Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 325–344
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Visco-resolvent algorithms for monotone operators and nonexpansive mappings

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Communicated by Y. J. Cho

Special Issue In Honor of Professor Ravi P. Agarwal

Abstract

Two new type of visco-resolvent algorithms for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive mapping in a Hilbert space are investigated. The algorithms consist of the zeros and the fixed points of the considered problems in which one operator is replaced with its resolvent and a viscosity term is added. Strong convergence of the algorithms are shown. As special cases, we can approach to the minimum norm common element of the zero of the sum of two monotone operators and the fixed point of a nonexpansive mapping without using the metric projection. Some applications are included.

Keywords: Monotone operator, nonexpansive mapping, zero point, fixed point, resolvent.
2010 MSC: 49J40, 47J20, 47H09, 65J15.

1. Introduction

Let H be a real Hilbert space. Let $A : H \to H$ be a single-valued nonlinear mapping and $B : H \to 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a zero $x \in H$ of the sum of two monotone operators A and B such that

$$0 \in A(x) + B(x), \tag{1.1}$$

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Received 2014-09-10

where 0 is the zero vector in H. The set of solutions of problem (1.1) is denoted by $(A+B)^{-1}(0)$. If $H = R^m$, then problem (1.1) becomes the generalized equation introduced by Robinson [22]. If $A = 0$, then problem (1.1) becomes the inclusion problem introduced by Rockafellar [24]. It is known that (1.1) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, game theory, etc. Also various types of variational inclusions problems have been extended and generalized. Recently, Zhang et al. [32] introduced a new iterative scheme for finding a common element of the set of solutions to the problem (1.1) and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [21] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping. Some related works, please see [3, 4, 7, 9, 10, 12, 13, 15, 16, 19, 20] and the references therein.

Recently, Takahashi et al. [27] introduced the following iterative algorithm for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive mapping

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S\big(\alpha_n x + (1-\alpha_n) J^B_{\lambda_n}(x_n - \lambda_n A x_n)\big) \tag{1.2}$$

for all $n \geq 0$. Under some assumptions, they proved that the sequence $\{x_n\}$ converges strongly to a point of $F(S) \cap (A+B)^{-1}0$.

Remark 1.1. We note that in their result, the authors added an additional assumption: the domain of B is included in C (The reader can refer to Lemma 4.3 in the last section for a possible example which satisfies this assumption). This assumption is indeed not restrict in order to guarantee $J^B_{\lambda_n}(x_n - \lambda_n A x_n) \in C$.

Remark 1.2. From the listed references, there exist a large number of problems which need to find the minimum norm solution, see, e.g., [11, 17, 25, 30, 31]. A useful path to circumvent this problem is to use projection. Bauschke and Browein [2] and Censor and Zenios [6] provide reviews of the field. The main difficult is in computation. Hence, it is an interesting problem of finding the minimum norm element without using the projection. We note that the algorithm (1.1) can not use to find the minimum norm element.

Motivated and inspired by the works in this field, we first suggest the following two algorithms without using projection:

$$x_t = J^B_\lambda\big(t\gamma f(x_t) + (1-t)Sx_t - \lambda ASx_t\big), \quad t \in (0,1)$$

and

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J^B_{\lambda_n}\big(\alpha_n \gamma f(x_n) + (1-\alpha_n) S x_n - \lambda_n A S x_n\big), \quad n \geq 0.$$

(Notice that these two algorithms are indeed well-defined (see the next section).) We will show the suggested algorithms converge strongly to to a point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x}))$ which solves the following variational inequality

$$\langle \gamma f(\tilde{x}) - \tilde{x}, \tilde{x} - z\rangle \geq 0, \quad \forall z \in F(S) \cap (A+B)^{-1}0.$$

As special cases, we can approach to the minimum norm element in $F(S) \cap (A+B)^{-1}0$ without using the metric projection. Some applications are also included.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping $A : C \to H$ is said to be α-inverse strongly-monotone iff

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$$

for some $\alpha > 0$ and for all $x, y \in C$. It is known that if A is α-inverse strongly-monotone, then $\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff

$$\langle x - y, u - v\rangle \geq 0$$

for all $x, y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to dom(B)$, which is called the resolvent of B for λ. It is known that the resolvent J_λ^B is firmly nonexpansive, i.e.,

$$\left\|J_\lambda^B x - J_\lambda^B y\right\|^2 \leq \left\langle J_\lambda^B x - J_\lambda^B y, x - y\right\rangle$$

for all $x, y \in C$ and $B^{-1}0 = F(J_\lambda^B)$ for all $\lambda > 0$.

The following resolvent identity is well-known: for $\lambda > 0$ and $\mu > 0$, there holds the identity

$$J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^B x\right), \quad x \in H. \tag{2.1}$$

We use the following notation:

- $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;
- $x_n \to x$ stands for the strong convergence of (x_n) to x.

We need the following lemmas for the next section.

Lemma 2.1. ([28]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $\lambda > 0$ be a constant. Then, we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in C.$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2. ([14]) *Let C be a closed convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Then $F(S)$ is a closed convex subset of C and the mapping $I - S$ is demiclosed at 0, i.e. whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, then $(I - S)x = 0$.*

Lemma 2.3. ([18]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments, that is, $F(x + ty) \to F(x)$ weakly as $t \to 0$. Then the variational inequality*

$$x^* \in C, \quad \langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

Lemma 2.4. ([26]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. ([29]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we will prove our main results.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H. Let $f : C \to H$ be a ρ-contraction and γ be a constant such that $0 < \gamma < \frac{1}{\rho}$. Let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself, such that $F(S)\cap(A+B)^{-1}0 \neq \emptyset$. Let λ be a constant satisfying $a \leq \lambda \leq b$ where $[a,b] \subset (0,2\alpha)$. For $t \in \big(0, 1-\frac{\lambda}{2\alpha}\big)$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = J_\lambda^B\big(t\gamma f(x_t) + (1-t)Sx_t - \lambda ASx_t\big). \tag{3.1}$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x}))$ which solves the following variational inequality

$$\langle \gamma f(\tilde{x}) - \tilde{x}, \tilde{x} - z\rangle \geq 0, \quad \forall z \in F(S)\cap(A+B)^{-1}0.$$

Proof. First, we show the net $\{x_t\}$ is well-defined. For any $t \in \big(0, 1-\frac{\lambda}{2\alpha}\big)$, we define a mapping $W := J_\lambda^B\big(t\gamma f + (1-t)S - \lambda AS\big)$. Note that J_λ^B, S and $I - \frac{\lambda}{1-t}A$ (see Lemma 2.1) are nonexpansive. For any $x, y \in C$, we have

$$\begin{aligned}
&\|Wx - Wy\| \\
= {}& \left\|J_\lambda^B\left(t\gamma f(x) + (1-t)\left(I - \frac{\lambda}{1-t}A\right)Sx\right) - J_\lambda^B\left(t\gamma f(y) + (1-t)\left(I - \frac{\lambda}{1-t}A\right)Sy\right)\right\| \\
\leq {}& \left\|t\gamma(f(x)-f(y)) + (1-t)\left[\left(I - \frac{\lambda}{1-t}A\right)Sx - \left(I - \frac{\lambda}{1-t}A\right)Sy\right]\right\| \\
\leq {}& t\gamma\|f(x)-f(y)\| + (1-t)\left\|(I - \frac{\lambda}{1-t}A)Sx - \left(I - \frac{\lambda}{1-t}A\right)Sy\right\| \\
\leq {}& t\gamma\rho\|x-y\| + (1-t)\|x-y\| \\
= {}& [1-(1-\gamma\rho)t]\|x-y\|,
\end{aligned}$$

which implies the mapping W is a contraction on C. We use x_t to denote the unique fixed point of W in C. Therefore, $\{x_t\}$ is well-defined.

