Journal of Nonlinear Science and Applications

Some formulas for the generalized Apostol-type polynomials and numbers

Wen-Kai Shao[a], Yuan He[b,*]

[a]Department of Mathematical Teaching and Research, Yibin Vocational & Technical College, 644003 Yibin, Sichuan, People's Republic of China.

[b]Faculty of Science, Kunming University of Science and Technology, 650500 Kunming, Yunnan, People's Republic of China.

Communicated by S. H. Rim

Abstract

In this paper, we perform a further investigation for the unified family of the generalized Apostol-Bernoulli, Euler and Genocchi polynomials and numbers introduced by El-Desouky and Gomaa (2014). By using the generating function methods and summation transform techniques, we establish some new formulas for this family of polynomials and numbers, and give some illustrative special cases. ©2016 All rights reserved.

Keywords: Apostol-Bernoulli polynomials and numbers, Apostol-Euler polynomials and numbers, Apostol-Genocchi polynomials and numbers, combinatorial identities.
2010 MSC: 11B68, 05A19.

1. Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{C}$ and $\mathbb{C}_0$ denotes the set of natural numbers, the set of non-negative integers, the set of complex numbers and the set of complex number excluding zero, respectively. For $\alpha, \lambda \in \mathbb{C}$, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ together with the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x; \lambda)$, the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x; \lambda)$ are defined by the following generating functions (see, e.g., [16, 18, 19]):

$$\left(\frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \tag{1.1}$$

*Corresponding author
Email addresses: `wksh_0@163.com` (Wen-Kai Shao), `hyyhe@aliyun.com,hyyhe@outlook.com` (Yuan He)

$$(|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log \lambda| \text{ when } \lambda \neq 1; 1^\alpha := 1),$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \tag{1.2}$$

$$(|t| < \pi \text{ when } \lambda = 1; |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1; 1^\alpha := 1)$$

and

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \tag{1.3}$$

$$(|t| < \pi \text{ when } \lambda = 1; |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1; 1^\alpha := 1).$$

Hence, the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given by

$$B_n(x) = \mathcal{B}_n^{(1)}(x; 1), \quad E_n(x) = \mathcal{E}_n^{(1)}(x; 1) \quad \text{and} \quad G_n(x) = \mathcal{G}_n^{(1)}(x; 1), \tag{1.4}$$

respectively. Meanwhile, the case $\alpha = 1$ in (1.1), (1.2) and (1.3) gives the Apostol-Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$, the Apostol-Euler polynomials $\mathcal{E}_n(x; \lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n(x; \lambda)$, respectively. In particular, $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0; \lambda)$, $\mathcal{E}_n(\lambda) = 2^n \mathcal{E}_n(1/2; \lambda)$ and $\mathcal{G}_n(\lambda) = \mathcal{G}_n(0; \lambda)$ are called the Apostol-Bernoulli numbers, the Apostol-Euler numbers and the Apostol-Genocchi numbers, respectively. Since the publication of the above works by Luo and Srivastava, numerous properties for the generalized Apostol-Bernoulli polynomials, Euler and Genocchi polynomials have been studied. We refer to [5, 7, 8, 15, 17, 20, 21, 25] for some related results on these Apostol-type polynomials and numbers.

In [24, 26], Ozden et al., constructed the following generating function:

$$\frac{2^{1-k} t^k e^{xt}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x; k, a, b) \frac{t^n}{n!}, \tag{1.5}$$

$$\left(|t| < 2\pi \text{ when } \beta = a; |t| < \log\left(\frac{\beta}{a}\right) \text{ when } \beta \neq a; k, \beta \in \mathbb{C}; a, b \in \mathbb{C}_0\right).$$

It is obvious that the polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$ can be regarded as a generalization and unification of the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Apostol-Genocchi polynomials. In [23], Ozarslan further gave an extension of (1.5) in the following ways:

$$\left(\frac{2^{1-k} t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x; k, a, b) \frac{t^n}{n!}, \tag{1.6}$$

$$\left(|t| < 2\pi \text{ when } \beta = a; |t| < \log\left(\frac{\beta}{a}\right) \text{ when } \beta \neq a; k, \beta \in \mathbb{C}; a, b \in \mathbb{C}_0; 1^\alpha := 1\right).$$

