Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 153–162
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Coupled fixed point theorems with respect to binary relations in metric spaces

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Communicated by P. Kumam

Abstract

In this paper we present a new extension of coupled fixed point theorems in metric spaces endowed with a reflexive binary relation that is not necessarily neither transitive nor antisymmetric. The key feature in this coupled fixed point theorems is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the binary relation. Next on the basis of the coupled fixed point theorems, we prove the existence and uniqueness of positive definite solutions of a nonlinear matrix equation.

Keywords: Coupled fixed point, reflexive relation, matrix equations, positive define solution.
2010 MSC: 47H10, 15A24, 54H25.

1. Introduction and Preliminaries

Existence and uniqueness of fixed point in partially ordered sets has been considered in [15], where some applications to matrix equations are presented. In [6], Bhaskar and Lakshmikantham defined the concept of coupled fixed point and discussed the existence and uniqueness of solution for a periodic boundary value problem. During the last few decades, many authors discussed on coupled fixed point results in various spaces and considered this concept to study nonlinear differential equations, nonlinear integral equations and matrix equations (see, [1, 8, 12, 13, 16, 17, 18, 19, 20]).

In this paper, we discuss some results on the existence and uniqueness of coupled fixed points in metric spaces endowed with a reflexive relation and some applications to nonlinear matrix equations. Throughout the paper X will be a topological space and R is a a reflexive relation on X. We start our consideration

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Received 2014-9-22

by giving a brief review of the definitions and basic properties of coupled fixed point in metric spaces. For more informations we refer to [5, 6].

Notation 1.1. *Let X be a nonempty set and let $f : X \times X \to X$ be a mapping. Then*

(i) *We will denote $f^0(x, y) = x$ and $f^n(x, y) = f\big(f^{n-1}(x, y), f^{n-1}(y, x)\big)$ for all $x, y \in X, n \in \mathbb{N}$.*

(ii) *The cartezian product of f and g is denoted by $f \times g$, and defined by*

$$f \times g(x, y) = \big(f(x, y), g(y, x)\big).$$

Definition 1.2. Let X be a nonempty set and let $f : X \times X \to X$ be a mapping. Then an element $(x, y) \in X \times X$ is called a coupled fixed point of f, if $f(x, y) = x$ and $f(y, x) = y$ and an element $x \in X$ is called a fixed point of f, if $f(x, x) = x$. We will denote the set of all the coupled fixed points of f by F^c_f and the set of all the fixed points of f by F_f.

2. Main results

In this section we will prove the coupled fixed point theorems with respect to a reflexive relation.

Definition 2.1. Let X be a topological space and let $f, g : X \times X \to X$ be two map. Then

(i) An element $(x, y) \in X \times X$ is called a coupled attractor basin element of f with respect to $(x^*, y^*) \in X \times X$, if $f^n(x, y) \to x^*$ and $f^n(y, x) \to y^*$, as $n \to \infty$ and an element $x \in X$ is called an attractor basin element of f with respect to $x^* \in X$, if $f^n(x, x) \to x^*$, as $n \to \infty$. We will denote the set of all the coupled attractor basin of f with respect to (x^*, y^*) by $A^c_f(x^*, y^*)$ and the set of all the attractor basin of f with respect to $x^* \in X$ by $A_f(x^*)$.

(ii) The mapping f is called orbitally continuous if $(x, y), (a, b) \in X \times X$ and $f^{n_k}(x, y) \to a$, $f^{n_k}(y, x) \to b$, as $k \to \infty$ imply $f^{n_k+1}(x, y) \to f(a, b)$ and $f^{n_k+1}(y, x) \to f(b, a)$ as $k \to \infty$.

(iii) The mapping f is called a Picard operator, if there exists $x^* \in X$ such that:

(1) $F_f = \{x^*\}$.

(2) $A_f(x^*) = X$.

Also f is called a weakly Picard operator, if the sequences $\{f^n(x, x)\}_{n\in\mathbb{N}}$ convergent for all $x \in X$ and the limits (which may depend on x) are a fixed point of f.

