{o(x)}\left({}_0^AD_x^{o(x)}\left\{e^{-ax}f(x)\right\}\right)(s)=sF\left(s+a\right)-f(0),$$

10. Distance Scaling

$$L_{o(x)}\left({}_0^A D_x^{o(x)}\left\{f\left(ax\right)\right\}\right)(s)=\frac{s}{a}F\left(\frac{s}{a}\right)-f(0).$$

Proof. Proof of 1: By definition, we have the following formula

$$L_{o(x)}\left(af\left(x\right)+bg\left(x\right)\right)(s)=\int_0^8\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{o(x,t)-1}e^{-st}\left(af\left(t\right)+bg\left(t\right)\right)dt.$$

Using the linearity of the integral, we obtain the following results

$$aL_{o(x)}\left(f\left(x\right)\right)(s)+bL_{o(x)}\left(g\left(x\right)\right)(s).$$

This completes the proof of property 1.

proof of 2: By definition, we have the following formula

$$\begin{aligned}&L_{o(x)}\left({}_0^A D_x^{o(x,t)}\left\{f\left(x-a\right).\delta(x-a)\right\}\right)(s)\\&=\int_0^8\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{1-o(x,t)}\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{o(x,t)-1}e^{-st}\left(f\left(t-a\right).\delta(t-a)\right)dt\\&=\int_0^8 e^{-st}f\left(t-a\right).\delta\left(t-a\right)dt.\end{aligned}$$

Using the properties of Laplace transform operator [5, 6], we obtain the requested result

$$L_{o(x)}\left({}_0^A D_x^{o(x,t)}\left\{f\left(x-a\right).\delta(x-a)\right\}\right)(s)=se^{-sa}F\left(s\right).$$

proof of 3: By definition, we have the following

$$\begin{aligned}&L_{o(x)}\left({}_0^A D_x^{o(x,t)}\left(\frac{df(x)}{dx}\right)\right)(s)\\&=\int_0^8\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{1-o(x,t)}\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{o(x,t)-1}e^{-st}\left(\left(\frac{df(t)}{dt}\right)\right)dt\\&=\int_0^8 e^{-st}\left(\left(\frac{df(t)}{dt}\right)\right)dt.\end{aligned}$$

Using the property of Laplace transform for first derivative, we obtain the requested results [5, 6]

$$L_{o(x)}\left({}_0^A D_x^{o(x,t)}\left(\frac{df(x)}{dx}\right)\right)(s)=s^2F\left(s\right)-sf\left(0\right)-f\left(0\right).$$

The proof of 4 and 5 are similar to the one above.

proof of 6: By definition, we have

$$L_{o(x)}\left({}_0^A D_x^{o(x)}\left(\int_0^x f(t)dt\right)\right)(s)=L_{o(x)}\left(\left(t+\frac{1}{\Gamma\left(1-o(x,t)\right)}\right)^{1-o(x,t)}\left(\int_0^t f(v)dv\right)'\right),$$

Due to the fundamental theorem of calculus, the right hand side can be transformed to

$$L_{o(x)}\left(\left(t+\frac{1}{\Gamma(1-o(x,t))}\right)^{1-o(x,t)}\left(\int_0^t f(v)dv\right)'\right)=L_{o(x)}\left(\left(t+\frac{1}{\Gamma(1-o(x,t))}\right)^{1-o(x,t)}f(t)\right)$$
$$=\int_0^8 e^{-st}(f(t))dt=F(s),$$

This completes the proof of 6.

Proof 7:

$$L_{o(x)}\left({}_0^AD_x^{o(x)}(f*g(x))\right)(s)=L_{o(x)}\left(\left(t+\frac{1}{\Gamma(1-o(x,t))}\right)^{1-o(x,t)}(f*g(x))'\right)$$
$$=\int_0^8\frac{d}{dt}f*g(t)\,e^{-st}dt=sF(s)\,G(s).$$

This completes the proof of 7. Note that items 8, 9 and 10 are obvious. □

Therefore, applying the above operator on both sides of equation (2.3), we obtain the following equation

$$L_{o(x,t)}\left\{\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}\right\}(u)=\frac{S}{T}\left(u^2S(r,u)-uS(r,0)-S(r,0)\right) \tag{3.2}$$