Recently, El-Desouky and Gomaa [6] introduced a new unified family $M_n^{(r)}(x; k, \overline{\alpha}_r)$ of the generalized Apostol-Bernoulli, Euler and Genocchi polynomials associated with a sequence of complex numbers $\overline{\alpha}_r = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ by the following generating function:

$$\frac{(-1)^r t^{rk} 2^{r(1-k)} e^{xt}}{(1 - \alpha_0 e^t)(1 - \alpha_1 e^t) \cdots (1 - \alpha_{r-1} e^t)} = \sum_{n=0}^{\infty} M_n^{(r)}(x; k, \overline{\alpha}_r) \frac{t^n}{n!}, \tag{1.7}$$

$$(|t| < 2\pi \text{ when } \alpha_i = 1; |t| < |\log(\alpha_i)| \text{ when } \alpha_i \neq 1, \forall i = 0, 1, \ldots, r-1; k, r \in \mathbb{C}),$$

and showed that the generating function in (1.7) can be used to give many types of polynomials including the above mentioned polynomials as special cases. Moreover, they discovered some interesting relationships between the unified family of polynomials $M_n^{(r)}(x; k, \overline{\alpha}_r)$, the unified family of numbers $M_n^{(r)}(k, \overline{\alpha}_r) =$

$M_n^{(r)}(0; k, \overline{\alpha}_r)$, the Stirling numbers, the generalized Laguerre polynomials, the Bernstein polynomials and the generalized Eulerian polynomials.

Motivated and inspired by the work of El-Desouky and Gomaa [6], we perform a further investigation for the unified family of polynomials $M_n^{(r)}(x; k, \overline{\alpha}_r)$ and the unified family of numbers $M_n^{(r)}(k, \overline{\alpha}_r)$. By using the generating function methods and summation transform techniques, we establish some new formulas for this family of polynomials and numbers, and give some illustrative special cases.

2. The statements of results

Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ be n indeterminates. We apply the familiar partial fraction decomposition and let

$$\frac{1}{(1 - a_1 z)(1 - a_2 z) \cdots (1 - a_n z)} = \sum_{k=1}^{n} \frac{G(k, l)}{1 - a_k z}. \tag{2.1}$$

To determine the coefficients $G(k, l)$ in (2.1). By multiplying both sides in the preceding identity by $(1 - a_k z)$ and take $z \to 1/a_k$, we obtain

$$G(k, l) = \lim_{z \to \frac{1}{a_k}} \frac{1 - a_k z}{(1 - a_1 z)(1 - a_2 z) \cdots (1 - a_n z)} = \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 - \frac{a_j}{a_k}\right)^{-1}. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\frac{1}{(1 - a_1 z)(1 - a_2 z) \cdots (1 - a_n z)} = \sum_{k=1}^{n} \frac{1}{1 - a_k z} \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 - \frac{a_j}{a_k}\right)^{-1}. \tag{2.3}$$

If we replace z by e^t, a_i by α_{i-1} for $1 \le i \le n$, and n by r in (2.3), we discover

$$\frac{(-1)^r t^{rk} 2^{r(1-k)} e^{xt}}{(1 - \alpha_0 e^t)(1 - \alpha_1 e^t) \cdots (1 - \alpha_{r-1} e^t)} = \sum_{i=0}^{r-1} \frac{(-1)^r t^{rk} 2^{r(1-k)} e^{xt}}{1 - \alpha_i e^t} \prod_{\substack{j=0 \\ j \neq i}}^{r-1} \left(1 - \frac{a_j}{a_i}\right)^{-1}, \tag{2.4}$$

which means

$$\frac{(-1)^r t^{rk} 2^{r(1-k)} e^{xt}}{(1 - \alpha_0 e^t)(1 - \alpha_1 e^t) \cdots (1 - \alpha_{r-1} e^t)} = \sum_{i=0}^{r-1} t^{(r-1)k} \frac{-t^k 2^{1-k} e^{xt}}{1 - \alpha_i e^t} \prod_{\substack{j=0 \\ j \neq i}}^{r-1} 2^{1-k} \left(\frac{a_j}{a_i} - 1\right)^{-1}. \tag{2.5}$$

If we apply (1.7) to (2.5) then we have

$$\sum_{n=0}^{\infty} M_n^{(r)}(x; k, \overline{\alpha}_r) \frac{t^n}{n!} = \sum_{i=0}^{r-1} \sum_{n=0}^{\infty} M_n^{(1)}(x; k, \alpha_i) \frac{t^{n+(r-1)k}}{n!} \prod_{\substack{j=0 \\ j \neq i}}^{r-1} 2^{1-k} \left(\frac{a_j}{a_i} - 1\right)^{-1}. \tag{2.6}$$

Thus, by comparing the coefficients of $t^{n+(r-1)k}/n!$ in (2.6), we get the following result.