Definition 2.2. Let X be nonempty set and let R be a reflexive relation on X, for every $(z, t) \in X \times X$ we define

$$X_R(z, t) = \big\{(x, y) \in X \times X : \ xRz \ \wedge \ tRy\big\}.$$

Note that $(x, y) \in X_R(z, t)$ if and only if $(t, z) \in X_R(y, x)$ and if $(x, y) \in R$, then $(x, y) \in X_R(y, x)$.

Definition 2.3. Let X be nonempty set and let R be a reflexive relation on X, $f : X \times X \to X$.

(i) We say that f has the mixed R-monotone property on X, if $f \times f(X_R(x, y)) \subseteq X_R(f \times f(x, y))$ for all $(x, y) \in X \times X$.

(ii) An element $(x, y) \in X \times X$ is called a R-coupled fixed point of f, if $f \times f(x, y) \in X_R(x, y)$.

(iii) A sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq X \times X$ is called a R-monotone sequence, if $(x_n, y_n) \in X_R(x_{n-1}, y_{n-1})$ for all $n \in \mathbb{N}$.

We begin with the following theorem that establishes the existence of a coupled fixed point for a orbitally continuous function $f : X \times X \to X$ with respect to a reflexive relation R on topological space X.

Theorem 2.4. *Let X be a topological space and R be a reflexive relation on X. Assume that $f : X \times X \to X$ is a mapping having the following property:*

(i) *For each $(x, y), (z, t) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(x, y), (z, t) \in X_R(u, v)$.*

(ii) *There exists (x_0, y_0), $(x^*, y^*) \in X \times X$ such that $(x_0, y_0) \in A_f(x^*, y^*)$.*

(iii) *For each $(x, y), (z, t) \in X \times X$ if $(x, y) \in X_R(z, t)$ and $(z, t) \in A_f(x^*, y^*)$ then $(x, y) \in A_f(x^*, y^*)$.*

Then $A_f(x^*, y^*) = X \times X$. Moreover, if f is orbitally continuous then, it is also a Picard operator and $F_f = \{x^*\}$.

Proof. Let $(x, y) \in X \times X$ be arbitrary, then from (i) there exists $(z, t) \in X \times X$ such that $(x, y), (x_0, y_0) \in X_R(z, t)$. From $(x_0, y_0) \in X_R(z, t)$ we have $(t, z) \in X_R(y_0, x_0)$ and from (ii) and (iii) we get that $(z, t) \in A_f(x^*, y^*)$, also from $(x, y) \in X_R(z, t), (z, t) \in A_f(x^*, y^*)$ and (iii) we obtain $(x, y) \in A_f(x^*, y^*)$, thus $A_f(x^*, y^*) = X \times X$. Now, let f be an orbitally continuous mapping, then (ii) follows that $f(x^*, y^*) = x^*, f(y^*, x^*) = y^*$. Also, from $(y^*, x^*) \in A_f(x^*, y^*)$ we get that $x^* = y^*$. Therefore $A_f(x^*) = X$, which this shows that the operator f is Picard. □

Remark 2.5. Note that the assumption (iii) in Theorem 2.4 is essential. To see this, let $X = \mathbb{N}$ with discrete topology τ. Suppose that R is the division relation on X and $f : X \times X \to X$ be defined by $f(x, y) = x$. Then for every $(x, y), (z, t) \in X \times X$, $(x, y), (z, t) \in X_R([x, z], (y, t))$, where $[., .]$ and $(., .)$ are the least common multiple and the greatest common divisor on X. Also, $A_f(x, y) = \{(x, y)\}$ for all $x, y \in X$ and there exists $(x, y) \in X_R(z, t)$ and $(z, t) \in A_f(a, b)$ such that $(x, y) \notin A_f(a, b)$. Moreover, f is continuous and $F_f = \mathbb{N}$, thus f is not a Picard operator.

In the following theorem we prove Theorem 2.1 in [6] for a orbitally continuous mapping with respect to a reflexive relation on the metric space X.