The above equation can be rearranged as follow

$$S(r,u)=\frac{T}{Su^2}\left\{L_{o(x,t)}\left\{\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}\right\}(u)\right\}+\left\{\frac{1}{u}+\frac{1}{u^2}\right\}S(r,0) \tag{3.3}$$

We next employ the inverse Laplace transform operator on both sides of the above equation to obtain

$$S(r,t)=\mathcal{L}^{-1}\left\{\frac{T}{Su^2}\left\{L_{o(x,t)}\left\{\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}\right\}(u)\right\}\right\}+\mathcal{L}^{-1}\left\{\left\{\frac{1}{u}+\frac{1}{u^2}\right\}S(r,0)\right\} \tag{3.4}$$

For simplicity, we put

$$g(r,t)=\mathcal{L}^{-1}\left\{\left\{\frac{1}{u}+\frac{1}{u^2}\right\}S(r,0)\right\}.$$

Then, from equation (3.4) one can construct a recursive formula that will be used to generate the special solution of equation (2.3). The recursive formula associate to equation (3.4) is

$$S_{n+1}(r,t)=\mathcal{L}^{-1}\left\{\frac{T}{Su^2}\left\{L_{o(x,t)}\left\{\frac{\partial^2 S_n(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S_n(r,t)}{\partial r}-\frac{S_n(r,t)}{B^2}\right\}(u)\right\}\right\}\ for\ n\geq 1 \tag{3.5}$$

$$S_0(r,t)=g(r,t)$$

Our next step is to prove the stability of used iteration method.

4. Uniqueness of the solution

Let assume by contradiction that, there exist two different special solutions $S_{sp1}(r,t)$ and $S_{sp2}(r,t)$. Let

$$G(S)={}_0^AD_\alpha^t(S(r,t))=\frac{\partial^2 S(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S(r,t)}{\partial r}-\frac{S(r,t)}{B^2}$$

The aim of our proof is to show that using the inner product that.

$$\|S_{sp1} - S_{sp2}\| \ll \epsilon$$

To achieve this, we evaluate $(G(S_{sp1}) - G(S_{sp2}), \quad w)$ for $w \in H = \{u, v/ \int uv < \infty\}$
However,

$$\begin{aligned} G(S_{sp1}) - G(S_{sp2}) =& \frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\} \\ &+ \frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\} + \frac{1}{B^2}\{S_{exp1}(r,t) - S_{exp2}(r,t)\} \end{aligned} \tag{4.1}$$

Thus,

$$\begin{aligned} (G(S_{sp1}) - G(S_{sp2}), \quad w) =& \left(\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) \\ &+ \left(\frac{1}{B^2}\{S_{exp1}(r,t) - S_{exp2}(r,t)\}, w\right) \end{aligned} \tag{4.2}$$

We shall evaluate the first component

$$\left(\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right).$$

In the real world problem, the level is bounded, that S_{sp1}, S_{sp2} are bounded, therefore we can find a positive constant M such that, $(S_{sp1}, S_{sp1}) < M^2$. It follows by the use of Schwartz inequality that

$$\left(\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w,\right) \le \left\|\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}\right\| \|w\| \tag{4.3}$$

However, we can find a positive constant ω_1, ω_2 such that

$$\left\|\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}\right\| \le \omega_1\omega_2 \|S_{exp2}(r,t) - S_{exp1}(r,t)\|$$

$$\left(\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, \ w\right) \le \omega_1\omega_2 \|S_{exp2}(r,t) - S_{exp1}(r,t)\| \|w\|$$

We next evaluate,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right)$$

It follows by the use of Schwartz inequality that,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) \le \left\|\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}\right\| \|w\| \tag{4.4}$$

However, we can find a positive constant O_1 such that

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) \le \frac{O_1}{r_1} \|S_{exp2}(r,t) - S_{exp1}(r,t)\| \|w\|,$$

$$\begin{aligned} &\left(\frac{\partial^2}{\partial r^2}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) + \left(\frac{1}{r}\frac{\partial}{\partial r}\{S_{exp2}(r,t) - S_{exp1}(r,t)\}, w\right) \\ &\quad + \left(\frac{1}{B^2}\{S_{exp1}(r,t) - S_{exp2}(r,t)\}, w\right) \le \left(\frac{O_1}{r_1} + \omega_1\omega_2 + \frac{1}{B^2}\right) \|S_{exp2}(r,t) - S_{exp1}(r,t)\| \|w\| \end{aligned}$$