Take any $z \in F(S)\cap(A+B)^{-1}0$. It is obvious that $z = Sz = J_\lambda^B(z - \lambda Az)$ for all $\lambda > 0$. So, we have

$$z = Sz = J_\lambda^B(z-\lambda Az) = J_\lambda^B\left(tz + (1-t)\left(I - \frac{\lambda}{1-t}A\right)Sz\right)$$

for all $t \in (0,1)$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned}
&\|x_t - z\| \\
= \;& \left\| J_\lambda^B\left(t\gamma f(x_t) + (1-t)\left(I - \frac{\lambda}{1-t}A\right)Sx_t\right) - z\right\| \\
= \;& \left\| J_\lambda^B\left(t\gamma f(x_t) + (1-t)\left(Sx_t - \frac{\lambda}{1-t}ASx_t\right)\right) - J_\lambda^B\left(tz + (1-t)\left(Sz - \frac{\lambda}{1-t}ASz\right)\right)\right\| \\
\le \;& \left\|\left(t\gamma f(x_t) + (1-t)\left(Sx_t - \frac{\lambda}{1-t}ASx_t\right)\right) - \left(tz + (1-t)\left(Sz - \frac{\lambda}{1-t}ASz\right)\right)\right\| \\
= \;& \left\|(1-t)\left(\left(Sx_t - \frac{\lambda}{1-t}ASx_t\right) - \left(Sz - \frac{\lambda}{1-t}ASz\right)\right) + t(\gamma f(x_t) - z)\right\| \\
\le \;& (1-t)\left\|\left(I - \frac{\lambda}{1-t}A\right)Sx_t - \left(I - \frac{\lambda}{1-t}A\right)Sz\right\| + t\gamma\|f(x_t) - f(z)\| + t\|\gamma f(z) - z\| \\
\le \;& (1-t)\|x_t - z\| + t\gamma\rho\|x_t - z\| + t\|\gamma f(z) - z\|.
\end{aligned} \tag{3.2}$$

It follows that

$$\|x_t - z\| \le \frac{1}{1-\gamma\rho}\|\gamma f(z) - z\|.$$

Therefore, $\{x_t\}$ is bounded. We deduce immediately that $\{f(x_t)\}$, $\{Ax_t\}$ $\{Sx_t\}$ and $\{ASx_t\}$ are also bounded.

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, from (3.2), we derive

$$\begin{aligned}
&\|x_t - z\|^2 \\
\le \;& \left\|(1-t)\left(\left(Sx_t - \frac{\lambda}{1-t}ASx_t\right) - \left(Sz - \frac{\lambda}{1-t}ASz\right)\right) + t(\gamma f(x_t) - z)\right\|^2 \\
\le \;& (1-t)\left\|\left(Sx_t - \frac{\lambda}{1-t}ASx_t\right) - \left(Sz - \frac{\lambda}{1-t}ASz\right)\right\|^2 + t\|\gamma f(x_t) - z\|^2 \\
= \;& (1-t)\left\|(Sx_t - Sz) - \frac{\lambda}{1-t}(ASx_t - ASz)\right\|^2 + t\|\gamma f(x_t) - z\|^2 \\
= \;& (1-t)\left(\|Sx_t - Sz\|^2 - \frac{2\lambda}{1-t}\langle ASx_t - ASz, Sx_t - Sz\rangle + \frac{\lambda^2}{(1-t)^2}\|ASx_t - ASz\|^2\right) \\
& + t\|\gamma f(x_t) - z\|^2 \\
\le \;& (1-t)\left(\|Sx_t - Sz\|^2 - \frac{2\alpha\lambda}{1-t}\|ASx_t - ASz\|^2 + \frac{\lambda^2}{(1-t)^2}\|ASx_t - ASz\|^2\right) \\
& + t\|\gamma f(x_t) - z\|^2 \\
= \;& (1-t)\left(\|Sx_t - Sz\|^2 + \frac{\lambda}{(1-t)^2}(\lambda - 2(1-t)\alpha)\|ASx_t - ASz\|^2\right) + t\|\gamma f(x_t) - z\|^2 \\
\le \;& (1-t)\|x_t - z\|^2 + \frac{\lambda}{1-t}(\lambda - 2(1-t)\alpha)\|ASx_t - ASz\|^2 + t\|\gamma f(x_t) - z\|^2.
\end{aligned} \tag{3.3}$$

So,

$$\frac{\lambda}{1-t}(2(1-t)\alpha-\lambda)\|ASx_t-Az\|^2\le t\|\gamma f(x_t)-z\|^2-t\|x_t-z\|^2\to 0.$$

By the assumption, we have $2(1-t)\alpha-\lambda>0$ for all $t\in(0,1-\frac{\lambda}{2\alpha})$. Then, we obtain

$$\lim_{t\to 0+}\|ASx_t-Az\|=0. \tag{3.4}$$

Next, we show $\|x_t-Sx_t\|\to 0$. By using the firm nonexpansivity of J_λ^B, we have

$$\begin{aligned}
&\|x_t-z\|^2\\
=\ &\left\|J_\lambda^B\big(t\gamma f(x_t)+(1-t)Sx_t-\lambda ASx_t\big)-z\right\|^2\\
=\ &\left\|J_\lambda^B\big(t\gamma f(x_t)+(1-t)Sx_t-\lambda ASx_t\big)-J_\lambda^B\big(z-\lambda Az\big)\right\|^2\\
\le\ &\big\langle t\gamma f(x_t)+(1-t)Sx_t-\lambda ASx_t-(z-\lambda Az),x_t-z\big\rangle\\
=\ &\frac{1}{2}\big(\|t\gamma f(x_t)+(1-t)Sx_t-\lambda ASx_t-(z-\lambda Az)\|^2+\|x_t-z\|^2\\
&-\left\|t\gamma f(x_t)+(1-t)Sx_t-\lambda(ASx_t-\lambda Az)-x_t\right\|^2\big).
\end{aligned}$$

By the nonexpansivity of $I-\frac{\lambda}{1-t}A$, we have

$$\begin{aligned}
&\|t\gamma f(x_t)+(1-t)Sx_t-\lambda ASx_t-(z-\lambda Az)\|^2\\
=\ &\left\|(1-t)\left(\left(Sx_t-\frac{\lambda}{1-t}ASx_t-\left(Sz-\frac{\lambda}{1-t}ASz\right)\right)\right)+t(\gamma f(x_t)-z)\right\|^2\\
\le\ &(1-t)\left\|\left(Sx_t-\frac{\lambda}{1-t}ASx_t-\left(Sz-\frac{\lambda}{1-t}ASz\right)\right\|^2+t\|\gamma f(x_t)-z\|^2\\
\le\ &(1-t)\|x_t-z\|^2+t\|\gamma f(x_t)-z\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_t-z\|^2\ \le\ &\frac{1}{2}\big((1-t)\|x_t-z\|^2+t\|\gamma f(x_t)-z\|^2+\|x_t-z\|^2\\
&-\left\|t\gamma f(x_t)+(1-t)Sx_t-\lambda(ASx_t-Az)-x_t\right\|^2\big).
\end{aligned}$$

