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A fixed point theorem in a lattice ordered semigroup cone valued cone metric spaces

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Abstract

In this paper, we introduce the notion of a cone which is a lattice ordered semigroup (l.o.s.g. cone) in a real Banach space, obtain certain preliminary results on such cones and obtain a fixed point theorem on a cone metric space with l.o.s.g. cone which eventually extends a result of Filipovic et. al. [M. Filipović, L. Paunović, S. Radenović and M. Rajović, Math. Compu. Model. 54 (2011), 1467–1472] to cone metric spaces equipped with l.o.s.g. cone.

Keywords: Cone metric space, Comparison function, Lattice ordered semigroup, l.o.s.g. cone, Coincidence point, Fixed point.
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1. Introduction

In 2007, Haung and Zhang [12] introduced the concept of cone metric spaces by replacing Banach space instead of the set of real numbers as the co-domain of a metric. Later many authors (see [1]-[8] and [10]-[29]) considered this concept and proved some fixed point theorems for contractive type mappings in cone metric spaces.

In 2010, J.R. Morales and E. Rojas [20] proved fixed point theorems of T- Kannan and T- Chatterjea contractions in cone metric spaces when the underlying cone is normal. Later M. Filipovic et.al.[11] proved these results without using the normality of the cone. In this paper, we introduce the notion of a cone

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which is a lattice ordered semigroup (briefly, l.o.s.g. cone) in a real Banach space and obtain certain basic properties of l.o.s.g. cones. We also prove a fixed point theorem (Theorem 3.1) for cone metric space with values in a l.o.s.g. cone.

We observe that our result is an extension and generalization of the result of Filipovic et.al. [11] to l.o.s.g. cone valued cone metric spaces. We also provide two examples to show that hypothesis in Theorem 3.1 can not be further relaxed. The following definitions and results will be needed in what follows.

Definition 1.1. [12] Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold.

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in R, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \bigcap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in Int\ P$ (Interior of P).

Definition 1.2. [12] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3. [11] Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

(i) a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$.
(ii) a convergent sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N such that for all $n \geq N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.4. [11] Let (X, d) be a cone metric space and $T : X \to X$ be a mapping. Then

(i) T is said to be continuous if $\lim\limits_{n\to\infty} x_n = x$ implies that $\lim\limits_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X.
(ii) T is said to be sequentially convergent if for every sequence $\{y_n\}$, $T(y_n)$ is convergent implies $\{y_n\}$ is also convergent.
(iii) T is said to be sub sequentially convergent, if for every sequence $\{y_n\}$, $T(y_n)$ is convergent implies $\{y_n\}$ has a convergent subsequence.

M. Filipovic et.al. [11] proved the following fixed point theorem.

Theorem 1.5. *[11] Let (X, d) be a complete cone metric space and P be a solid cone (that is, $IntP \neq \phi$), in addition let $T : X \to X$ be a one to one continuous mapping and $f : X \to X$ a T- Hardy- Rogers contraction that is, there exist $a_i \geq 0, i = \overline{1, 5}$ with $\sum\limits_{i=1}^{5} a_i < 1$ such that for all $x, y \in X$*

$$d(Tfx, Tfy) \leq a_1 d(Tx, Ty) + a_2 d(Tx, Tfx) + a_3 d(Ty, Tfy) + a_4 d(Tx, Tfy) + a_5 d(Ty, Tfx)$$

Then

(1) For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is Cauchy.
(2) There is $v_{x_0} \in X$ such that $\lim\limits_{n\to\infty} Tf^n x_0 = v_{x_0}$.
(3) If T is sub sequentially convergent then $\{f^n x_0\}$ has a convergent subsequence.
(4) There is a unique $u_{x_0} \in X$ such that $fu_{x_0} = u_{x_0}$.
(5) If T is sequentially convergent then for each $x_0 \in X$ the iterate sequence $\{f^n x_0\}$ converges to u_{x_0}.

2. Preliminary results on lattices

Before going to prove the main result we need the following definitions and lemmas on lattices and lattice ordered semigroups.

Definition 2.1. [9] A lattice is a partially ordered set S in which any two elements $a, b \in S$ have the supremum $(a \cup b)$ and the infimum $(a \cap b)$.