Theorem 2.1. *Let* $k, n \in \mathbb{N}_0$. *Then, for* $r \in \mathbb{N}$,

$$M_{n+(r-1)k}^{(r)}(x; k, \overline{\alpha}_r) = \frac{(n + (r-1)k)!}{n!} \sum_{i=0}^{r-1} M_n^{(1)}(x; k, \alpha_i) \prod_{\substack{j=0 \\ j \neq i}}^{r-1} 2^{1-k} \left(\frac{\alpha_j}{\alpha_i} - 1\right)^{-1},$$

where $\alpha_i \neq \alpha_j$ *when* $i \neq j$, $\forall i, j = 0, 1, \ldots, r - 1$.

We now recall Euler's elementary and beautiful idea in the discovery of his famous Pentagonal Number Theorem: for infinite number of complex numbers $x_1, x_2, x_3, \cdots$, (see, e.g., [1, 3])

$$(1 + x_1)(1 + x_2)(1 + x_3) \cdots = (1 + x_1) + x_2(1 + x_1) + x_3(1 + x_1)(1 + x_2) + \cdots. \tag{2.7}$$

We shall make use of the finite form of (2.7) to establish another formulas for the unified family of polynomials $M_n^{(r)}(x; k, \overline{\alpha}_r)$. Clearly, the finite form of (2.7) can be written as

$$
\begin{aligned}
(1 + x_1)(1 + x_2)(1 + x_3) \cdots (1 + x_r) =& (1 + x_1) + x_2(1 + x_1) + x_3(1 + x_1)(1 + x_2) \\
&+ \cdots + x_r(1 + x_1)(1 + x_2) \cdots (1 + x_{r-1}) \quad (r \in \mathbb{N}).
\end{aligned} \tag{2.8}
$$

It is trivial to see that for $1 \le i \le r$, substituting $x_i - 1$ for x_i in (2.8) gives

$$x_1 \cdots x_r - 1 = \sum_{i=1}^{r}(x_i - 1)x_1 \cdots x_{i-1}, \tag{2.9}$$

where the product $x_1 \cdots x_{i-1}$ is considered to be equal to 1 when $i = 1$. If we take $x_i = \alpha_{i-1}e^t$ for $1 \le i \le r$ in (2.9) then we have

$$\alpha_0\alpha_1 \cdots \alpha_{r-1}e^{rt} - 1 = \sum_{i=0}^{r-1}(\alpha_i e^t - 1)\prod_{j=0}^{i-1}\alpha_j e^t. \tag{2.10}$$

It follows from (2.10) that

$$\frac{1}{(1 - \alpha_0 e^t)(1 - \alpha_1 e^t) \cdots (1 - \alpha_{r-1}e^t)} = \frac{1}{\alpha_0\alpha_1 \cdots \alpha_{r-1}e^{rt} - 1}\sum_{i=0}^{r-1}\prod_{l=0}^{i-1}\frac{\alpha_l e^t}{\alpha_l e^t - 1}\prod_{l=i+1}^{r-1}\frac{1}{\alpha_l e^t - 1}. \tag{2.11}$$

Hence, multiplying both sides of the above identity by $t^{rk}2^{r(1-k)}e^{xt}$ gives

$$\frac{(-1)^r t^{rk}2^{r(1-k)}e^{xt}}{(1 - \alpha_0 e^t)(1 - \alpha_1 e^t) \cdots (1 - \alpha_{r-1}e^t)} = \frac{1}{r^k} \cdot \frac{-(rt)^k 2^{1-k}e^{\frac{x}{r} \cdot rt}}{1 - \alpha_0\alpha_1 \cdots \alpha_{r-1}e^{rt}}\sum_{i=0}^{r-1}\prod_{l=0}^{i-1}\alpha_l \frac{-t^k 2^{1-k}e^t}{1 - \alpha_l e^t}\prod_{l=i+1}^{r-1}\frac{-t^k 2^{1-k}}{1 - \alpha_l e^t}. \tag{2.12}$$