Subsequently $S(r,t)$ the exact solution converges to $S_{sp1}, \quad S_{sp2}$ then we can find two large number N, M such that

$$\|S - S_{sp1}\| \ll \frac{\epsilon}{2\left(\frac{O_1}{r_1} + \omega_1\omega_2 + \frac{1}{B^2}\right)\|w\|} \; for \; N \; and \, \|S - S_{sp2}\| \ll \frac{\epsilon}{2\left(\frac{O_1}{r_1} + \omega_1\omega_2 + \frac{1}{B^2}\right)\|w\|} \; for \; M$$

And then,

$$\|(S_{sp1} - S_{sp2})\| \leq \|S - S_{sp2}\| + \|S - S_{sp1}\|$$

Consider $m = \max(N, M)$, then

$$\|(S_{sp1} - S_{sp2})\| \ll \frac{\epsilon}{\left(\frac{O_1}{r_1} + \omega_1\omega_2 + \frac{1}{B^2}\right)\|w\|} \tag{4.5}$$

Replacing the above in (4.5), we arrive at

$$\left(G\left(S_{sp1}\right) - G\left(S_{sp2}\right), \quad w\right) \ll \epsilon \tag{4.6}$$

Now with ϵ extremely very small, we have that,

$$\|(S_{sp1} - S_{sp2})\| = 0 \Longrightarrow S_{sp1} = S_{sp2}.$$

This completes the proof. We shall next present the stability of the method

5. Stability analysis of the used method

The stability of method for solving an equation is very important component of analysis since it shows the strength of the method for solving that equation. To achieve this, we need to make use of the inner product and the operator G

$$\left(G\left(S\right) - G\left(S_1\right), \quad S - S_1\right)$$

for any $u, v \in H$ constructed in (4.1). In particular we aim to show that, we can find a positive number k such that

$$\left(G\left(S\right) - G\left(S_1\right), \; S - S_1\right) \leq L\|S - S_1\|^2$$

Proof. First we have that,

$$\begin{aligned}\left(G\left(S\right) - G\left(S_1\right), \; S - S_1\right) = &\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right) - S_1\left(r,t\right)\right\}, S - S_1\right) \\ &+ \left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right) - S_1\left(r,t\right)\right\}, S - S_1\right) \\ &+ \left(\frac{1}{B^2}\left\{S\left(r,t\right) - S_1\left(r,t\right)\right\}, S - S_1\right)\end{aligned}$$

We shall evaluate the first component

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right) - S_1\left(r,t\right)\right\}, S\left(r,t\right) - S_1\left(r,t\right)\right)$$

It follows by the use of Schwartz inequality that

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)\leq\left\|\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\}\right\|\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\| \tag{5.1}$$

However, we can find a positive constant $f_1,\ f_2$ such that

$$\left\|\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\}\right\|\leq f_1f_2\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|$$

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},\ S\left(r,t\right)-S_1\left(r,t\right)\right)\leq f_1f_2\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|$$

We next evaluate,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)$$

It follows by the use of Schwartz inequality that,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)\leq\left\|\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\}\right\|\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\| \tag{5.2}$$

However, we can find a positive constant g_1 such that

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)\leq\frac{g_1}{r_1}\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|,$$

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)&+\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)\\&+\left(\frac{1}{B^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},S\left(r,t\right)-S_1\left(r,t\right)\right)\\&\leq\left(\frac{g_1}{r_1}+f_1f_2+\frac{1}{B^2}\right)\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\end{aligned}$$

Thus,

$$\left(G\left(S\right)-G\left(S_1\right),\ S-S_1\right)\leq\left(\frac{g_1}{r_1}+f_1f_2+\frac{1}{B^2}\right)\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|^2$$

Let $\left(\frac{g_1}{r_1}+f_1f_2+\frac{1}{B^2}\right)=L$ and thus

$$\left(G\left(S\right)-G\left(S_1\right),\ S-S_1\right)\leq L\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|^2$$