Definition 2.2. Let $(S, +)$ be a semi group and $(S, \cup, \cap)$ be a lattice. Then $(S, \cup, \cap, +)$ is called a lattice ordered semi group if it satisfies the following conditions:

(i) $a + (b \cup c) = (a + b) \cup (a + c)$; $(a \cup b) + c = (a + c) \cup (b + c)$

(ii) $a + (b \cap c) = (a + b) \cap (a + c)$; $(a \cap b) + c = (a + c) \cap (b + c)$ for all $a, b, c \in S$.

Definition 2.3. [24] Let $f, g : X \to X$ be two mappings. If $w = f(x) = g(x)$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 2.4. [24] Let (X, d) be a cone metric space with cone P. A non decreasing function $\varphi : P \to P$ is called a comparison function if it satisfies

(i) $\varphi(0) = 0$ and $0 < \varphi(x) < x$ for all $x \in P \setminus \{0\}$

(ii) If $x \in Int\ P$ then $x - \varphi(x) \in Int\ P$

(iii) $\lim\limits_{n\to\infty} \varphi^n(x) = 0$ for all $x \in P \setminus \{0\}$.

Lemma 2.5. *[26] Let (X, d) be a cone metric space with cone P. Assume that P is a lattice. Let φ be a comparison function satisfying*

(i) $\varphi : P \to P$ is a lattice homomorphism i.e. $\varphi(a \cup b) = \varphi(a) \cup \varphi(b)\ \forall\ a, b \in P$

(ii) $0 \le a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for every $x \in P$

Then $a, b \in P$ and $b \le \varphi(a \cup b) \Rightarrow b \le \varphi(a)$.

Proof. Suppose $a, b \in P$ and $b \le \varphi(a \cup b) = \varphi(a) \cup \varphi(b)$

$$\text{Then } b \le \varphi(a) \cup \varphi(b) \tag{2.5.1}$$

Claim: For any positive integer k, $b \le \varphi(a) \cup \varphi^k(b)$

The result is true for $k = 1$ by (2.5.1).

Assume it to be true for k. Then $b \le \varphi(a) \cup \varphi^k(b)$

$$\begin{aligned} \text{Now }\ \varphi(b) &\le \varphi(\varphi(a) \cup \varphi^k(b)) \\ &= \varphi^2(a) \cup \varphi^{k+1}(b) \\ &\le \varphi(a) \cup \varphi^{k+1}(b) \end{aligned}$$

So that $b \le \varphi(a) \cup \varphi^{k+1}(b)$

$\therefore$ By induction for every positive integer k, we have $b \le \varphi(a) \cup \varphi^k(b)$

Thus our claim is established.

Now letting $k \to \infty$ and using (ii) we get $b \le \varphi(a)$ □

Lemma 2.6. *[26] Let P be a cone in E. Suppose $(P, \le)$ is a lattice. Suppose $a, b \in P$, and $\alpha \in R$. Then $\alpha \ge 0 \Rightarrow (\alpha a) \cup (\alpha b) = \alpha(a \cup b)$*

Proof. We may suppose that $\alpha > 0$

$$\begin{aligned} \text{Now } 0 \le a \le a \cup b \text{ and } \alpha > 0 &\Rightarrow \alpha((a \cup b) - a) \ge 0 \\ &\Rightarrow \alpha(a \cup b) - \alpha a \ge 0 \\ &\Rightarrow \alpha(a \cup b) \ge \alpha a \end{aligned}$$

$$\text{Similarly } \alpha(a \cup b) \ge \alpha b$$

$$\therefore \alpha(a \cup b) \ge (\alpha a) \cup (\alpha b)$$

Further

$$\begin{aligned} \alpha a \leq x \text{ and } \alpha b \leq x &\Rightarrow a \leq \frac{1}{\alpha}x \text{ and } b \leq \frac{1}{\alpha}x \\ &\Rightarrow a \cup b \leq \frac{1}{\alpha}x \\ &\Rightarrow \alpha(a \cup b) \leq x \end{aligned}$$

$$\therefore (\alpha a) \cup (\alpha b) = \alpha(a \cup b)$$

□

Lemma 2.7. *[26] Let P be a cone in E. Suppose $(P, \leq, +)$ is a lattice ordered semigroup. Then $a, b \in P \Rightarrow a \cup b \cup (\frac{a+b}{2}) = a \cup b$*

Proof.