If we apply (1.7) to (2.12), we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} M_n^{(r)}(x; k, \overline{\alpha}_r)\frac{t^n}{n!} =& \frac{1}{r^k}\sum_{i=0}^{r-1}\sum_{j_i=0}^{\infty} M_{j_i}^{(1)}\left(\frac{x}{r}; k, \alpha_0\alpha_1 \cdots \alpha_{r-1}\right)\frac{(rt)^{j_i}}{j_i!} \\
&\times \prod_{l=0}^{i-1}\alpha_l \sum_{j_l=0}^{\infty} M_{j_l}^{(1)}(1; k, \alpha_l)\frac{t^{j_l}}{j_l!}\prod_{l=i+1}^{r-1}\sum_{j_l=0}^{\infty} M_{j_l}^{(1)}(k, \alpha_l)\frac{t^{j_l}}{j_l!},
\end{aligned} \tag{2.13}
$$

which together with the Cauchy product yields

$$
\begin{aligned}
\sum_{n=0}^{\infty} M_n^{(r)}(x; k, \overline{\alpha}_r)\frac{t^n}{n!} =& \frac{1}{r^k}\sum_{i=0}^{r-1}\sum_{n=0}^{\infty}\left(\sum_{\substack{j_0+j_1+\cdots+j_{r-1}=n \\ j_0,j_1,\ldots,j_{r-1}\ge 0}}\binom{n}{j_0, j_1, \ldots, j_{r-1}}r^{j_i} \right. \\
&\left. \times M_{j_i}^{(1)}\left(\frac{x}{r}; k, \alpha_0\alpha_1 \cdots \alpha_{r-1}\right)\prod_{l=0}^{i-1}\alpha_l M_{j_l}^{(1)}(1; k, \alpha_l)\prod_{l=i+1}^{r-1} M_{j_l}^{(1)}(k, \alpha_l)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.14}
$$

where $\binom{n}{r_1, r_2, \ldots, r_k}$ denotes by the multinomial coefficient given by

$$\binom{n}{r_1, r_2 \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdots r_k!} \quad (n, r_1, r_2, \ldots, r_k \in \mathbb{N}_0). \tag{2.15}$$

Thus, by comparing the coefficients of $t^n/n!$ in (2.14), we obtain the following result.

Theorem 2.2. *Let $k, n \in \mathbb{N}_0$ and $r \in \mathbb{N}$. If $\beta = \alpha_0 \alpha_1 \cdots \alpha_{r-1}$ then*

$$M_n^{(r)}(x; k, \overline{\alpha}_r) = \frac{1}{r^k} \sum_{i=0}^{r-1} \sum_{\substack{j_0 + j_1 + \cdots + j_{r-1} = n \\ j_0, j_1, \ldots, j_{r-1} \geq 0}} \binom{n}{j_0, j_1, \ldots, j_{r-1}} r^{j_i} M_{j_i}^{(1)}\left(\frac{x}{r}; k, \beta\right)$$

$$\times \prod_{l=0}^{i-1} \alpha_l M_{j_l}^{(1)}(1; k, \alpha_l) \prod_{l=i+1}^{r-1} M_{j_l}^{(1)}(k, \alpha_l).$$

It follows that we show some special cases of Theorem 2.2. Since $M_n^{(r)}(x; k, \overline{\alpha}_r)$ satisfies the difference equation (see, e.g., [6, Theorem 3.3]):

$$\alpha_{r-1} M_n^{(r)}(x+1; k, \overline{\alpha}_r) - M_n^{(r)}(x; k, \overline{\alpha}_r) = 2^{1-k} \frac{n!}{(n-k)!} M_{n-k}^{(r-1)}(x; k, \overline{\alpha}_{r-1}), \tag{2.16}$$

which implies the following difference equation for $M_n^{(1)}(x; k, \alpha)$:

$$\alpha M_n^{(1)}(x+1; k, \alpha) - M_n^{(1)}(x; k, \alpha) = 2^{1-k} x^{n-k} \frac{n!}{(n-k)!}. \tag{2.17}$$

So from (2.17), we have

$$\prod_{l=0}^{i-1} \alpha_l M_{j_l}^{(1)}(x+1; k, \alpha_l) = \sum_{T \subseteq \{0, 1, \ldots, i-1\}} \prod_{l \in T} M_{j_l}^{(1)}(x; k, \alpha_l) \prod_{l \in \overline{T}} 2^{1-k} x^{j_l - k} \frac{j_l!}{(j_l - k)!}. \tag{2.18}$$

Thus, by taking $x = 0$ in (2.18), in light of Theorem 2.2, we get the following result.