Theorem 2.6. *Let (X, d) be a metric space and R be a reflexive relation on X. If $f : X \times X \to X$ is a mapping such that:*

(i) *f having the mixed R-monotone property on X.*

(ii) *(X, d) be a complete metric space.*

(iii) *f having a R-coupled fixed point, i.e., there exists $(x_0, y_0) \in X \times X$ such that $f \times f(x_0, y_0) \in X_R(x_0, y_0)$.*

(iv) *There exists $k \in [0, 1)$ such that:*

$$d(f(x, y), f(z, t)) \leq \frac{k}{2}[d(x, z) + d(y, t)], \quad \forall (x, y) \in X_R(z, t).$$

(v) *f is an orbitally continuous mapping.*

Then:

(a) *There exist $x^*, y^* \in X$ such that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.*

(b) *The sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = f(y_n, x_n)$ converge respectively to x^* and y^*.*

(c) *The error estimation is given by:*

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]$$

Proof. Since $f \times f(x_0, y_0) \in X_R(x_0, y_0)$, so from (i) it follows that

$$(f^2(x_0, y_0), f^2(y_0, x_0)) \in X_R(f(x_0, y_0), f(y_0, x_0)).$$

Further, we can easily verify that for any $n \in \mathbb{N}$

$$(f^n(x_0, y_0), f^n(y_0, x_0)) \in X_R(f^{n-1}(x_0, y_0), f^{n-1}(y_0, x_0)). \tag{2.1}$$

Now, we claim that, for $n \in \mathbb{N}$

$$\begin{aligned} d(f^{n+1}(x_0, y_0), f^n(x_0, y_0)) &\leq \frac{k^n}{2}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)] \\ d(f^{n+1}(y_0, x_0), f^n(y_0, x_0)) &\leq \frac{k^n}{2}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]. \end{aligned} \tag{2.2}$$

Indeed, for $n = 1$, using (iii) and (iv), we get

$$\begin{aligned} d(f^2(x_0, y_0), f(x_0, y_0)) &= d(f(f(x_0, y_0), f(y_0, x_0)), f(x_0, y_0)) \\ &\leq \frac{k}{2}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]. \end{aligned}$$

Similarly.

$$\begin{aligned} d(f^2(y_0, x_0), f(y_0, x_0)) &= d\big(f(y_0, x_0), f^2(y_0, x_0)\big) \\ &= d(f(y_0, x_0), f(f(y_0, x_0), f(x_0, y_0))) \\ &\leq \frac{k}{2}[d(f(y_0, x_0), y_0) + d(f(x_0, y_0), x_0)]. \end{aligned}$$

Now, assume that (2.2) holds. Using (iv) we get

$$\begin{aligned} &d(f^{n+2}(x_0, y_0), f^{n+1}(x_0, y_0)) \\ &\quad = d(f(f^{n+1}(x_0, y_0), f^{n+1}(y_0, x_0)), f(f^n(x_0, y_0), f^n(y_0, x_0))) \\ &\quad \leq \frac{k}{2}[d(f^{n+1}(x_0, y_0), f^n(x_0, y_0)) + d(f^{n+1}(y_0, x_0), f^n(y_0, x_0))] \\ &\quad \leq \frac{k^{n+1}}{2}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]. \end{aligned}$$

Similarly, we can show that

$$d(f^{n+2}(y_0, x_0), f^{n+1}(x_0, y_0)) \leq \frac{k^{n+1}}{2}[d(f(y_0, x_0), y_0) + d(f(x_0, y_0), x_0)].$$

This implies that $\{f^n(x_0, y_0)\}_{n\in\mathbb{N}}$ and $\{f^n(y_0, x_0)\}_{n\in\mathbb{N}}$ are Cauchy sequences in X. Because, if $m > n$, then

$$\begin{aligned} d(f^m(x_0, y_0), f^n(x_0, y_0)) &\leq \sum_{j=n}^{m-1} d(f^{j+1}(x_0, y_0), f^j(x_0, y_0)) \\ &\leq \frac{\sum_{j=n}^{m-1} k^j}{2}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)] \\ &= \frac{k^n - k^m}{2(1-k)}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)] \\ &< \frac{k^n}{2(1-k)}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]. \end{aligned}$$

Similarly, we can verify that $\{f^n(y_0, x_0)\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Since X is complete, there exist $x^*, y^* \in X$ such that $f^n(x_0, y_0) \to x^*$ and $f^n(y_0, x_0) \to y^*$, as $n \to \infty$. Now the conclusion of theorem follows from the orbitally continuous of f. □