That is,

$$\begin{aligned}
&\|x_t-z\|^2\\
\le\ &(1-t)\|x_t-z\|^2+t\|\gamma f(x_t)-z\|^2\\
&-\left\|t\gamma f(x_t)+(1-t)Sx_t-x_t-\lambda(ASx_t-Az)\right\|^2\\
=\ &(1-t)\|x_t-z\|^2+t\|\gamma f(x_t)-z\|^2-\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|^2\\
&+2\lambda\big\langle t\gamma f(x_t)+(1-t)Sx_t-x_t,ASx_t-Az\big\rangle-\lambda^2\|ASx_t-Az\|^2\\
\le\ &(1-t)\|x_t-z\|^2+t\|\gamma f(x_t)-z\|^2-\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|^2\\
&+2\lambda\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|\|ASx_t-Az\|.
\end{aligned}$$

Hence,

$$\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|^2 \le t\|\gamma f(x_t)-z\|^2+2\lambda\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|\|ASx_t-Az\|.$$

Since $\|ASx_t-Az\|\to 0$, we deduce

$$\lim_{t\to 0+}\|t\gamma f(x_t)+(1-t)Sx_t-x_t\|=0.$$

Which implies that

$$\lim_{t\to 0+}\|x_t-Sx_t\|=0. \tag{3.5}$$

From (3.2), we have

$$\begin{aligned}
&\|x_t-z\|^2\\
\le\ & \left\|(1-t)\left(\left(Sx_t-\frac{\lambda}{1-t}ASx_t\right)-\left(z-\frac{\lambda}{1-t}Az\right)\right)+t(\gamma f(x_t)-z)\right\|^2\\
=\ & (1-t)^2\left\|\left(Sx_t-\frac{\lambda}{1-t}ASx_t\right)-\left(z-\frac{\lambda}{1-t}Az\right)\right\|^2\\
+\ & 2t(1-t)\left\langle\gamma f(x_t)-z,\left(Sx_t-\frac{\lambda}{1-t}ASx_t\right)-\left(z-\frac{\lambda}{1-t}Az\right)\right\rangle+t^2\|\gamma f(x_t)-z\|^2\\
\le\ & (1-t)^2\|x_t-z\|^2+2t(1-t)\left\langle\gamma f(x_t)-z,Sx_t-\frac{\lambda}{1-t}(ASx_t-ASz)-z\right\rangle\\
&+t^2\|\gamma f(x_t)-z\|^2\\
=\ & (1-t)^2\|x_t-z\|^2+2t(1-t)\gamma\left\langle f(x_t)-f(z),Sx_t-\frac{\lambda}{1-t}(ASx_t-ASz)-z\right\rangle\\
&+2t(1-t)\left\langle\gamma f(z)-z,Sx_t-\frac{\lambda}{1-t}(ASx_t-ASz)-z\right\rangle+t^2\|\gamma f(x_t)-z\|^2\\
\le\ & (1-t)^2\|x_t-z\|^2+2t(1-t)\gamma\|f(x_t)-f(z)\|\left(\|Sx_t-z\|+\left\|\frac{\lambda}{1-t}(ASx_t-ASz)\right\|\right)\\
&+2t(1-t)\left\langle\gamma f(z)-z,Sx_t-\frac{\lambda}{1-t}(ASx_t-ASz)-z\right\rangle+t^2\|\gamma f(x_t)-z\|^2\\
\le\ & (1-t)^2\|x_t-z\|^2+2t(1-t)\gamma\rho\|x_t-z\|^2+2t\lambda\gamma\rho\|x_t-z\|\|ASx_t-ASz\|\\
&+2t(1-t)\left\langle\gamma f(z)-z,Sx_t-\frac{\lambda}{1-t}(ASx_t-ASz)-z\right\rangle+t^2\|\gamma f(x_t)-z\|^2\\
\le\ & [1-2(1-\gamma\rho)t]\|x_t-z\|^2+2t\left\{(1-t)\left\langle\gamma f(z)-z,Sx_t-\frac{\lambda}{1-t}(ASx_t-Az)-z\right\rangle\right.\\
&\left.+\frac{t}{2}(\|\gamma f(x_t)-z\|^2+\|x_t-z\|^2)+\lambda\gamma\rho\|x_t-z\|\|ASx_t-ASz\|\right\}.
\end{aligned}$$

It follows that

$$\|x_t-z\|^2$$

$$\begin{aligned} &\leq \frac{1}{1-\gamma\rho}\Bigg(\Big\langle \gamma f(z)-z, Sx_t-\frac{\lambda}{1-t}(ASx_t-Az)-z\Big\rangle \\ &\quad +\frac{t}{2}(\|\gamma f(x_t)-z\|^2+\|x_t-z\|^2)+t\|\gamma f(z)-z\|\Big\|Sx_t-\frac{\lambda}{1-t}(ASx_t-Az)-z\Big\| \\ &\quad +\lambda\gamma\rho\|x_t-z\|\|ASx_t-ASz\|\Bigg) \\ &\leq \frac{1}{1-\gamma\rho}\Big\langle \gamma f(z)-z, Sx_t-z\Big\rangle+(t+\|ASx_t-Az\|)M, \end{aligned} \tag{3.6}$$

where M is some constant such that

$$\sup\frac{1}{1-\gamma\rho}\Bigg\{\frac{1}{2}(\|\gamma f(x_t)-z\|^2+\|x_t-z\|^2), \|\gamma f(z)-z\|\Big\|Sx_t-\frac{\lambda}{1-t}(ASx_t-Az)-z\Big\|,$$
$$\lambda\gamma\rho\|x_t-z\|, t\in\left(0,1-\frac{\lambda}{2\alpha}\right)\Bigg\}\leq M.$$

Next we show that $\{x_t\}$ is relatively norm-compact as $t\to 0+$. Assume $\{t_n\}\subset(0,1-\frac{\lambda}{2\alpha})$ is such that $t_n\to 0+$ as $n\to\infty$. Put $x_n:=x_{t_n}$. From (3.6), we have

$$\|x_n-z\|^2\leq\frac{1}{1-\gamma\rho}\langle\gamma f(z)-z, Sx_n-z\rangle+(t_n+\|ASx_n-Az\|)M. \tag{3.7}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j}\rightharpoonup\tilde{x}\in C$. Hence, $x_{n_j}-\frac{\lambda}{1-t_{n_j}}(ASx_{n_j}-Az)\rightharpoonup\tilde{x}$ because of $\|ASx_n-Az\|\to 0$ by (3.4). From (3.5), we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.8}$$

We can use Lemma 2.2 to (3.8) to deduce $\tilde{x}\in F(S)$. Further, we show that $\tilde{x}$ is also in $(A+B)^{-1}0$. Let $v\in Bu$. Note that $x_n=J_\lambda^B\big(t_n\gamma f(x_n)+(1-t_n)Sx_n-\lambda ASx_n\big)$ for all n. Then, we have

$$\begin{aligned} &t_n\gamma f(x_n)+(1-t_n)Sx_n-\lambda ASx_n\in(I+\lambda B)x_n \\ &\Longrightarrow\quad \frac{t_n\gamma f(x_n)}{\lambda}+\frac{1-t_n}{\lambda}Sx_n-ASx_n-\frac{x_n}{\lambda}\in Bx_n. \end{aligned}$$