Corollary 2.3. *Let $k, n \in \mathbb{N}_0$. Then, for $r \in \mathbb{N}$ with $n \geq k(r-1)$,*

$$M_n^{(r)}(x; k, \overline{\alpha}_r) = \frac{1}{r^k} \sum_{i=0}^{r-1} \binom{r}{i} 2^{(1-k)(r-1-i)} \sum_{\substack{j_0 + j_1 + \cdots + j_i = n - k(r-1-i) \\ j_0, j_1, \ldots, j_i \geq 0}} \binom{n}{j_0, j_1, \ldots, j_i} r^{j_i}$$

$$\times M_{j_i}^{(1)}\left(\frac{x}{r}; k, \alpha_0 \alpha_1 \cdots \alpha_{r-1}\right) \prod_{l=0}^{i-1} M_{j_l}^{(1)}(k, \alpha_l).$$

It becomes obvious that the case $x = 0$ in Corollary 2.3 gives a relationship for the unified family of the generalized Apostol-Bernoulli, Euler and Genocchi numbers as follows,

$$M_n^{(r)}(k, \overline{\alpha}_r) = \frac{1}{r^k} \sum_{i=0}^{r-1} \binom{r}{i} 2^{(1-k)(r-1-i)} \sum_{\substack{j_0 + j_1 + \cdots + j_i = n - k(r-1-i) \\ j_0, j_1, \ldots, j_i \geq 0}} \binom{n}{j_0, j_1, \ldots, j_i} r^{j_i}$$

$$\times M_{j_i}^{(1)}(k, \alpha_0 \alpha_1 \cdots \alpha_{r-1}) \prod_{l=0}^{i-1} M_{j_l}^{(1)}(k, \alpha_l). \tag{2.19}$$

If we take $r = 2$ in Corollary 2.3 then for $k, n \in \mathbb{N}_0$ with $0 \leq k \leq n$,

$$M_n^{(2)}(x; k, \overline{\alpha}_2) = 2^{n+1-3k} \binom{n}{k} M_{n-k}^{(1)}\left(\frac{x}{2}; k, \alpha_0 \alpha_1\right) + 2^{1-k} \sum_{i=0}^{n} \binom{n}{i} 2^i M_i^{(1)}\left(\frac{x}{2}; k, \alpha_0 \alpha_1\right) M_{n-i}^{(1)}(k, \alpha_0). \tag{2.20}$$

On the other hand, let $r \in \mathbb{N}$. If we take $\alpha_i = (\frac{\beta}{a})^b$ for $0 \leq i \leq r - 1$ in (1.7), one can get that for $n \in \mathbb{N}_0$,

$$M_n^{(r)}\left(x; k, \left(\frac{\beta}{a}\right)^b\right) = a^{br} \mathcal{Y}_{n,\beta}^{(r)}(x; k, a, b). \tag{2.21}$$

It follows from Corollary 2.3 and (2.21) that

$$\mathcal{Y}_{n,\beta}^{(r)}(x;k,a,b) = \frac{1}{r^k} \sum_{i=0}^{r-1} \binom{r}{i} 2^{(1-k)(r-1-i)} a^{bi} \sum_{\substack{j_0+j_1+\cdots+j_i=n-k(r-1-i) \\ j_0,j_1,\ldots,j_i \geq 0}} \binom{n}{j_0,j_1,\ldots,j_i} r^{j_i}$$

$$\times \mathcal{Y}_{j_i,\beta^r}\left(\frac{x}{r};k,a^r,b\right) \prod_{l=0}^{i-1} \mathcal{Y}_{j_l,\beta}(0;k,a,b). \tag{2.22}$$

If we take $k=1$ and $\alpha_i = \lambda$ for $0 \leq i \leq r-1$ in (1.7), one can get that for $n \in \mathbb{N}_0$,

$$M_n^{(r)}(x;1,\lambda) = \mathcal{B}_n^{(r)}(x;\lambda). \tag{2.23}$$

Hence, by applying (2.23) to Corollary 2.3, we obtain

$$\mathcal{B}_n^{(r)}(x;\lambda) = \frac{1}{r} \sum_{i=0}^{r-1} \binom{r}{i} \sum_{\substack{j_0+j_1+\cdots+j_i=n-(r-1-i) \\ j_0,j_1,\ldots,j_i \geq 0}} \binom{n}{j_0,j_1,\ldots,j_i} r^{j_i} \mathcal{B}_{j_i}\left(\frac{x}{r};\lambda^r\right) \prod_{l=0}^{i-1} \mathcal{B}_{j_l}(\lambda). \tag{2.24}$$