Example 2.7. Let $X = \mathbb{R}$ with $d(x,y) = |x-y|$ and consider the relation R on X by

$$xRy \Leftrightarrow x^2 + x = y^2 + y.$$

Let $f : X \times X \to X$ be defined by $f(x,y) = x^2 + x - 1$. Then for any $(x,y) \in X \times X$,

$$X_R(x,y) = \{(x,y), (x,-y-1), (-x-1,y), (-x-1,-y-1)\}$$
$$f \times f\big(X_R(x,y)\big) = \{f \times f(x,y)\} \subseteq X_R\big(f \times f(x,y)\big).$$

Thus f having the mixed R-monotone property on X. Moreover, f is continuous and there exists point $(1,-2) \in X \times X$ such that $f \times f(1,-2) \in X_R(1,-2)$. So, the hypothesis of Theorem 2.6 is satisfies. Therefore, we conclude that f has a coupled fixed point in $X \times X$. This coupled fixed points are $(x,y) = (1,1), (1,-1), (-1,1), (-1,-1)$.

Remark 2.8. Note that the assumption (iv), i.e., having a R-coupled fixed point for f in Theorem 2.6 is essential. To see this, let $X = [1,\infty)$ with $d(x,y) = |x-y|$ and consider the relation R on X by $xRy \Leftrightarrow x + \frac{1}{x} = y + \frac{1}{y}$. Let $f : X \times X \to X$ be defined by $f(x,y) = x + \frac{1}{x}$. Then f has no coupled fixed point and for any $(x,y) \in X \times X$,

$$X_R(x,y) = \{(x,y), (x,\frac{1}{y}), (\frac{1}{x},y), (\frac{1}{x},\frac{1}{y})\}$$
$$f \times f\big(X_R(x,y)\big) = \{f \times f(x,y)\} \subseteq X_R\big(f \times f(x,y)\big).$$

This shows that f having the mixed R-monotone property on X. Moreover, f is continuous and $f \times f(x,y) \notin X_R(x,y)$ for all $(x,y) \in X \times X$. Also, the hypothesis of Theorem 2.6 is satisfies.

The following theorem is an extension of Theorem 2.4 in [6] for a orbitally continuous mapping with respect to a reflexive relation on X.

Theorem 2.9. *In addition to the hypothesis of Theorem 2.6, suppose that for every $(x,y), (z,t) \in X \times X$ there exists $(u,v) \in X \times X$ such that $(x,y), (z,t) \in X_R(u,v)$. Then f is a Picard operator.*

Proof. According to the proof of Theorem 2.6, there exist $x^*, y^* \in X$ such that $f(x^*,y^*) = x^*$ and $f(y^*,x^*) = y^*$. Now, we show that $A_f(x^*,y^*) = X \times X$. Let $(x,y) \in X \times X$ be arbitrary, then (i) implies that there exists $(u,v) \in X \times X$ such that $(x,y), (x_0,y_0) \in X_R(u,v)$. From $(x_0,y_0) \in X_R(u,v)$ and (ii) it follows that for $n \in \mathbb{N}$

$$(f^n(x_0,y_0), f^n(y_0,x_0)) \in X_R(f^n(u,v), f^n(v,u)).$$

Also by using (v) we have

$$d(f^n(x_0,y_0), f^n(u,v)) \leq \frac{k^n}{2}[d(x_0,u) + d(y_0,v)],$$
$$d(f^n(y_0,x_0), f^n(v,u)) \leq \frac{k^n}{2}[d(x_0,u) + d(y_0,v)].$$

From this and the fact that $(x_0,y_0) \in A_f(x^*,y^*)$, it follows that $(u,v) \in A_f(x^*,y^*)$. Also, from $(x,y) \in X_R(u,v)$ we get that $(x,y) \in A_f(x^*,y^*)$, which this implies that $A_f(x^*,y^*) = X \times X$. Now as the proof of Theorem 2.4 we obtain that f is a Picard operator. □

Theorem 2.6 is still valid for a mapping without the orbitally continuous property, assuming an additional hypothesis on X. The following theorem is an extension of Theorem 2.2 in [6] with respect to a reflexive relation on X.