Since B is monotone, we have, for $(u,v)\in B$,

$$\begin{aligned} &\Big\langle\frac{t_n\gamma f(x_n)}{\lambda}+\frac{1-t_n}{\lambda}Sx_n-ASx_n-\frac{x_n}{\lambda}-v, x_n-u\Big\rangle\geq 0 \\ &\Longrightarrow\quad \langle t_n\gamma f(x_n)+(1-t_n)Sx_n-\lambda ASx_n-x_n-\lambda v, x_n-u\rangle\geq 0 \\ &\Longrightarrow\quad \langle ASx_n+v, x_n-u\rangle\leq\frac{1}{\lambda}\langle Sx_n-x_n, x_n-u\rangle-\frac{t_n}{\lambda}\langle Sx_n-\gamma f(x_n), x_n-u\rangle \\ &\Longrightarrow\quad \langle AS\tilde{x}+v, x_n-u\rangle\leq\frac{1}{\lambda}\langle Sx_n-x_n, x_n-u\rangle-\frac{t_n}{\lambda}\langle Sx_n-\gamma f(x_n), x_n-u\rangle \\ &\qquad\quad +\langle AS\tilde{x}-ASx_n, x_n-u\rangle \\ &\Longrightarrow\quad \langle AS\tilde{x}+v, x_n-u\rangle\leq\frac{1}{\lambda}\|Sx_n-x_n\|\|x_n-u\|+\frac{t_n}{\lambda}\|Sx_n-\gamma f(x_n)\|\|x_n-u\| \\ &\qquad\quad +\|AS\tilde{x}-ASx_n\|\|x_n-u\|. \end{aligned}$$

It follows that

$$\begin{aligned}&\langle AS\tilde{x}+v,\tilde{x}-u\rangle\\ \leq\ & \frac{1}{\lambda}\|Sx_{n_j}-x_{n_j}\|\|x_{n_j}-u\|+\frac{t_{n_j}}{\lambda}\|Sx_{n_j}-\gamma f(x_{n_j})\|\|x_{n_j}-u\|\\ &+\|AS\tilde{x}-ASx_{n_j}\|\|x_{n_j}-u\|+\langle AS\tilde{x}+v,\tilde{x}-x_{n_j}\rangle. \end{aligned}\tag{3.9}$$

Since

$$\langle x_{n_j}-\tilde{x},ASx_{n_j}-AS\tilde{x}\rangle\geq\alpha\|ASx_{n_j}-AS\tilde{x}\|^2,$$

$ASx_{n_j}\to ASz$ and $x_{n_j}\rightharpoonup\tilde{x}$, we have $ASx_{n_j}\to AS\tilde{x}$. We also observe that $t_n\to0$ and $\|Sx_n-x_n\|\to0$. Then, from (3.9), we derive

$$\langle AS\tilde{x}+v,\tilde{x}-u\rangle\leq0.$$

That is, $\langle -A\tilde{x}-v,\tilde{x}-u\rangle\geq0$. Since B is maximal monotone, we have $-A\tilde{x}\in B\tilde{x}$. This shows that $0\in(A+B)\tilde{x}$. Hence, we have $\tilde{x}\in F(S)\cap(A+B)^{-1}0$. Therefore we can substitute $\tilde{x}$ for z in (3.7) to get

$$\|x_n-\tilde{x}\|^2\leq\frac{1}{1-\gamma\rho}\big\langle\gamma f(\tilde{x})-\tilde{x},Sx_n-\tilde{x}\big\rangle+(t_n+\|ASx_n-A\tilde{x}\|)M.$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n\to\tilde{x}$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t\to0+$.

Now we return to (3.7) and take the limit as $n\to\infty$ to get

$$\|\tilde{x}-z\|^2\leq\frac{1}{1-\gamma\rho}\langle\gamma f(z)-z,\tilde{x}-z\rangle,\quad\forall z\in F(S)\cap(A+B)^{-1}0.$$

In particular, $\tilde{x}$ solves the following variational inequality

$$\tilde{x}\in F(S)\cap(A+B)^{-1}0,\ \langle\gamma f(z)-z,\tilde{x}-z\rangle\geq0,$$

for all $z\in F(S)\cap(A+B)^{-1}0$, or the equivalent dual variational inequality (see Lemma 2.3)

$$\tilde{x}\in F(S)\cap(A+B)^{-1}0,\ \langle\gamma f(\tilde{x})-\tilde{x},\tilde{x}-z\rangle\geq0,$$

for all $z\in F(S)\cap(A+B)^{-1}0$. Hence,

$$\tilde{x}=P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x})).$$

Clearly this is sufficient to conclude that the entire net $\{x_t\}$ converges to $\tilde{x}$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H. Let $f:C\to H$ be a ρ-contraction and γ be a constant such that $0<\gamma<\frac{1}{\rho}$. Let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ and let S be a nonexpansive mapping of C into itself, such that*

$$F(S)\cap(A+B)^{-1}0\neq\emptyset.$$

For given $x_0\in C$, let $\{x_n\}\subset C$ be a sequence generated by

$$x_{n+1}=\beta_nx_n+(1-\beta_n)J_{\lambda_n}^B\big(\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_nASx_n\big)\tag{3.10}$$

for all $n\geq0$, where $\{\lambda_n\}\subset(0,2\alpha)$, $\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset(0,1)$ satisfy

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_n\alpha_n=\infty$;
(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;
(iii) $a(1-\alpha_n)\leq\lambda_n\leq b(1-\alpha_n)$ *where* $[a,b]\subset(0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)=0$.

Then $\{x_n\}$ *generated by* (3.10) *converges strongly to a point* $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x}))$ *which solves the following variational inequality*

$$\langle \gamma f(\tilde{x}) - \tilde{x}, \tilde{x} - z\rangle \geq 0,$$

for all $z \in F(S) \cap (A+B)^{-1}0$.

Proof. Pick up $z \in F(S) \cap (A+B)^{-1}0$. It is obvious that

$$z = J_{\lambda_n}^B(z - \lambda_n Az) = J_{\lambda_n}^B\big(\alpha_n z + (1-\alpha_n)(z - \lambda_n Az/(1-\alpha_n))\big)$$

for all $n \geq 0$. Since J_λ^B, S and $I - \frac{\lambda}{1-\alpha_n}A$ are nonexpansive for all $\lambda > 0$ and n, we have