The case $\lambda = 1$ in (2.24) gives the corresponding expression for the classical Bernoulli polynomials of order r ($r \in \mathbb{N}$) defined by Nörlund [22]. See also [4, 9, 10] for a further exposition on the classical Bernoulli polynomials of order r. If we take $k=0$ and $\alpha_i = -\lambda$ for $0 \leq i \leq r-1$ in (1.7), one can get that for $n \in \mathbb{N}_0$,

$$M_n^{(r)}(x;0,-\lambda) = (-1)^r \mathcal{E}_n^{(r)}(x;\lambda), \tag{2.25}$$

which together with Corollary 2.3 yields

$$\mathcal{E}_n^{(r)}(x;\lambda) = \sum_{i=0}^{r-1} \binom{r}{i} (-2)^{r-1-i} \sum_{\substack{j_0+j_1+\cdots+j_i=n \\ j_0,j_1,\ldots,j_i \geq 0}} \binom{n}{j_0,j_1,\ldots,j_i} r^{j_i} \mathcal{E}_{j_i}\left(\frac{x}{r};(-1)^{r-1}\lambda^r\right) \prod_{l=0}^{i-1} \mathcal{E}_{j_l}(0,\lambda). \tag{2.26}$$

In a similar consideration, if we take $k=1$ and $\alpha_i = -\lambda$ for $0 \leq i \leq r-1$ in (1.7), one can get that for $n \in \mathbb{N}_0$,

$$M_n^{(r)}(x;1,-\lambda) = \left(\frac{-1}{2}\right)^r \mathcal{G}_n^{(r)}(x;\lambda). \tag{2.27}$$

Thus, applying (2.27) to Corollary 2.3 gives

$$\mathcal{G}_n^{(r)}(x;\lambda) = \frac{1}{r} \sum_{i=0}^{r-1} \binom{r}{i} (-2)^{r-1-i} \sum_{\substack{j_0+j_1+\cdots+j_i=n-(r-1-i) \\ j_0,j_1,\ldots,j_i \geq 0}} \binom{n}{j_0,j_1,\ldots,j_i} r^{j_i} \tag{2.28}$$

$$\times \mathcal{G}_{j_i}\left(\frac{x}{r};(-1)^{r-1}\lambda^r\right) \prod_{l=0}^{i-1} \mathcal{G}_{j_l}(\lambda). \tag{2.29}$$

In the following we present some formulas between the unified family of numbers $M_n^{(r)}(k,\overline{\alpha}_r)$ and the generalized Eulerian polynomials considered by Araci et al., [2]. We take $x_i = \alpha_{i-1}e^{t\ln b}$ for $1 \leq i \leq r$ in (2.9) to obtain

$$\alpha_0\alpha_1\cdots\alpha_{r-1}e^{rt\ln b} - 1 = \sum_{i=0}^{r-1}(\alpha_i e^{t\ln b} - 1)\prod_{j=0}^{i-1}\alpha_j e^{t\ln b}. \tag{2.30}$$

It follows from (2.30) that

$$\prod_{l=0}^{r-1}\frac{1}{\alpha_l e^{t\ln b} - 1} = \frac{1}{\alpha_0\alpha_1\cdots\alpha_{r-1}e^{rt\ln b} - 1}\sum_{i=0}^{r-1}\prod_{l=0}^{i-1}\frac{\alpha_l e^{t\ln b}}{\alpha_l e^{t\ln b} - 1} \times \prod_{l=i+1}^{r-1}\frac{1}{\alpha_l e^{t\ln b} - 1}. \tag{2.31}$$

By multiplying both sides of the above identity by $(t \ln b)^{rk} 2^{r(1-k)}$, we get

$$
\frac{(-1)^r (t \ln b)^{rk} 2^{r(1-k)}}{(1 - \alpha_0 e^{t \ln b})(1 - \alpha_1 e^{t \ln b}) \cdots (1 - \alpha_{r-1} e^{t \ln b})}
$$
$$
= \frac{(t \ln b)^{rk} 2^{r(1-k)}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} e^{rt \ln b} - 1} \sum_{i=0}^{r-1} (-1)^i \prod_{l=0}^{i-1} \frac{1}{\frac{1}{\alpha_l} e^{-t \ln b} - 1} \prod_{l=i+1}^{r-1} \frac{1}{\alpha_l e^{t \ln b} - 1}. \tag{2.32}
$$