Theorem 2.10. *Let (X,d) be a metric space and R be a reflexive relation on X. Assume that $f : X \times X \to X$ is a mapping having the following property:*

(*i*) *f having the mixed R-monotone property on X.*

(*ii*) *(X, d) be a complete metric space.*

(*iii*) *f having a R-coupled fixed point, i.e., there exists $(x_0, y_0) \in X \times X$ such that $f \times f(x_0, y_0) \in X_R(x_0, y_0)$.*

(*iv*) *There exists $k \in [0, 1)$ such that:*

$$d(f(x, y), f(z, t)) \leq \frac{k}{2}[d(x, z) + d(y, t)], \quad \forall (x, y) \in X_R(z, t).$$

(*v*) *If a R-monotone sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \to (x, y)$, then $(x_n, y_n) \in X_R(x, y)$ for all $n \in \mathbb{N}$.*

Then:

(*a*) *There exist $x^*, y^* \in X$ such that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.*

(*b*) *The sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ defined by $x_{n+1} = f(x_n, y_n)$ and $y_{n+1} = f(y_n, x_n)$ converge respectively to x^* and y^*.*

(*c*) *The error estimation is given by:*

$$\max_{n \in \mathbb{N}}\{d(x_n, x^*), d(y_n, y^*)\} \leq \frac{k^n}{2(1-k)}[d(f(x_0, y_0), x_0) + d(f(y_0, x_0), y_0)]$$

Proof. Following the proof of Theorem 2.6, we only have to show that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$. Since $f^n(x_0, y_0) \to x^*$ and $f^n(y_0, x_0) \to y^*$, using (v), we get

$$\begin{aligned} d(f(x^*, y^*), x^*) &\leq d(f(x^*, y^*), f^{n+1}(x_0, y_0)) + d(f^{n+1}(x_0, y_0), x^*) \\ &= d(f(x^*, y^*), f(f^n(x_0, y_0), f^n(y_0, x_0))) + d(f^{n+1}(x_0, y_0), x^*) \\ &\leq \frac{k}{2}[d(x^*, f^n(x_0, y_0)) + d(y^*, f^n(y_0, x_0))] + d(f^{n+1}(x_0, y_0), x^*) \to 0, \quad (n \to \infty). \end{aligned}$$

This implies that $f(x^*, y^*) = x^*$. Similar to the previous case, we can prove $f(y^*, x^*) = y^*$. □

Alternatively, if we know that in Theorem 2.6 (resp. Theorem 2.10), the element $(x_0, y_0) \in X \times X$ is such that $(x_0, y_0) \in R$, then we can also demonstrate that the components x^* and y^* of the coupled fixed point are indeed the same.

Theorem 2.11. *In addition to the hypothesis of Theorem 2.6 (resp. Theorem 2.10), suppose that $(x_0, y_0) \in X \times X$ is such taht $(x_0, y_0) \in R$. Then $x^* = y^*$.*

Proof. If $(x_0, y_0) \in R$, then $(x_0, y_0) \in X_R(y_0, x_0)$, so from the mixed R-monotone property of f, it follows that $(f(x_0, y_0), f(y_0, x_0)) \in X_R(f(y_0, x_0), f(x_0, y_0))$. Further, we can easily verify that for any $n \in \mathbb{N}$,

$$(f^{n-1}(x_0, y_0), f^{n-1}(y_0, x_0)) \in X_R(f^{n-1}(y_0, x_0), f^{n-1}(x_0, y_0)).$$

Also by using the contractivity property of f, we obtain

$$\begin{aligned} d(f^n(x_0, y_0), f^n(y_0, x_0)) &= d(f(f^{n-1}(x_0, y_0), f^{n-1}(y_0, x_0)), f(f^{n-1}(y_0, x_0), f^{n-1}(x_0, y_0))) \\ &\leq kd(f^{n-1}(x_0, y_0), f^{n-1}(y_0, x_0)) \leq \cdots \leq k^n d(x_0, y_0) \to 0, \quad (n \to \infty). \end{aligned}$$