$$\begin{aligned}
&\big\|J_{\lambda_n}^B\big(\alpha_n\gamma f(x_n) + (1-\alpha_n)Sx_n - \lambda_n ASx_n\big) - z\big\| \\
= \ & \left\|J_{\lambda_n}^B\left(\alpha_n\gamma f(x_n) + (1-\alpha_n)\left(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\right)\right)\right. \\
&\left. - J_{\lambda_n}^B\left(\alpha_n z + (1-\alpha_n)\left(z - \frac{\lambda_n}{1-\alpha_n}Az\right)\right)\right\| \\
\leq \ & \left\|\left(\alpha_n\gamma f(x_n) + (1-\alpha_n\left(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\right)\right)\right. \\
&\left. - \left(\alpha_n z + (1-\alpha_n)\left(z - \frac{\lambda_n}{1-\alpha_n}Az\right)\right)\right\|^2 \\
= \ & \left\|(1-\alpha_n)\left(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n - \left(z - \frac{\lambda_n}{1-\alpha_n}Az\right) + \alpha_n(\gamma f(x_n) - z)\right\|\right. \qquad (3.11) \\
\leq \ & (1-\alpha_n)\|x_n - z\| + \alpha_n\|\gamma f(x_n) - \gamma f(z)\| + \alpha_n\|\gamma f(z) - z\| \\
\leq \ & [1 - (1-\gamma\rho)\alpha_n]\|x_n - z\| + \alpha_n\|\gamma f(z) - z\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \beta_n\|x_n - z\| + (1-\beta_n)[1 - (1-\gamma\rho)\alpha_n]\|x_n - z\| + (1-\beta_n)\alpha_n\|\gamma f(z) - z\| \\
&= [1 - (1-\gamma\rho)\alpha_n(1-\beta_n)]\|x_n - z\| + (1-\beta_n)\alpha_n\|\gamma f(z) - z\|.
\end{aligned}$$

By induction, we have

$$\|x_{n+1} - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{1-\gamma\rho}\|\gamma f(z) - z\|\right\}.$$

Therefore, $\{x_n\}$ is bounded. Since A is α-inverse strongly monotone, it is $\frac{1}{\alpha}$-Lipschitz continuous. We deduce immediately that $\{f(x_n)\}$, $\{Sx_n\}$ and $\{ASx_n\}$ are also bounded. Set $u_n = \alpha_n\gamma f(x_n) + (1-\alpha_n)Sx_n - \lambda_n ASx_n$ and $y_n = J_{\lambda_n}^B u_n$ for all $n \geq 0$. Noticing that $J_{\lambda_n}^B$ is nonexpansive, we can check easily that $\{u_n\}$ and $\{y_n\}$ are bounded.

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, from (3.11), we derive

$$\begin{aligned}
&\left\|(1-\alpha_n)\left(\left(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\right) - \left(z - \frac{\lambda_n}{1-\alpha_n}Az\right)\right) + \alpha_n(\gamma f(x_n) - z)\right\|^2 \\
\leq \ & (1-\alpha_n)\left\|\left(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\right) - \left(z - \frac{\lambda_n}{1-\alpha_n}Az\right)\right\|^2 + \alpha_n\|\gamma f(x_n) - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\left\|(Sx_n - z) - \frac{\lambda_n}{1-\alpha_n}(ASx_n - Az)\right\|^2 + \alpha_n\|\gamma f(x_n) - z\|^2 \\
&= (1-\alpha_n)\left(\|Sx_n - z\|^2 - \frac{2\lambda_n}{1-\alpha_n}\langle ASx_n - Az, Sx_n - z\rangle + \frac{\lambda_n^2}{(1-\alpha_n)^2}\|ASx_n - Az\|^2\right) \\
&\quad +\alpha_n\|\gamma f(x_n) - z\|^2 \\
&\le (1-\alpha_n)\left(\|x_n - z\|^2 - \frac{2\alpha\lambda_n}{1-\alpha_n}\|ASx_n - Az\|^2 + \frac{\lambda_n^2}{(1-\alpha_n)^2}\|ASx_n - Az\|^2\right) \\
&\quad +\alpha_n\|\gamma f(x_n) - z\|^2 \\
&= (1-\alpha_n)\left(\|x_n - z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2}(\lambda_n - 2(1-\alpha_n)\alpha)\|ASx_n - Az\|^2\right) \\
&\quad +\alpha_n\|\gamma f(x_n) - z\|^2. \qquad (3.12)
\end{aligned}
$$

By condition (iii), we get

$$\lambda_n - 2(1-\alpha_n)\alpha \le 0$$

for all $n \ge 0$. Then, from (3.11) and (3.12), we obtain

$$
\begin{aligned}
\left\|J_{\lambda_n}^B u_n - z\right\|^2 &\le (1-\alpha_n)\left(\|x_n - z\|^2 + \frac{\lambda_n}{(1-\alpha_n)^2}(\lambda_n - 2(1-\alpha_n)\alpha)\|ASx_n - Az\|^2\right) \\
&\quad +\alpha_n\|\gamma f(x_n) - z\|^2. \qquad (3.13)
\end{aligned}
$$

From (3.10), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\beta_n(x_n - z) + (1-\beta_n)\big(J_{\lambda_n}^B u_n - z\big)\right\|^2 \\
&\le \beta_n\|x_n - z\|^2 + (1-\beta_n)\left\|J_{\lambda_n}^B u_n - z\right\|^2. \qquad (3.14)
\end{aligned}
$$

We can rewrite (3.10) as $x_{n+1} = \beta_n x_n + (1-\beta_n)y_n$ for all $n \ge 0$. Next, we estimate $\|x_{n+1} - x_n\|$. In fact, we have

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \left\|J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_n}^B u_n\right\| \\
&\le \left\|J_{\lambda_{n+1}}^B u_{n+1} - J_{\lambda_{n+1}}^B u_n\right\| + \left\|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\right\| \\
&\le \left\|\big(\alpha_{n+1}\gamma f(x_{n+1}) + (1-\alpha_{n+1})Sx_{n+1} - \lambda_{n+1}ASx_{n+1}\big)\right. \\
&\quad \left. -\big(\alpha_n\gamma f(x_n) + (1-\alpha_n)Sx_n - \lambda_n ASx_n\big)\right\| + \left\|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\right\| \\
&= \|(I-\lambda_{n+1}A)Sx_{n+1} - (I-\lambda_{n+1}A)Sx_n + (\lambda_n - \lambda_{n+1})ASx_n \\
&\quad +\alpha_n(Sx_n - \gamma f(x_n)) - \alpha_{n+1}(Sx_{n+1} - \gamma f(x_{n+1}))\| + \|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\| \\
&\le \|(I-\lambda_{n+1}A)Sx_{n+1} - (I-\lambda_{n+1}A)Sx_n\| + |\lambda_{n+1} - \lambda_n|\|ASx_n\| \\
&\quad +\alpha_n\|Sx_n - \gamma f(x_n)\| + \alpha_{n+1}\|Sx_{n+1} - \gamma f(x_{n+1})\| + \|J_{\lambda_{n+1}}^B u_n - J_{\lambda_n}^B u_n\|.
\end{aligned}
$$

Since $I - \lambda_{n+1}A$ is nonexpansive for $\lambda_{n+1} \in (0, 2\alpha)$, we have

$$\|(I-\lambda_{n+1}A)Sx_{n+1} - (I-\lambda_{n+1}A)Sx_n\|$$

$$\leq \|Sx_{n+1} - Sx_n\| \leq \|x_{n+1} - x_n\|.$$

By the resolvent identity (2.1), we have

$$J^B_{\lambda_{n+1}} u_n = J^B_{\lambda_n}\bigg(\frac{\lambda_n}{\lambda_{n+1}} u_n + \Big(1 - \frac{\lambda_n}{\lambda_{n+1}}\Big) J^B_{\lambda_{n+1}} u_n\bigg).$$