The next step is to prove that $\left(G\left(S\right)-G\left(S_1\right),\ S-S_1\right)\leq H\left\|S-S_1\right\|\left\|W\right\|$
Thus,

$$\begin{aligned}\left(G\left(S\right)-G\left(S_1\right),\quad W\right)=\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)+\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)\\+\left(\frac{1}{B^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)\end{aligned} \tag{5.3}$$

We shall evaluate the first component

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)$$

It follows by the use of Schwartz inequality that,

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)\leq\left\|\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\}\right\|\left\|W\right\|$$

However, we can find a positive constant m_1, m_2 such that

$$\left\|\frac{\partial^2}{\partial r^2}\left\{S_{exp2}\left(r,t\right)-S_{exp1}\left(r,t\right)\right\}\right\|\leq m_1m_2\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|$$

$$\left(\frac{\partial^2}{\partial r^2}\left\{S_{exp2}\left(r,t\right)-S_{exp1}\left(r,t\right)\right\},\ W\right)\leq m_1m_2\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|W\right\|$$

We next evaluate,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)$$

It follows by the use of Schwartz inequality that,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)\leq\left\|\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\}\right\|\left\|W\right\| \tag{5.4}$$

However, we can find a positive constant N_1 such that

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)\leq\frac{N_1}{r_1}\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|W\right\|$$

However putting all these equation together, we obtain the following

$$\left(\frac{\partial^2}{\partial r^2}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)+\left(\frac{1}{r}\frac{\partial}{\partial r}\left\{S\left(r,t\right)-S_1\left(r,t\right)\right\},W\right)+\left(\frac{1}{B^2}\left\{S_1\left(r,t\right)-S\left(r,t\right)\right\},W\right)$$
$$\leq\left(\frac{N_1}{r_1}+m_1m_2+\frac{1}{B^2}\right)\left\|S\left(r,t\right)-S_1\left(r,t\right)\right\|\left\|W\right\|$$

Taking $\left(\frac{N_1}{r_1}+m_1m_2+\frac{1}{B^2}\right)=H$
Then

$$\left(G\left(S\right)-G\left(S_1\right),\quad S-S_1\right)\leq H\left\|S-S_1\right\|\left\|W\right\| \tag{5.5}$$

This completes the proof. □

6. Numerical simulations

We devote this part to the numerical simulation, to achieve this we first propose an algorithm that will be used for numerical simulations.

Algorithm 1

Input

$$S_0\left(r,t\right)=g(r,t)$$

as preliminary input,

- $i-$number terms in the rough calculation
- Output $S_{Ap}(r,t)$, the approximate solution

Step 1: Put $S_0\,(r,t)=g(r,t)$ and $S_{Ap}(r,t)=\ S_{Ap}(r,t)$
Step 2: for $i=1$ to $n-1$ do step 3, step 4 and step 5

$$S_{n+1}\,(r,t)=\mathcal{L}^{-1}\left\{\frac{T}{Su^2}\left\{L_{o(x,t)}\ \left\{\frac{\partial^2 S_n\,(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial S_n\,(r,t)}{\partial r}-\frac{S_n\,(r,t)}{B^2}\right\}(u)\right\}\right\}$$

Step 3: compute

$$\beta_{1(n+1)}\,(r,t)=\beta_{1(n)}\,(r,t)+\ S_{Ap}(r,t)$$

Step 4: Compute:

$$S_{Ap}(r,t)\ \ =\ \ S_{Ap}(r,t)\ \ \ +\beta_{1(n+1)}\,(r,t)$$

Stop.
The overhead method shall be employed to yield the numerical replication of the physical problem under investigation as indicate in the following figure 1 and 2. The considered equation is subjected to the following conditions. The equation (2.1) is subjected to the following initial and boundary conditions:

$$\Phi\,(r,0)=\Phi_0,\qquad \lim_{r\to 8}\Phi(r,t)\quad=\quad\Phi_0\qquad\quad Q=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}r_b^{n-1}Kd^{3-n}\partial_r\Phi\,(r_b,t)\,.$$