$$\begin{aligned} a \cup b \cup \left(\frac{a+b}{2}\right) &= a \cup \left(\frac{b}{2} + \frac{b}{2}\right) \cup \left(\frac{a}{2} + \frac{b}{2}\right) \\ &= a \cup \left(\frac{b}{2} + \left(\frac{a}{2} \cup \frac{b}{2}\right)\right) \text{ (since } P \text{ is a lattice ordered semigroup)} \\ &= \left(\frac{a}{2} + \frac{a}{2}\right) \cup \left(\frac{b}{2} + \left(\frac{a}{2} \cup \frac{b}{2}\right)\right) \\ &\leq \left(\frac{a}{2} + \left(\frac{a}{2} \cup \frac{b}{2}\right)\right) \cup \left(\frac{b}{2} + \left(\frac{a}{2} \cup \frac{b}{2}\right)\right) \\ &= \left(\frac{a}{2} \cup \frac{b}{2}\right) + \left(\frac{a}{2} \cup \frac{b}{2}\right) \\ &= \frac{(a \cup b)}{2} + \frac{(a \cup b)}{2} \text{ (By lemma 2.6)} \\ &= a \cup b \\ &\leq a \cup b \cup \left(\frac{a+b}{2}\right) \end{aligned}$$

$$\therefore a \cup b \cup \left(\frac{a+b}{2}\right) = a \cup b$$

□

Lemma 2.8. *[26] Let (X, d) be a cone metric space with cone P. Assume that P is a lattice ordered semi group. Let φ be a comparison function satisfying*

(i) $\varphi : P \to P$ is a lattice homomorphism. i.e. $\varphi(a \cup b) = \varphi(a) \cup \varphi(b)\ \forall\ a, b \in P$
(ii) $0 \leq a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for all $x \in P$.

Then $a, b \in P$ and $b \leq \varphi(a \cup b \cup (\frac{a+b}{2})) \Rightarrow b \leq \varphi(a)$

Proof.

$$\begin{aligned} b &\leq \varphi\Big(a \cup b \cup (\frac{a+b}{2})\Big) \\ &= \varphi(a \cup b) \text{ (By Lemma 2.7)} \end{aligned}$$

$$\Rightarrow b \leq \varphi(a) \text{ (By Lemma 2.5)}$$

□

3. Main results

Now we state and prove our main result.

Theorem 3.1. *Suppose P is a cone in a real Banach space E such that*

(1) P is a lattice ordered semigroup
(2) $0 \leq a_n$ and $a_n \to 0 \Rightarrow x \cup a_n \to x$ for all $x \in P$.

Suppose φ is a comparison function such that $\varphi : P \to P$ is a lattice homomorphism and $\sum \varphi^n(t)$ converges on X for $t \in P$. Suppose (X, d) is a complete cone metric space, $T : X \to X$ is a continuous mapping and $f : X \to X$ is a mapping such that, for some comparision function φ,

$$d(Tf(x), Tf(y)) \leq \varphi\Big(max\Big\{d(Tx, Ty), d(Tx, Tf(x)), d(Ty, Tf(y)), \frac{d(Tx, Tf(y)) + d(Ty, Tf(x))}{2}\Big\}\Big) \quad (3.1.1)$$

for all $x, y \in P$. Then

(3.1.2) $x_0 \in X \Rightarrow \{Tf^n x_0\}$ is a Cauchy sequence and hence converges
(3.1.3) If $f^n x = x$ for some $n \geq 1$ then T is constant on the sequence $x, fx, f^2x, \cdots$
In other words $Tx = Tfx = Tf^2x = \cdots$
(3.1.4) If T is sub sequentially convergent then
(i) Tf and T have a coincidence point.
(ii) Tf and T have a unique point of coincidence
(iii) If further T is one to one then f has a unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence of iterates by $x_n = f^n x_0$ for $n = 1, 2, 3, \cdots$
Now, from $(3.1.1)$ we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(Tf^n x_0, Tf^{n+1} x_0) \\ &= d(Tff^{n-1} x_0, Tff^n x_0) \\ &\leq \varphi\Big(max\Big\{d(Tf^{n-1}x_0, Tf^n x_0), d(Tf^{n-1}x_0, Tf^n x_0), \\ &\quad d(Tf^n x_0, Tf^{n+1} x_0), \frac{d(Tf^{n-1}x_0, Tf^{n+1}x_0) + d(Tf^n x_0, Tf^n x_0)}{2}\Big\}\Big) \\ &\leq \varphi\Big(max\Big\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \frac{d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})}{2}\Big\}\Big) \end{aligned}$$