This implies that

$$x^* = \lim_{n \to \infty} f^n(x_0, y_0) = \lim_{n \to \infty} f^n(y_0, x_0) = y^*.$$

□

3. An application

In this section, on the basis of the coupled fixed point theorems in section 2, we study the nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(X) B_j, \tag{3.1}$$

where Q is a positive definite matrix, A_i, B_j are arbitrary $n \times n$ matrices and $\mathcal{G}, \mathcal{K}$ are two continuous order-preserving maps from $\mathcal{H}(n)$ into $\mathcal{P}(n)$ such that $\mathcal{G}(0) = \mathcal{K}(0) = 0$. In this section we will use the following notations: $\mathcal{M}(n)$ denotes the set of all $n \times n$ complex matrices, $\mathcal{H}(n) \subset \mathcal{M}(n)$ the set of all $n \times n$ Hermitian matrices and $\mathcal{P}(n) \subset \mathcal{H}(n)$ is the set of all $n \times n$ positive definite matrices. Instead of $X \in \mathcal{P}(n)$ we will also write $X > 0$. Furthermore, $X \geq 0$ means that X is positive semidefinite. Moreover, in $\mathcal{H}(n)$, if we define $X \geq Y$ $(X > Y)$ as $X \geq Y$ $(X > Y)$. Then $\mathcal{H}(n)$ is a partially ordered set and for every $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound. Therefore, for any $(X, Y), (A, B) \in \mathcal{H}(n) \times \mathcal{H}(n)$ there exists $(U, V) \in \mathcal{H}(n) \times \mathcal{H}(n)$ such that $(X, Y), (A, B) \in \mathcal{H}(n)_{\leq}(U, V)$. We also denote by $\|.\|$ the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$ where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. We will use the metric induced by the trace norm $\|.\|_1$ defined by $\|A\|_1 = \sum_{j=1}^{n} s_j(A)$, where $s_j(A)$, $j = 1, \cdots, n$, are the singular values of A. The set $\mathcal{H}(n)$ endowed with this norm is a complete metric space. In [4, 5, 9, 15], the authors considered matrix equations and established the existence and uniqueness of positive definite solutions. Matrix equations of type Eq.(3.1) often arise from many areas, such as ladder networks [2, 3], dynamic programming [10, 14], control theory [7, 11].

The following lemmas will be useful in the study of the matrix equations, which is taken from [15].

Lemma 3.1. *Let $A \geq 0$ and $B \geq 0$ be $n \times n$ matrices, then $0 \leq tr(AB) \leq \|A\| tr(B)$.*

Lemma 3.2. *Let $A \in \mathcal{H}(n)$ satisfy $A < I$, then $\|A\| < 1$.*

In total of this section if, we define the mapping $\mathcal{F} : \mathcal{H}(n) \times \mathcal{H}(n) \to \mathcal{H}(n)$ by

$$\mathcal{F}(X, Y) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(Y) B_j, \qquad \forall X, Y \in \mathcal{P}(n), \tag{3.2}$$

where $Q \in \mathcal{P}(n)$, $A_i, B_j \in \mathcal{M}(n)$ and $\mathcal{G}, \mathcal{K}$ are two continuous order-preserving maps. Then $\mathcal{F}$ is well defined and having the mixed $\leq$-monotone property on $\mathcal{H}(n)$ and the fixed points of $\mathcal{F}$ are the solutions of Eq.(3.1). In the following theorem we first discuss existence of a coupled fixed point of $\mathcal{F}$ in $\mathcal{H}(n) \times \mathcal{H}(n)$.

Theorem 3.3. *Let $Q \in \mathcal{P}(n)$. Assume there is a positive number M such that:*

(i) For every $(X, Y) \in \mathcal{H}(n)_{\leq}(U, V)$

$$\big|tr(\mathcal{G}(U) - \mathcal{G}(X))\big| \leq \frac{1}{M}\big|tr(U - X)\big|,$$
$$\big|tr(\mathcal{K}(Y) - \mathcal{K}(V))\big| \leq \frac{1}{M}\big|tr(Y - V)\big|.$$

(ii) $\sum_{i=1}^{m} A_i A_i^* < \frac{M}{2} I_n$ and $\sum_{j=1}^{k} B_j B_j^* < \frac{M}{2} I_n$.

