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Additive ρ-functional inequalities in normed spaces

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Abstract

In this paper, we solve the additive ρ-functional inequalities

$$\|f(x+y)-f(x)-f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right)\right\| \tag{1}$$

and

$$\left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \|\rho(f(x+y)-f(x)-f(y))\|, \tag{2}$$

where ρ is a number with $|\rho|<1$. Using the fixed point method, we prove the Hyers-Ulam stability of the additive functional inequalities (1) and (2) in normed spaces.

Keywords: Additive ρ-functional inequality, fixed point, Hyers-Ulam stability.
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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization

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of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the *Jensen equation*.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 14, 15, 19, 21, 22, 23, 24, 25, 27, 28]).

In [10], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [8, 20]. Gilányi [11] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [16] proved the Hyers-Ulam stability of additive functional inequalities.

Lemma 1.1. (Banach fixed point theorem) *Let (S, d) be a complete metric space and let $T : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, there exists a positive integer n_0 such that*

(1) $d(T^n x, T^{n+1}x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence* $\{T^n x\}$ *converges to a fixed point* y^* *of* T;
(3) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in S \mid d(T^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-\alpha} d(y, Ty)$ *for all* $y \in Y$.

Since we define the metric d as generalized metric in order to use this lemma in the proof of the problem we extend the lemma.

Lemma 1.2 ([6])**.** *Let (S, d) be a complete generalized metric space and let $J : S \to S$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in S$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in S \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 17]).

In Section 2, we solve the additive ρ-functional inequality (1) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (1) in normed spaces.

In Section 3, we solve the additive ρ-functional inequality (2) and prove the Hyers-Ulam stability of the additive ρ-functional inequality (2) in normed spaces.

Throughout this paper, assume that X is a normed space and Y is a Banach space. Let ρ be a number with $|\rho| < 1$.

2. Hyers-Ulam stability of the additive ρ-functional inequality (1): a fixed point approach

We solve the additive ρ-functional inequality (1) in normed spaces.

Lemma 2.1. *A mapping $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| \tag{2.1}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1). Letting $x = y = 0$ in (2.1), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| = |\rho| \|f(x+y) - f(x) - f(y)\|$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a, b) \le 2\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}\right) \tag{2.3}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \left\| \rho \left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right) \right\| + \varphi(x, y) \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2 - 2\alpha} \varphi(x, x)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g, h) := \inf\{\mu \in \mathbb{R}_+ : \|g - h\| \le \mu\varphi(x, x), x \in S\}.$$

It is easy to show that (S, d) is complete. Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{2} g(2x). \tag{2.5}$$

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then from (2.3) and (2.5), we get

$$\|Jg(x) - Jh(x)\| = \left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\| \leq \frac{1}{2} \varepsilon \varphi(2x, 2x) \leq \alpha \varepsilon \varphi(x, x).$$

This means $d(Jg, Jh) \leq \alpha d(g, h)$. So the function $J : S \to S$ is a contractive mapping such that

$$d(Jg, Jh) \leq \alpha d(g, h)$$

for $0 \leq \alpha < 1$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x)$$

and so

$$\|f(x) - Jf(x)\| \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in X$. Thus we get $d(f, Jf) \leq \frac{1}{2}$.

By Lemma 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\,(2a) = 2A(a) \tag{2.6}$$

for all $a \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(a) - A(a)\| \leq \mu \varphi\,(a, a)$$

for all $a \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l \to \infty} \frac{1}{2^l} f\left(2^l a\right) = A(a)$$

for all $a \in X$;

(3) $d(f, A) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{2 - 2\alpha}.$$

So

$$\|f(a) - A(a)\| \leq \frac{1}{2 - 2\alpha} \varphi(a, a)$$

for all $a \in X$.

Then

$$\begin{aligned}
\|A(x + y) - A(x) - A(y)\| &= \lim_{l \to \infty} \left\| \frac{1}{2^l} (f(2^l (x + y)) - f(2^l x) - f(2^l y)) \right\| \\
&\leq \lim_{l \to \infty} \left\| \frac{\rho}{2^l} (2f(2^{l-1}(x + y)) - f(2^l x) - f(2^l y)) \right\| + \lim_{l \to \infty} \frac{1}{2^l} \varphi(2^l x, 2^l y) \\
&= \left\| \rho \left(2A \left(\frac{x + y}{2} \right) - A(x) - A(y) \right) \right\|
\end{aligned}$$

for all $x, y \in X$. Hence

$$A\,(x + y) = A(x) + A(y)$$

for all x, y. So $A : X \to Y$ is additive. □

Remark 2.3. We could prove the same statement with the same manner in spite of replacing the condition $\varphi(a,b) \leq 2\alpha\varphi\left(\frac{a}{2},\frac{b}{2}\right)$ into $\varphi(a,b) \leq \frac{1}{2}\alpha\varphi\left(2a,2b\right)$ by defining J such that $Jg(x) = 2g(\frac{x}{2})$ instead of $Jg(x) = \frac{1}{2}g(2x)$. It could be also applied to Theorem 3.2.

Corollary 2.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2-2^r|}\|x\|^r$$

for all $x \in X$.

3. Hyers-Ulam stability of the additive ρ-functional inequality (2): a fixed point approach

We solve the additive ρ-functional inequality (2) in normed spaces.

Lemma 3.1. *A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|\rho\left(f(x+y) - f(x) - f(y)\right)\| \tag{3.1}$$

for all $x, y \in X$ if and only if $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\|f(x+y) - f(x) - f(y)\| = \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq |\rho|\|f(x+y) - f(x) - f(y)\|$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2) in Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(a,b) \leq 2\alpha\varphi\left(\frac{a}{2},\frac{b}{2}\right)$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \|\rho(f(x+y) - f(x) - f(y))\| + \varphi(x,y)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2-2\alpha}\varphi(x,0)$$

for all $x \in X$.

Proof. Consider the set

$$S := \{h : X \to Y\}$$

and let d be the generalized metric on S:

$$d(g, h) := \inf\{\mu \in \mathbb{R}_+ : \|g - h\| \le \mu\varphi(x, 0), x \in S\}.$$

Let J be the linear mapping from S to S such that

$$Jg(x) := \frac{1}{2}g(2x).$$

Let $A : X \to Y$ be defined as in the proof of Theorem 2.2. Then

$$\begin{aligned}\left\|2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right\| &= \lim_{l\to\infty}\left\|\frac{1}{2^l}(2f(2^{l-1}(x+y)) - f(2^l x) - f(2^l y))\right\| \\ &\le \lim_{l\to\infty}\left\|\frac{\rho}{2^l}(f(2^l(x+y)) - f(2^l x) - f(2^l y))\right\| + \lim_{l\to\infty}\frac{1}{2^l}\varphi(2^l x, 2^l y) \\ &= \|\rho(A(x+y) - A(x) - A(y))\|\end{aligned}$$

for all $x, y \in X$. Hence

$$A(x + y) = A(x) + A(y)$$

for all x, y. So $A : X \to Y$ is additive. □

Corollary 3.3. *Let $r \neq 1$ and θ be nonnegative real numbers, and Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \le \|\rho(f(x+y) - f(x) - f(y))\| + \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\theta}{|2 - 2^r|}\|x\|^r$$

for all $x \in X$.

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