Observe that

$$
\frac{1}{\alpha e^{t \ln b} - 1} = \frac{1}{\alpha - 1} \cdot \frac{1 - \frac{1}{\alpha}}{e^{\frac{\alpha}{\alpha - 1} t \ln b (1 - \frac{1}{\alpha})} - \frac{1}{\alpha}}. \tag{2.33}
$$

Hence, with the help of (2.33), (2.32) can be rewritten as

$$
\frac{(-1)^r (t \ln b)^{rk} 2^{r(1-k)}}{(1 - \alpha_0 e^{t \ln b})(1 - \alpha_1 e^{t \ln b}) \cdots (1 - \alpha_{r-1} e^{t \ln b})}
$$
$$
= \frac{(t \ln b)^{rk} 2^{r(1-k)}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} \cdot \frac{1 - \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}}{e^{\frac{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} rt \ln b (1 - \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{r-1}})} - \frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}}
$$
$$
\times \sum_{i=0}^{r-1} (-1)^i \prod_{l=0}^{i-1} \frac{1}{\frac{1}{\alpha_l} - 1} \cdot \frac{1 - \alpha_l}{e^{\frac{-1}{1 - \alpha_l} t \ln b (1 - \alpha_l)} - \alpha_l}
$$
$$
\times \prod_{l=i+1}^{r-1} \frac{1}{\alpha_l - 1} \cdot \frac{1 - \frac{1}{\alpha_l}}{e^{\frac{\alpha_l}{\alpha_l - 1} t \ln b (1 - \frac{1}{\alpha_l})} - \frac{1}{\alpha_l}}. \tag{2.34}
$$

Since the generalized Eulerian polynomials $A_n(a, b)$ can be expressed by the generating function (see, e.g., [2])

$$
\frac{1 - a}{e^{t(1-a) \ln b} - a} = \sum_{n=0}^{\infty} A_n(a, b) \frac{t^n}{n!}, \tag{2.35}
$$

so by applying (1.7) and (2.35) to (2.34), we have

$$
\sum_{n=0}^{\infty} M_n^{(r)}(k, \overline{\alpha}_r) \frac{(\ln b)^n t^{n-rk}}{n!} = \frac{2^{r(1-k)} (\ln b)^{rk}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} \sum_{i=0}^{r-1} (-1)^i \sum_{j_i=0}^{\infty} A_{j_i} \left(\frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}, b \right) \frac{\left(\frac{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} rt \right)^{j_i}}{j_i!}
$$
$$
\times \prod_{l=0}^{i-1} \frac{1}{\frac{1}{\alpha_l} - 1} \sum_{j_l=0}^{\infty} A_{j_l}(\alpha_l, b) \frac{\left(\frac{-1}{1 - \alpha_l} t \right)^{j_l}}{j_l!} \times \prod_{l=i+1}^{r-1} \frac{1}{\alpha_l - 1} \sum_{j_l=0}^{\infty} A_{j_l} \left(\frac{1}{\alpha_l}, b \right) \frac{\left(\frac{\alpha_l}{\alpha_l - 1} t \right)^{j_l}}{j_l!}, \tag{2.36}
$$

It follows from (2.36) and the Cauchy product that

$$
\sum_{n=0}^{\infty} M_n^{(r)}(k, \overline{\alpha}_r) \frac{(\ln b)^n t^{n-rk}}{n!} = \frac{2^{r(1-k)} (\ln b)^{rk}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} \sum_{i=0}^{r-1} (-1)^i \sum_{n=0}^{\infty} \left(\sum_{\substack{j_0 + j_1 + \cdots + j_{r-1} = n \\ j_0, j_1, \ldots, j_{r-1} \geq 0}} \binom{n}{j_0, j_1, \ldots, j_{r-1}} r^{j_i} \right.
$$
$$
\times \left(\frac{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}{\alpha_0 \alpha_1 \cdots \alpha_{r-1} - 1} \right)^{j_i} A_{j_i} \left(\frac{1}{\alpha_0 \alpha_1 \cdots \alpha_{r-1}}, b \right)
$$
$$
\times \prod_{l=0}^{i-1} \frac{1}{\frac{1}{\alpha_l} - 1} \left(\frac{1}{\alpha_l - 1} \right)^{j_l} A_{j_l}(\alpha_l, b)
$$
$$
\left. \times \prod_{l=i+1}^{r-1} \frac{1}{\alpha_l - 1} \left(\frac{\alpha_l}{\alpha_l - 1} \right)^{j_l} A_{j_l} \left(\frac{1}{\alpha_l}, b \right) \right) \frac{t^n}{n!}. \tag{2.37}
$$

Thus, by comparing the coefficients of $t^n / n!$ in (2.37), we get the following result.