It follows that

$$\begin{aligned}
&\|J^B_{\lambda_{n+1}} u_n - J^B_{\lambda_n} u_n\| \\
= &\left\| J^B_{\lambda_n}\bigg(\frac{\lambda_n}{\lambda_{n+1}} u_n + (1 - \frac{\lambda_n}{\lambda_{n+1}}) J^B_{\lambda_{n+1}} u_n\bigg) - J^B_{\lambda_n} u_n\right\| \\
\leq &\left\| \bigg(\frac{\lambda_n}{\lambda_{n+1}} u_n + (1 - \frac{\lambda_n}{\lambda_{n+1}}) J^B_{\lambda_{n+1}} u_n\bigg) - u_n\right\| \\
\leq &\frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|u_n - J^B_{\lambda_{n+1}} u_n\|.
\end{aligned}$$

So,

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq\ & \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|ASx_n\| + \alpha_n \|Sx_n - \gamma f(x_n)\| \\
&+\alpha_{n+1}\|Sx_{n+1} - \gamma f(x_{n+1})\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|u_n - J^B_{\lambda_{n+1}} u_n\|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq\ & |\lambda_{n+1} - \lambda_n| \|ASx_n\| + \alpha_n \|Sx_n - \gamma f(x_n)\| \\
&+\alpha_{n+1}\|Sx_{n+1} - \gamma f(x_{n+1})\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} \|u_n - J^B_{\lambda_{n+1}} u_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $\lambda_{n+1} - \lambda_n \to 0$ and $\liminf_{n\to\infty} \lambda_n > 0$, we obtain

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.4, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.15}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|y_n - x_n\| = 0.$$

From (3.13) and (3.14), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
\leq\ &\beta_n \|x_n - z\|^2 + (1 - \beta_n)\big\|J^B_{\lambda_n} u_n - z\big\|^2 \\
\leq\ &(1 - \beta_n)\bigg\{(1 - \alpha_n)\bigg(\|x_n - z\|^2 + \frac{\lambda_n}{(1 - \alpha_n)^2}(\lambda_n - 2(1 - \alpha_n)\alpha)\|ASx_n - Az\|^2\bigg) \\
&+\alpha_n \|\gamma f(x_n) - z\|^2\bigg\} + \beta_n \|x_n - z\|^2 \\
=\ &[1 - (1 - \beta_n)\alpha_n]\|x_n - z\|^2 + \frac{(1 - \beta_n)\lambda_n}{1 - \alpha_n}(\lambda_n - 2(1 - \alpha_n)\alpha)\|ASx_n - Az\|^2
\end{aligned}$$

$$
\begin{aligned}
&+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
\leq\ & \|x_n-z\|^2+\frac{(1-\beta_n)\lambda_n}{1-\alpha_n}(\lambda_n-2(1-\alpha_n)\alpha)\|ASx_n-Az\|^2 \\
&+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&\frac{(1-\beta_n)\lambda_n}{1-\alpha_n}(2(1-\alpha_n)\alpha-\lambda_n)\|ASx_n-Az\|^2 \\
\leq\ & \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
\leq\ & (\|x_n-z\|-\|x_{n+1}-z\|)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2.
\end{aligned}
$$

Since

$$\lim_{n\to\infty}\alpha_n=0,$$

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$$

and

$$\liminf_{n\to\infty}\frac{(1-\beta_n)\lambda_n}{1-\alpha_n}(2(1-\alpha_n)\alpha-\lambda_n)>0,$$

we have

$$\lim_{n\to\infty}\|ASx_n-Az\|=0. \tag{3.16}$$

Next, we show $\|x_n-Sx_n\|\to 0$. By using the firm nonexpansivity of $J^B_{\lambda_n}$, we have

$$
\begin{aligned}
\big\|J^B_{\lambda_n}u_n-z\big\|^2 =\ & \big\|J^B_{\lambda_n}\big(\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_nASx_n\big)-J^B_{\lambda_n}\big(z-\lambda_nAz\big)\big\|^2 \\
\leq\ & \big\langle\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_nASx_n-(z-\lambda_nAz),J^B_{\lambda_n}u_n-z\big\rangle \\
=\ & \frac{1}{2}\big(\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_nASx_n-(z-\lambda_nAz)\|^2+\big\|J^B_{\lambda_n}u_n-z\big\|^2 \\
&-\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_n(ASx_n-Az)-J^B_{\lambda_n}u_n\big\|^2\big).
\end{aligned}
$$

From condition (iii) and the α-inverse strongly monotonicity of A, we know that $I-\lambda_nA/(1-\alpha_n)$ is nonexpansive. Hence

$$
\begin{aligned}
&\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-\lambda_nASx_n-(z-\lambda_nAz)\|^2 \\
=\ & \left\|(1-\alpha_n)\left(Sx_n-\frac{\lambda_n}{1-\alpha_n}ASx_n-\left(z-\frac{\lambda_n}{1-\alpha_n}Az\right)\right)+\alpha_n(\gamma f(x_n)-z)\right\|^2 \\
\leq\ & (1-\alpha_n)\left\|Sx_n-\frac{\lambda_n}{1-\alpha_n}ASx_n-\left(z-\frac{\lambda_n}{1-\alpha_n}Az\right)\right\|^2+\alpha_n\|\gamma f(x_n)-z\|^2 \\
\leq\ & (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|\gamma f(x_n)-z\|^2.
\end{aligned}
$$

Thus,

$$\big\|J^B_{\lambda_n}u_n-z\big\|^2$$

$$\begin{aligned}
&\leq \frac{1}{2}\big((1-\alpha_n)\|x_n-z\|^2+\alpha_n\|\gamma f(x_n)-z\|^2+\big\|J^B_{\lambda_n}u_n-z\big\|^2 \\
&\quad -\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n-\lambda_n(ASx_n-Az)\big\|^2\big).
\end{aligned}$$

That is,

$$\begin{aligned}
&\big\|J^B_{\lambda_n}u_n-z\big\|^2 \\
\leq\ & (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|\gamma f(x_n)-z\|^2 \\
& -\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n-\lambda_n(ASx_n-Az)\big\|^2 \\
=\ & (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|\gamma f(x_n)-z\|^2-\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|^2 \\
& +2\lambda_n\big\langle\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n, ASx_n-Az\big\rangle-\lambda_n^2\|ASx_n-Az\|^2 \\
\leq\ & (1-\alpha_n)\|x_n-z\|^2+\alpha_n\|\gamma f(x_n)-z\|^2-\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|^2 \\
& +2\lambda_n\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|\|ASx_n-Az\|.
\end{aligned}$$

This together with (3.14) imply that

$$\begin{aligned}
\|x_{n+1}-z\|^2 \ \leq\ & \beta_n\|x_n-z\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-z\|^2+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
& -(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|^2 \\
& +2\lambda_n(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|\|ASx_n-Az\| \\
=\ & [1-(1-\beta_n)\alpha_n]\|x_n-z\|^2+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
& -(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|^2 \\
& +2\lambda_n(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|\|ASx_n-Az\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
& (1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|^2 \\
\leq\ & \|x_n-z\|^2-\|x_{n+1}-z\|^2-(1-\beta_n)\alpha_n\|x_n-z\|^2+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
& +2\lambda_n(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|\|ASx_n-Az\| \\
\leq\ & (\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|\gamma f(x_n)-z\|^2 \\
& +2\lambda_n(1-\beta_n)\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|\|ASx_n-Az\|.
\end{aligned}$$

Since $\limsup_{n\to\infty}\beta_n<1$, $\|x_{n+1}-x_n\|\to 0$, $\alpha_n\to 0$ and $\|ASx_n-Az\|\to 0$ (by (3.16)), we deduce

$$\lim_{n\to\infty}\big\|\alpha_n\gamma f(x_n)+(1-\alpha_n)Sx_n-J^B_{\lambda_n}u_n\big\|=0.$$

This implies that

$$\lim_{n\to\infty}\big\|Sx_n-J^B_{\lambda_n}u_n\big\|=0. \tag{3.17}$$

Combining (3.15) and (3.17), we get

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{3.18}$$

Put $\tilde{x} = \lim_{t\to 0+} x_t = P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x}))$ where x_t is the net defined by (1). We will finally show that $x_n \to \tilde{x}$.