Hence by Lemma 2.8,

$$d(Tx_n, Tx_{n+1}) \leq \varphi\big(d(Tx_{n-1}, Tx_n)\big)$$

Consequently $d(Tx_n, Tx_{n+1}) \leq \varphi^n(d(Tx_0, Tx_1))$
For $\epsilon \gg 0$ choose a natural number n_0 and a real number δ such that

$$\epsilon - \varphi(\epsilon) + \{u \in E : ||u|| < \delta\} \subset IntP$$

Now there exists n_0 such that

$$||\sum_{m=n}^{n+k} \varphi^m(d(Tx_0, Tx_1))|| < \delta \quad \text{for all } n \geq n_0 \text{ and } k = 1, 2, \cdots$$

$$\sum_{m=n}^{n+k} \varphi^m(d(Tx_0, Tx_1)) \ll \epsilon - \varphi(\epsilon) < \epsilon \quad (3.1.5)$$

for all $n \geq n_0$ and $k = 1, 2, \cdots$.

$$\begin{aligned} d(Tx_n, Tx_{n+k}) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{n+k-1}, Tx_{n+k}) \\ &\leq \varphi^n(d(Tx_0, Tx_1)) + \varphi^{n+1}(d(Tx_0, Tx_1)) + \cdots + \varphi^{n+k-1}(d(Tx_0, Tx_1)) \\ &\ll \epsilon - \varphi(\epsilon) \text{ (By (3.1.5))} \\ &< \epsilon \end{aligned}$$

Thus $\{Tf^n x_0\}$ is a Cauchy sequence. Since (X, d) is a complete cone metric space, there is $v_{x_0} \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v_{x_0} \tag{3.1.6}$$

Thus (*3.1.2*) is established.
To prove (*3.1.3*), we use induction on n.

That is, if $f^n x = x$ for some $n \geq 1$, T is constant on the sequence $x, fx, f^2x, \cdots$.

If $n = 1, fx = x$, and hence $Tf^n x = Tx$ for $n = 1, 2, 3, \cdots$

Thus the result is true when $n = 1$

Assume that $f^n x = x$ for some $n \geq 1$ implies T is constant on the sequence $x, fx, f^2x, \cdots$

Suppose $f^{n+1}y = y$. If $f^n y = y$ then by induction T is a constant on the sequence $y, fy, f^2y, \cdots$

Now suppose $f^n y \neq y = f^{n+1}y$ and suppose $d(Ty, Tfy) \neq 0$. Then by *(3.1.1)*, we have

$$\begin{aligned} d(Ty, Tfy) &= d(Tff^n y, Tf^2 f^n y) \\ &= d(Tff^n y, Tff^{n+1} y) \\ &\leq \varphi\Big(max\Big\{d(Tf^n y, Tf^{n+1}y), d(Tf^n y, Tf^{n+1}y), d(Tf^{n+1}y, Tf^{n+2}y), \\ &\qquad \frac{d(Tf^n y, Tf^{n+2}y) + d(Tf^{n+1}y, Tf^{n+1}y)}{2}\Big\}\Big) \\ &\leq \varphi\Big(max\Big\{d(Tf^n y, Ty), \frac{d(Tf^n y, Ty)}{2}\Big\}\Big) \\ &\leq \varphi\big(d(Tf^n y, Tf^{n+1}y)\big) \\ &\leq \varphi^n(d(Ty, Tfy)) \text{ (Since } \phi(t) \leq t, \ \forall\, t \geq 0) \\ &< d(Ty, Tfy) \end{aligned}$$

a contradiction.
$\therefore Ty = Tfy$
That is, T is a constant on the sequence $x, fx, f^2x, \cdots$
Thus, by induction, (*3.1.3*) is established.