(iii) $\sum_{i=1}^{m} A_i^* \mathcal{G}(2Q) A_i < Q$ and $\sum_{j=1}^{k} B_j^* \mathcal{K}(2Q) B_j < Q$.

Then, there exist $X^*, Y^* \in \mathcal{H}(n)$ such that $\mathcal{F}(X^*, Y^*) = X^*$ and $\mathcal{F}(Y^*, X^*) = Y^*$.

Proof. Let $(X,Y) \in \mathcal{H}(n)_{\leq}(U,V)$. Then $\mathcal{G}(X) \leq \mathcal{G}(U)$ and $\mathcal{K}(Y) \geq \mathcal{K}(V)$. Therefore

$$\begin{aligned}
\|\mathcal{F}(U,V) - \mathcal{F}(X,Y)\|_1 &= tr\big(\mathcal{F}(U,V) - \mathcal{F}(X,Y)\big) \\
&= \sum_{i=1}^{m} tr\big(A_i^*(\mathcal{G}(U) - \mathcal{G}(X))A_i\big) + \sum_{j=1}^{k} tr\big(B_j^*(\mathcal{K}(Y) - \mathcal{K}(V))B_j\big) \\
&= \sum_{i=1}^{m} tr\big(A_iA_i^*(\mathcal{G}(U) - \mathcal{G}(X))\big) + \sum_{j=1}^{k} tr\big(B_jB_j^*(\mathcal{K}(Y) - \mathcal{K}(V))\big) \\
&= tr\Big(\big(\sum_{i=1}^{m} A_iA_i^*\big)\big(\mathcal{G}(U) - \mathcal{G}(X)\big)\Big) + tr\Big(\big(\sum_{j=1}^{k} B_jB_j^*\big)\big(\mathcal{K}(Y) - \mathcal{K}(V)\big)\Big) \\
&\leq \Big\|\sum_{i=1}^{m} A_iA_i^*\Big\| \|\mathcal{G}(U) - \mathcal{G}(X)\|_1 + \Big\|\sum_{i=1}^{k} B_jB_j^*\Big\| \|\mathcal{K}(Y) - \mathcal{K}(V)\|_1 \\
&\leq \frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{M}\|U - X\|_1 + \frac{\|\sum_{i=1}^{k} B_jB_j^*\|}{M}\|Y - V\|_1 \\
&\leq \frac{\lambda}{2}\big(\|U - X\|_1 + \|Y - V\|_1\big),
\end{aligned}$$

where $\lambda = 2\max\Big\{\frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{M}, \frac{\|\sum_{i=1}^{k} B_jB_j^*\|}{M}\Big\}$. From (ii) and Lemma 3.2, we have $\lambda < 1$. Thus, the contractive condition of Theorem 2.6 is satisfied for all $(X,Y) \in \mathcal{H}(n)_{\leq}(U,V)$. Moreover, $\mathcal{F}$ has the mixed $\leq$-monotone property on $\mathcal{H}(n)$ and from (iii), we have $\mathcal{F} \times \mathcal{F}(2Q,0) \in \mathcal{H}(n)_{\leq}(2Q,0)$. Now from Theorem 2.6, there exist $X^*, Y^* \in \mathcal{H}(n)$ such that $\mathcal{F}(X^*,Y^*) = X^*$ and $\mathcal{F}(Y^*,X^*) = Y^*$. □

Theorem 3.4. *Let $Q \in \mathcal{P}(n)$ and $\sum_{i=1}^{m} A_i^*\mathcal{G}(2Q)A_i < Q$ and $\sum_{j=1}^{k} B_j^*\mathcal{K}(2Q)B_j < Q$. Then Eq.(3.1) has at least one positive definite solution in $[\mathcal{F}(0,2Q), \mathcal{F}(2Q,0)]$.*