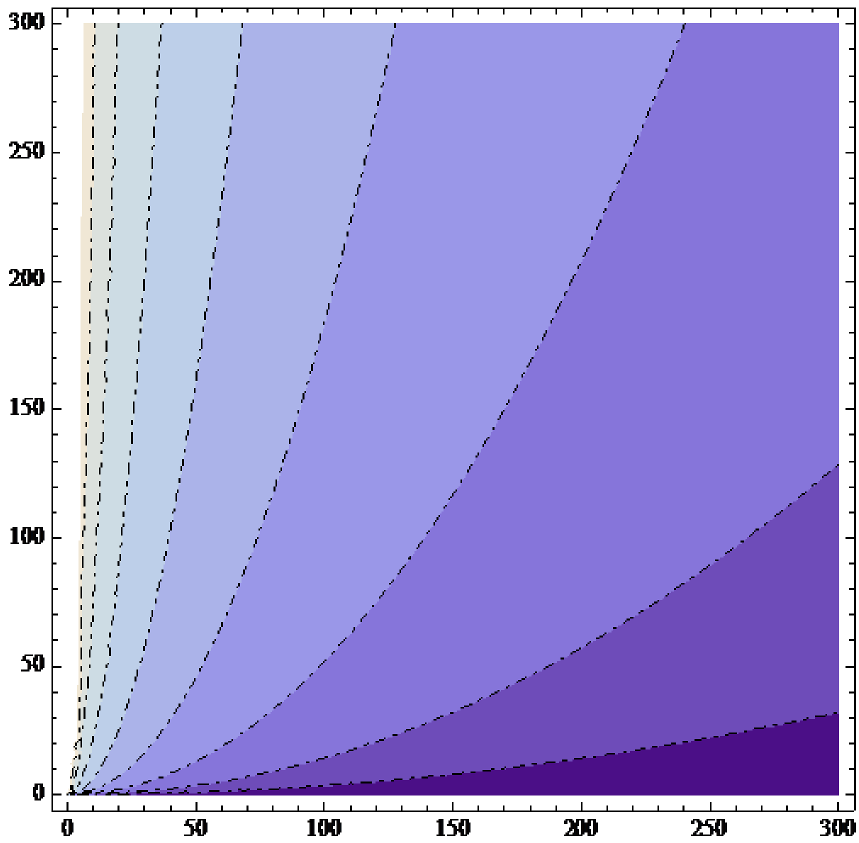

Figure 1: Contour plot of the proposed solution as function of space and time. This shows the wave in change of level of water within the confined aquifer during the pumping test on one side of the well for $\mathbf{o}\,(\mathbf{r},\mathbf{t})=\mathbf{0.8}\sin(\mathbf{r},\mathbf{t})$