Now we prove (*3.1.4*). Suppose T is sub sequentially convergent. Then $\{f^n x_0\}$ has a convergent subsequence. So there are u_{x_0} and (x_{n_i}) such that
$\lim_{i\to\infty} f^{n_i} x_0 = u_{x_0}$. Since T is continuous

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu_{x_0} \tag{3.1.7}$$

From (3.1.6) and (3.1.7) we have $Tu_{x_0} = v_{x_0}$.
Now we show that $Tfu_{x_0} = Tu_{x_0}$.
Suppose $Tfu_{x_0} \neq Tu_{x_0}$. Then there exists M such that for $i \geq M$ we have

$$d(Tu_{x_0}, Tf^{n_i - 1}x_0) < \frac{d(Tu_{x_0}, Tfu_{x_0})}{2} = \epsilon \ (say)$$

$$d(Tf^{n_i-1}x_0, Tf^{n_i}x_0) < \frac{d(Tu_{x_0}, Tfu_{x_0})}{2} = \epsilon$$

$$d(Tu_{x_0}, Tf^{n_i-1}x_0) < \frac{d(Tu_{x_0}, Tfu_{x_0})}{2} = \epsilon$$

and

$$d(Tf^{n_i-1}x_0, Tfu_{x_0}) < d(Tu_{x_0}, Tfu_{x_0}) + \epsilon = 3\epsilon \ (say)$$

By using (*3.1.1*) we have

$$\begin{aligned}
d(Tfu_{x_0}, Tf^{n_i}x_0) &= d(Tfu_{x_0}, Tff^{n_i-1}x_0) \\
&\leq \varphi\Big(max\Big\{d(Tu_{x_0}, Tf^{n_i-1}x_0), d(Tu_{x_0}, Tfu_{x_0}), \\
&\quad d(Tf^{n_i-1}x_0, Tf^{n_i}x_0), \frac{d(Tu_{x_0}, Tf^{n_i}x_0) + d(Tf^{n_i-1}x_0, Tfu_{x_0})}{2}\Big\}\Big) \\
&\leq \varphi\Big(max\Big\{\epsilon, 2\epsilon, \epsilon, \frac{\epsilon + 3\epsilon}{2}\Big\}\Big) \\
&= \varphi(max\{\epsilon, 2\epsilon\}) \\
&= \varphi(d(Tu_{x_0}, Tfu_{x_0}))
\end{aligned}$$

$\therefore d(Tfu_{x_0}, Tf^{n_i}x_0) \leq \varphi(d(Tfu_{x_0}, Tu_{x_0}))$ for $i \geq M$
On letting $i \to \infty$ we get

$$d(Tfu_{x_0}, Tu_{x_0}) \leq \varphi(d(Tfu_{x_0}, Tu_{x_0})), \text{ a contradiction}$$

$$\therefore Tfu_{x_0} = Tu_{x_0}$$

Consequently u_{x_0} is a coincidence point of T and Tf.

Thus (*3.1.4*)(*i*) is established.

Suppose x and y are coincidence points of Tf and T and $Tx \neq Ty$. Then

$$\begin{aligned}
d(Tx, Ty) = d(Tfx, Tfy) &\leq \varphi\Big(max\Big\{d(Tx, Ty), d(Tx, Tf(x)), d(Ty, Tf(y)), \\
&\quad \frac{d(Tx, Tf(y)) + d(Ty, Tf(x))}{2}\Big\}\Big) \\
&\leq \varphi(d(Tx, Ty)) \\
&< d(Tx, Ty)
\end{aligned}$$

a contradiction
$\therefore Tfx = Tx = Ty = Tfy$.
Hence Tf and T have unique point of coincidence.

Thus (*3.1.4*)(*ii*) is established.

Now suppose further that T is one to one. Then clearly, by (*3.1.4*)(*i*) and (*3.1.4*)(*ii*) f has a unique fixed point. Thus (*3.1.4*)(*iii*) is established.

Thus the theorem is completely proved. □

Note: If T is not one to one then f may not have unique fixed point when T is sub sequentially convergent even in a metric space.

Example 3.2. Let $X = R$ with the usual metric $T \equiv 0$ and $f = I$. Then T is not sub sequentially convergent and *(3.1.4)* fails.

Example 3.3. Let $X = \{0, 1\}$ with usual metric, $T0 = T1 = 0, f(0) = 0, f(1) = 1$ then T is sub sequentially convergent but not one to one and *(3.1.4)(iii)* fails.

Remark 3.4. By setting $\varphi(t) = \lambda t,\ 0 \leq \lambda < 1$ in Theorem 3.1 we have the generalized version of Theorem 1.5 when the underlying cone is a lattice ordered semigroup.

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