Proof. Define the mapping $\mathcal{S} : \mathcal{P}(n) \to \mathcal{H}(n)$ by

$$\mathcal{S}(X) = Q + \sum_{i=1}^{m} A_i^*\mathcal{G}(X)A_i - \sum_{j=1}^{k} B_j^*\mathcal{K}(X)B_j, \qquad \forall X \in \mathcal{P}(n).$$

Now, we claim that $\mathcal{S}([\mathcal{F}(0,2Q), \mathcal{F}(2Q,0)]) \subseteq [\mathcal{F}(0,2Q), \mathcal{F}(2Q,0)]$. Indeed, if $\mathcal{F}(0,2Q) \leq X \leq \mathcal{F}(2Q,0)$, then we have $X \leq 2Q$. Applying $\mathcal{G}, \mathcal{K}$, we can easily obtain that

$$\begin{aligned}
\sum_{j=1}^{k} B_j^*\big(\mathcal{K}(X) - \mathcal{K}(2Q)\big)B_j &\leq \sum_{i=1}^{m} A_i^*\mathcal{G}(X)A_i \\
\sum_{i=1}^{m} A_i^*\big(\mathcal{G}(X) - \mathcal{G}(2Q)\big)A_i &\leq \sum_{j=1}^{k} B_j^*\mathcal{K}(X)B_j.
\end{aligned}$$

This implies that, $\mathcal{S}$ maps the compact convex set $[\mathcal{F}(0,2Q), \mathcal{F}(2Q,0)]$ into itself. Since $\mathcal{S}$ is continuous, it follows from Schauder's fixed point theorem that $\mathcal{S}$ has at least one fixed point in this set. However, fixed points of $\mathcal{S}$ are solutions of Eq.(3.1). □

Theorem 3.5. *Under the assumptions Theorem 3.3, the Eq.(3.1) has an unique solution $\widehat{X} \in \mathcal{H}(n)$.*

Proof. Since for every $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound, for any $(X,Y), (A,B) \in \mathcal{H}(n) \times \mathcal{H}(n)$ there exists $(U,V) \in \mathcal{H}(n) \times \mathcal{H}(n)$ such that $(X,Y), (A,B) \in \mathcal{H}(n)_{\leq}(U,V)$. Therefore, we deduce from Theorem 2.9 that $X^*, Y^* \in \mathcal{H}(n)$ in Theorem 3.3 is unique and $X^* = Y^* = \widehat{X}$. □

Theorem 3.6. *Let $Q \in \mathcal{P}(n)$. Then under the assumptions Theorem 3.3,*

(i) Eq.(3.1) has an unique positive definite solution $\widehat{X} \in [\mathcal{F}(0, 2Q), \mathcal{F}(2Q, 0)]$.

(ii) The sequences $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ defined by $X_0 = 2Q$, $Y_0 = 0$ and

$$X_n = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X_{n-1}) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(Y_{n-1}) B_j,$$

$$Y_n = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(Y_{n-1}) A_i - \sum_{j=1}^{k} B_j^* \mathcal{K}(X_{n-1}) B_j,$$

converge to $\widehat{X}$ and the error estimation is given by

$$\max_{n\in\mathbb{N}} \left\{ \|X_n - \widehat{X}\|_1, \|Y_n - \widehat{X}\|_1 \right\} \leq \frac{\lambda^n}{2(1-\lambda)} (\|X_1 - X_0\|_1 + \|Y_1 - Y_0\|_1),$$

for all $n \in \mathbb{N}$, where $\lambda = 2\max\left\{ \frac{\left\| \sum_{i=1}^{m} A_i A_i^* \right\|}{M}, \frac{\left\| \sum_{i=1}^{k} B_j B_j^* \right\|}{M} \right\}$.

Proof. By Theorem 3.4, Eq.(3.1) has at least one positive definite solution in $[\mathcal{F}(0, 2Q), \mathcal{F}(2Q, 0)]$ and by Theorem 3.5 this equation having a unique solution in $\mathcal{H}(n)$. Thus this solution must be in this set. Further, the proof of (ii) follows from part (c) of Theorem 2.6. □

Acknowledgements

The authors would like to thank the editors and anonymous reviewers for their constructive comments and suggestions. This work was supported by the Islamic Azad University, Central Tehran Branch.

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