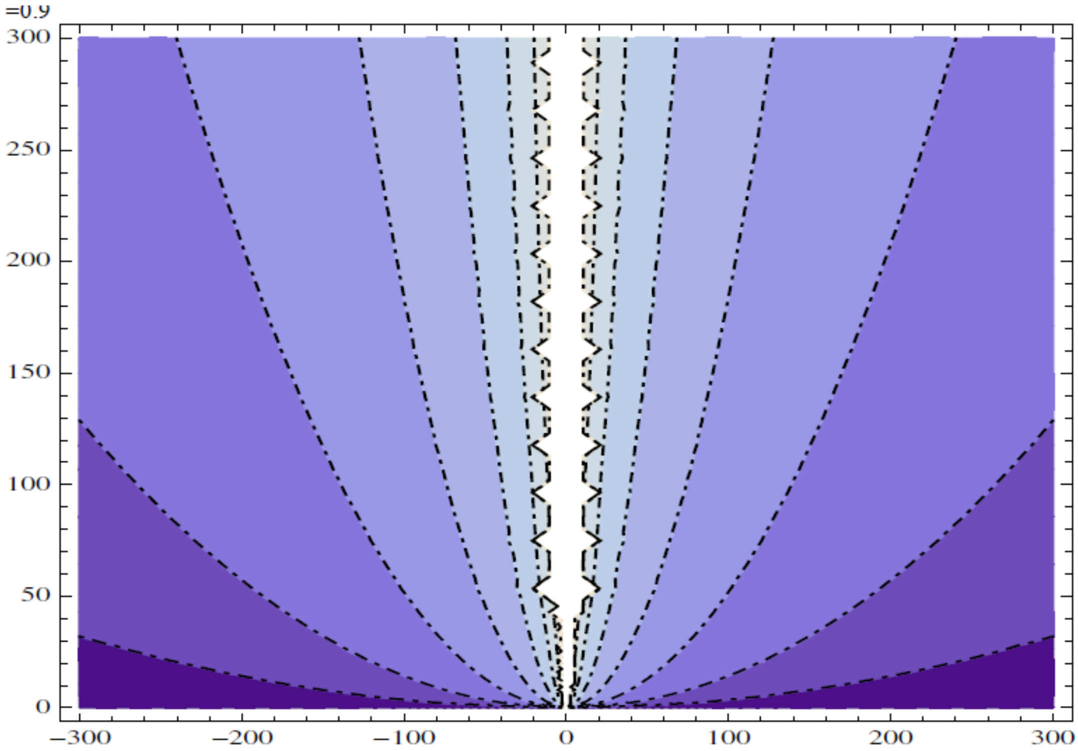

Figure 2: Contour plot of the proposed solution as function of space and time. This shows the wave in change of level of water within the confined aquifer during the pumping test around the well for $\mathbf{o}(\mathbf{r},\mathbf{t}) = \mathbf{0.8}\sin(\mathbf{r},\mathbf{t})$

7. Conclusion

A new derivative that takes into account the complexity of the physical phenomena was used to enhance the model describing the movement of groundwater flowing within a leaky aquifer. We made use of a new operator called $o(x,t)$-Laplace transform together with the concept of iterative method to solve the new groundwater equation. We have showed in detail stability analysis of the used method together with the uniqueness of the special solution. We presented the numerical simulations.

Conflict of interest:

The authors declare there is no conflict of interest for this paper

Acknowledgements:

The authors would like to thank the editor and the anonymous reviewers for their valuable suggestions tower the enhancement of this paper.

References

[1] A. Atangana, *Drawdown in prolate spheroidal-spherical coordinates obtained via Green's function and perturbation methods*, Commun. Nonlinear Sci. Num. Simul., **19** (2014), 1259–1269. 1

[2] A. Atangana, *Analytical solutions for the recovery tests after constant-discharge tests in confined aquifers*, Water SA, **40** (2014), 595–600. 1

[3] A. Atangana, B. Necdet, *The use of fractional order derivative to predict the groundwater flow*, Math. Probl. Eng., **2013** (2013), 9 pages. 1, 2

[4] J. B. Brackenridge, *The key to Newton's dynamics. The Kepler problem and the Principia*, University of California Press, Berkeley, (1995). 1

[5] M. A. B. Deakin, *The development of the Laplace transform*, Arch. Hist. Exact Sci., **26** (1982), 351–381. 2.1, 3

[6] M. A. B. Deakin, *The development of the Laplace transform*, Arch. Hist. Exact Sci., **25** (1981), 343–390. 3

[7] E. F. Doungmo Goufo, *A mathematical analysis of fractional fragmentation dynamics with growth*, J. Funct. Spaces, **2014** (2014), 7 pages. 1

[8] E. F. Doungmo Goufo, *A biomathematical view on the fractional dynamics of cellulose degradation*, Fract. Cal. Appl. Anal., (in press). 1

[9] E. F. Doungmo Goufo, R. Maritz, J. Munganga, *Some properties of Kermack-McKendrick epidemic model with fractional derivative and nonlinear incidence*, Adv. Difference Equ., **2014** (2014), 9 pages. 1

[10] V. J. Katz, *A History of Mathematics: An Introduction*, Harper Collins, New York, (1993). 1

[11] G. W. F. V. Leibniz, C. I. Gerhardt, *(trans.) The Early Mathematical Manuscripts of Leibniz*, Open Court Publishing, Chicago, (1920). 1

[12] I. Newton, *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World*, tr. A. Motte, rev. Florian Cajori. University of California Press, Berkeley, (1934). 1

[13] H. J. Ramey, *Well loss function and the skin effect: A review. In: Narasimhan TN (ed.) Recent Trends in Hydrogeology*, Geo. Soc. Amer. Special paper, **189** (1982), 265–272. 1

[14] C. V. Theis, *The relation between the lowering of the Piezometric surface and the rate and duration of discharge of well using ground-water storage*, Trans, Amer. Geophys. Union, **16** (1935), 519–524. 1