Setting $v_n = x_n - \frac{\lambda_n}{1-\alpha_n}(ASx_n - A\tilde{x})$ for all n. Taking $z = \tilde{x}$ in (3.16) to get $\|ASx_n - A\tilde{x}\| \to 0$. First, we prove $\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_n - \tilde{x}\rangle \le 0$. We take a subsequence $\{Sx_{n_i}\}$ of $\{Sx_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_n - \tilde{x}\rangle = \lim_{i\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_{n_i} - \tilde{x}\rangle.$$

It is clear that $\{Sx_{n_i}\}$ is bounded due to the boundedness of $\{Sx_n\}$ and $\|ASx_n - A\tilde{x}\| \to 0$. Then, there exists a subsequence $\{Sx_{n_{i_j}}\}$ of $\{Sx_{n_i}\}$ which converges weakly to some point $w \in C$. Hence, $\{x_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ also converge weakly to w because of $\|Sx_{n_{i_j}} - x_{n_{i_j}}\| \to 0$ and $\|x_{n_{i_j}} - y_{n_{i_j}}\| \to 0$. By the demi-closedness principle of the nonexpansive mapping (see Lemma 2.2) and (3.18), we deduce $w \in F(S)$. Furthermore, by the similar argument as that of Theorem 3.1, we can show that w is also in $(A+B)^{-1}0$. Hence, we have $w \in F(S)\cap(A+B)^{-1}0$. This implies that

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_n - \tilde{x}\rangle = \lim_{j\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_{n_{i_j}} - \tilde{x}\rangle = \langle \gamma f(\tilde{x}) - \tilde{x}, w - \tilde{x}\rangle.$$

Note that $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(\gamma f(\tilde{x}))$. Then, $\langle \gamma f(\tilde{x}) - \tilde{x}, w - \tilde{x}\rangle \le 0, w \in F(S)\cap(A+B)^{-1}0$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_n - \tilde{x}\rangle \le 0.$$

From (3.10), we have

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
\le\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|J^B_{\lambda_n}u_n - \tilde{x}\|^2 \\
=\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|J^B_{\lambda_n}u_n - J^B_{\lambda_n}(\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
\le\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|u_n - (\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
=\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|\alpha_n\gamma f(x_n) + (1-\alpha_n)Sx_n - \lambda_n ASx_n - (\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
=\ & (1-\beta_n)\Bigg\|(1-\alpha_n)\Bigg(\Big(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\Big) - \Big(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\Big)\Bigg) \\
&+\alpha_n(\gamma f(x_n) - \tilde{x})\Bigg\|^2 + \beta_n\|x_n - \tilde{x}\|^2 \\
=\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\Bigg((1-\alpha_n)^2\Bigg\|\Big(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\Big) - \Big(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\Big)\Bigg\|^2 \\
&+2\alpha_n(1-\alpha_n)\Big\langle \gamma f(x_n) - \tilde{x}, \Big(Sx_n - \frac{\lambda_n}{1-\alpha_n}ASx_n\Big) - \Big(\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x}\Big)\Big\rangle \\
&+\alpha_n^2\|\gamma f(x_n) - \tilde{x}\|^2\Bigg) \\
\le\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\big((1-\alpha_n)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n\lambda_n\langle \gamma f(x_n) - \tilde{x}, ASx_n - A\tilde{x}\rangle \\
&+2\alpha_n(1-\alpha_n)\gamma\langle f(x_n) - f(\tilde{x}), Sx_n - \tilde{x}\rangle + 2\alpha_n(1-\alpha_n)\langle \gamma f(\tilde{x}) - \tilde{x}, Sx_n - \tilde{x}\rangle \\
&+\alpha_n^2\|\gamma f(x_n) - \tilde{x}\|^2\big) \\
\le\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\big((1-\alpha_n)^2\|x_n - \tilde{x}\|^2 + 2\alpha_n\lambda_n\|\gamma f(x_n) - \tilde{x}\|\|ASx_n - A\tilde{x}\|
\end{aligned}$$

$$
\begin{aligned}
&+2\alpha_n(1-\alpha_n)\gamma\rho\|x_n-\tilde{x}\|^2+2\alpha_n(1-\alpha_n)\langle\gamma f(\tilde{x})-\tilde{x},Sx_n-\tilde{x}\rangle+\alpha_n^2\|\gamma f(x_n)-\tilde{x}\|^2\big)\\
\le\;& [1-2(1-\beta_n)(1-\gamma\rho)\alpha_n]\|x_n-\tilde{x}\|^2+2\alpha_n(1-\beta_n)\lambda_n\|\gamma f(x_n)-\tilde{x}\|\|ASx_n-A\tilde{x}\|\\
&+2\alpha_n(1-\beta_n)(1-\alpha_n)\langle\gamma f(\tilde{x})-\tilde{x},Sx_n-\tilde{x}\rangle+(1-\beta_n)\alpha_n^2(\|\gamma f(x_n)-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2)\\
=\;& [1-2(1-\beta_n)(1-\gamma\rho)\alpha_n]\|x_n-\tilde{x}\|^2\\
&+2(1-\beta_n)(1-\gamma\rho)\alpha_n\Big\{\frac{\lambda_n}{1-\gamma\rho}\|\gamma f(x_n)-\tilde{x}\|\|ASx_n-A\tilde{x}\|\\
&+\frac{1-\alpha_n}{1-\gamma\rho}\langle\gamma f(\tilde{x})-\tilde{x},Sx_n-\tilde{x}\rangle+\frac{\alpha_n}{2(1-\gamma\rho)}(\|\gamma f(x_n)-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2)\Big\}.
\end{aligned}
$$

It is clear that $\sum_n 2(1-\beta_n)(1-\gamma\rho)\alpha_n=\infty$ and

$$
\begin{aligned}
&\limsup_{n\to\infty}\Big\{\frac{\lambda_n}{1-\gamma\rho}\|\gamma f(x_n)-\tilde{x}\|\|ASx_n-A\tilde{x}\|+\frac{1-\alpha_n}{1-\gamma\rho}\langle\gamma f(\tilde{x})-\tilde{x},Sx_n-\tilde{x}\rangle\\
&+\frac{\alpha_n}{2(1-\gamma\rho)}(\|\gamma f(x_n)-\tilde{x}\|^2+\|x_n-\tilde{x}\|^2)\Big\}\le 0.
\end{aligned}
$$

We can therefore apply Lemma 2.5 to conclude that $x_n\to\tilde{x}$. This completes the proof. □

Corollary 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H. Let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ and let S be a nonexpansive mapping of C into itself, such that $F(S)\cap(A+B)^{-1}0\neq\emptyset$. Let λ be a constant satisfying $a\le\lambda\le b$ where $[a,b]\subset(0,2\alpha)$. For $t\in\big(0,1-\frac{\lambda}{2\alpha}\big)$, let $\{x_t\}\subset C$ be a net generated by*

$$x_t=J_\lambda^B\big((1-t)Sx_t-\lambda ASx_t\big).$$

Then the net $\{x_t\}$ converges strongly, as $t\to 0+$, to a point $\tilde{x}=P_{F(S)\cap(A+B)^{-1}0}(0)$ which is the minimum norm element in $F(S)\cap(A+B)^{-1}0$.