Theorem 2.4. *Let* $k, n \in \mathbb{N}_0$ *and* $r \in \mathbb{N}$. *If* $\beta = \alpha_0 \alpha_1 \cdots \alpha_{r-1}$ *then*

$$
M^{(r)}_{n+rk}(k, \overline{\alpha}_r) = \frac{(n+rk)! \cdot 2^{r(1-k)}}{n! \cdot (\ln b)^n} \sum_{i=0}^{r-1} \sum_{\substack{j_0+j_1+\cdots+j_{r-1}=n \\ j_0,j_1,\ldots,j_{r-1} \geq 0}} \binom{n}{j_0, j_1, \ldots, j_{r-1}} r^{j_i}
$$

$$
\times \frac{\beta^{j_i}}{(\beta-1)^{j_i+1}} A_{j_i}\left(\frac{1}{\beta}, b\right) \prod_{l=0}^{i-1} \frac{\alpha_l}{(\alpha_l-1)^{j_l+1}} A_{j_l}(\alpha_l, b) \prod_{l=i+1}^{r-1} \frac{\alpha_l^{j_l}}{(\alpha_l-1)^{j_l+1}} A_{j_l}\left(\frac{1}{\alpha_l}, b\right).
$$

It is easily seen that if we take $\alpha_i = -1$ for $0 \leq i \leq r-1$ in Theorem 2.4 then

$$
M^{(r)}_{n+rk}(k, \overline{\alpha}_r) = \frac{(-1)^{n+r}}{2^{n+rk}} \cdot \frac{(n+rk)!}{n! \cdot (\ln b)^n} \sum_{i=0}^{r-1} \sum_{\substack{j_0+j_1+\cdots+j_{r-1}=n \\ j_0,j_1,\ldots,j_{r-1} \geq 0}} \binom{n}{j_0, j_1, \ldots, j_{r-1}} r^{j_i}
$$

$$
\times (-1)^{j_i} A_{j_i}(-1, b) \prod_{l=0}^{i-1} -A_{j_l}(-1, b) \prod_{l=i+1}^{r-1} (-1)^{j_l} A_{j_l}(-1, b) \quad (2 \nmid r).
\tag{2.38}
$$

Note that for non-negative integer n (see, e.g., [2])

$$
A_n(-1, b) = \frac{2^{n+1}(\ln b)^{n+1}(1 - 2^{n+1})}{n+1} B_{n+1},
\tag{2.39}
$$

and

$$
A_n(-1, b) = \frac{2^{n+1}(\ln b)^{n+1}}{n+1} G_{n+1}.
\tag{2.40}
$$

It follows from (2.38), (2.39) and (2.40) that the unified family of the generalized Apostol-Bernoulli, Euler and Genocchi numbers due to El-Desouky and Gomaa [6] can be explicitly expressed by the classical Bernoulli numbers or the classical Genocchi numbers.

3. Concluding remarks

To conclude this paper, we remark that the topics explored in this paper are closely related to the Frobenius-Euler polynomials of order r $(r \in \mathbb{N})$ given by the generating function:

$$
\left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{xt} = e^{H^{(r)}(x|\lambda)t} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!},
$$

$$
(|t| < \pi \text{ when } \lambda = -1; \ |t| < \log(1/\lambda) \text{ when } \lambda \neq -1),
$$

with the usual convention about replacing $(H^{(r)}(x|\lambda))^n$ by $H_n^{(r)}(x|\lambda)$ (see, e.g., [11, 12, 13, 14]). In particular, the case $x = 0$ in the above generating function gives the Frobenius-Euler numbers of order r denoted by $H_n^{(r)}(\lambda) = H_n^{(r)}(0|\lambda)$. If one uses the same methods described in the second section, the corresponding similar results on the Frobenius-Euler polynomials and numbers of order r to Theorems 2.1, 2.2, 2.4 and Corollary 2.3 can be also established. The details are left as further investigations for the interested readers.

Acknowledgements

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