Corollary 3.4. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ such that $(A+B)^{-1}0\neq\emptyset$. Let λ be a constant satisfying $a\le\lambda\le b$ where $[a,b]\subset(0,2\alpha)$. For $t\in(0,1-\frac{\lambda}{2\alpha})$, let $\{x_t\}\subset C$ be a net generated by*

$$x_t=J_\lambda^B\big((1-t)x_t-\lambda Ax_t\big).$$

Then the net $\{x_t\}$ converges strongly, as $t\to 0+$, to a point $\tilde{x}=P_{(A+B)^{-1}0}(0)$ which is the minimum norm element in $(A+B)^{-1}0$.

Corollary 3.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly-monotone mapping of C into H. Let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B=(I+\lambda B)^{-1}$ be the resolvent of B for $\lambda>0$ and let S be a nonexpansive mapping of C into itself, such that $F(S)\cap(A+B)^{-1}0\neq\emptyset$. For given $x_0\in C$, let $\{x_n\}\subset C$ be a sequence generated by*

$$x_{n+1}=\beta_nx_n+(1-\beta_n)J_{\lambda_n}^B\big((1-\alpha_n)Sx_n-\lambda_nASx_n\big)$$

for all $n\ge 0$, where $\{\lambda_n\}\subset(0,2\alpha)$,$\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset(0,1)$ satisfy

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_n\alpha_n=\infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a,b] \subset (0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ *converges strongly to a point* $\tilde{x} = P_{F(S)\cap(A+B)^{-1}0}(0)$ *which is the minimum norm element in* $F(S) \cap (A+B)^{-1}0$.

Corollary 3.6. *Let* C *be a closed and convex subset of a real Hilbert space* H. *Let* A *be an* α*-inverse strongly-monotone mapping of* C *into* H *and let* B *be a maximal monotone operator on* H, *such that the domain of* B *is included in* C. *Let* $J_\lambda^B = (I+\lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$ *such that* $(A+B)^{-1}0 \neq \emptyset$. *For given* $x_0 \in C$, *let* $\{x_n\} \subset C$ *be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\lambda_n}^B\big((1-\alpha_n)x_n - \lambda_n A x_n\big)$$

for all $n \geq 0$, *where* $\{\lambda_n\} \subset (0,2\alpha)$,$\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a,b] \subset (0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ *converges strongly to a point* $\tilde{x} = P_{(A+B)^{-1}0}(0)$ *which is the minimum norm element in* $(A+B)^{-1}0$.

Remark 3.7. The present paper provides some methods which do not use projection for finding the minimum norm solution problem.

4. Applications

Next, we consider the problem for finding the minimum norm solution of a mathematical model related to equilibrium problems. Let C be a nonempty, closed and convex subset of a Hilbert space and let $G : C \times C \to R$ be a bifunction satisfying the following conditions:

(E1) $G(x,x) = 0$ for all $x \in C$;
(E2) G is monotone, i.e., $G(x,y) + G(y,x) \leq 0$ for all $x,y \in C$;
(E3) for all $x,y,z \in C$, $\limsup_{t\downarrow 0} G(tz+(1-t)x, y) \leq G(x,y)$;
(E4) for all $x \in C$, $G(x,\cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\tilde{x} \in C$ such that

$$G(\tilde{x}, y) \geq 0 \tag{4.1}$$

for all $y \in C$. The set of such solutions $\tilde{x}$ is denoted by $EP(G)$. The following lemma appears implicitly in Blum and Oettli [5]:

Lemma 4.1. *Let* C *be a nonempty, closed and convex subset of* H *and let* G *be a bifunction of* $C \times C$ *into* R *satisfying (E1)-(E4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$G(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma was given in Combettes and Hirstoaga [8]:

Lemma 4.2. *Assume that* $G : C \times C \to R$ *satisfies* (E1)-(E4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \Big\{z \in C : G(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\Big\}$$

for all $x \in H$. *Then, the following hold:*

(1) T_r *is single-valued;*
(2) T_r *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$*,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(G)$;
(4) $EP(G)$ *is closed and convex.*

We call such T_r the resolvent of G for $r > 0$. Using Lemmas 4.1 and 4.2, we have the following lemma. See [1] for a more general result.

Lemma 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $G : C \times C \to R$ satisfy* (E1)-(E4)*. Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{\{z \in H : G(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then, $EP(G) = A_G^{-1}(0)$ and A_G is a maximal monotone operator with $dom(A_G) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of A_G; i.e.,

$$T_r x = (I + rA_G)^{-1}x.$$

Form Lemma 4.3, Theorems 3.1 and 3.2, we have the following results.

Theorem 4.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying* (E1)-(E4) *and let T_r be the resolvent of G for $r > 0$. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap EP(G) \neq \emptyset$. For $t \in (0, 1)$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = T_r\big((1 - t)Sx_t\big), \quad t \in (0, 1).$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{F(S)\cap EP(G)}(0)$ which is the minimum norm element in $F(S) \cap EP(G)$.

Corollary 4.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying* (E1)-(E4) *and let T_r be the resolvent of G for $r > 0$. Suppose $EP(G) \neq \emptyset$. For $t \in (0, 1)$, let $\{x_t\} \subset C$ be a net generated by*

$$x_t = T_r\big((1 - t)x_t\big), \quad t \in (0, 1).$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = P_{EP(G)}(0)$ which is the minimum norm element in $EP(G)$.

Theorem 4.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying* (E1)-(E4) *and let T_λ be the resolvent of G for $\lambda > 0$. Let S be a nonexpansive mapping from C into itself such that $F(S) \cap EP(G) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T_{\lambda_n}\big((1 - \alpha_n)Sx_n\big)$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, \infty)$,$\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $a \leq \lambda_n \leq b$ *where* $[a, b] \subset (0, \infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = P_{F(S)\cap EP(G)}(0)$ which is the minimum norm element in $F(S) \cap EP(G)$.

Corollary 4.7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying* (E1)-(E4) *and let T_λ be the resolvent of G for $\lambda > 0$. Suppose $EP(G) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{\lambda_n}\big((1-\alpha_n)x_n\big)$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,\infty)$,$\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $a \leq \lambda_n \leq b$ *where* $[a, b] \subset (0,\infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = P_{EP(G)}(0)$ which is the minimum norm element in $EP(G)$.

Acknowledgements

Li-Jun Zhu was supported in part by NSFC of China (61362033) and NZ13087.

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