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Solution and stability of a reciprocal type functional equation in several variables

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Abstract

In this paper, we obtain the general solution and investigate the generalized Hyers-Ulam stability of a reciprocal type functional equation in several variables of the form

$$\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}\left[\prod_{j=2,j\neq i}^{m} r(x_1+x_j)\right]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)\left[\prod_{j=2,j\neq i}^{m} r(x_j)\right] + (m-1)\prod_{i=2}^{m} r(x_i)}$$

where m is a positive integer with $m \geq 3$. ©2014 All rights reserved.

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1. Introduction

About seventy years ago, Ulam [13] raised the well known stability problem of functional equations. In the next year, Ulam's problem was partially answered by Hyers [4] in Banach spaces. T. Aoki [1] generalized Hyers' theorem for additive mappings in the year 1950. In the year 1978, a generalized version of the theorem of Hyers for approximately linear mappings was given by Th.M. Rassias [12]. During 1982-1989, J.M. Rassias ([5], [6], [7]) treated the Ulam-Gavruta-Rassias stability on linear and non-linear mappings and generalized

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Hyers' result. In 1994, a further generalization of the Th.M. Rassias' theorem was obtained by P. Gavruta [3], who replaced the bound $\theta\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\phi(x,y)$.

In the year 2010, K. Ravi and B.V. Senthil Kumar [8] investigated the generalized Hyers-Ulam stability for the reciprocal functional equation

$$r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.1}$$

where $r : X \to Y$ is a mapping on the spaces of non-zero real numbers. The reciprocal function $r(x) = \frac{c}{x}$ is a solution of the functional equation (1.1).
Later, K. Ravi, J.M. Rassias and B.V. Senthil Kumar [10] introduced the reciprocal difference functional equation

$$r\left(\frac{x+y}{2}\right) - r(x+y) = \frac{r(x)r(y)}{r(x)+r(y)} \tag{1.2}$$

and the reciprocal adjoint functional equation

$$r\left(\frac{x+y}{2}\right) + r(x+y) = \frac{3r(x)r(y)}{r(x)+r(y)} \tag{1.3}$$

and investigated the generalized Hyers-Ulam stability for the above two functional equations (1.2) and (1.3).
Recently, K. Ravi, J.M. Rassias and B.V. Senthil Kumar [9] discussed the generalized Hyers-Ulam stability for the generalized reciprocal functional equation

$$r\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=1}^{m}\left[\alpha_i\left(\prod_{j=1, j\neq i}^{m} r(x_j)\right)\right]} \tag{1.4}$$

for arbitrary but fixed real numbers $\alpha_i \neq 0$ for $i = 1, 2, \ldots, m$, so that $0 < \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m = \sum_{i=1}^{m} \alpha_i \neq 1$ and $r : X \to Y$ with X and Y are the sets of non-zero real numbers.
Very recently, K. Ravi, E. Thandapani and B.V. Senthil Kumar [11] obtained the general solution and investigated the generalized Hyers-Ulam stability for the reciprocal type functional equations

$$r((k_1-k_2)x + (k_1-k_2)y) = \frac{r(k_1x-k_2y)r(k_1y-k_2x)}{r(k_1x-k_2y)+r(k_1y-k_2x)} \tag{1.5}$$

where k_1 and k_2 are any integers with $k_1 \neq k_2$ and

$$r((k_1+k_2)x + (k_1+k_2)y) = \frac{r(k_1x+k_2y)r(k_1y+k_2x)}{r(k_1x+k_2y)+r(k_1y+k_2x)} \tag{1.6}$$

where k_1 and k_2 are any integers with $k_1 \neq -k_2$.
In this paper, we obtain the solution and investigate the generalized Hyers-Ulam stability for a reciprocal type functional equation in several variables of the form

$$\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}\left[\prod_{j=2, j\neq i}^{m} r(x_1+x_j)\right]} = \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)\left[\prod_{j=2, j\neq i}^{m} r(x_j)\right] + (m-1)\prod_{i=2}^{m} r(x_i)} \tag{1.7}$$

where m is a positive integer with $m \geq 3$.
Throughout this paper, we assume that X is the set of non-zero real numbers. For convenience, we define the difference operator $D_m r : X^m \to \mathbb{R}$ such that

$$D_m r(x_1, x_2, \ldots, x_m) = \frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=2}^{m}\left[\prod_{j=2, j\neq i}^{m} r(x_1+x_j)\right]} - \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)\left[\prod_{j=2, j\neq i}^{m} r(x_j)\right] + (m-1)\prod_{i=2}^{m} r(x_i)}$$

for $x_1, x_2, \dots, x_m \in X$.
In the following results, we will set $\frac{0^{m-1}}{0^{m-2}} = 0$ for $m \geq 3$ and assume

$$\sum_{i=2}^{m}\left[\prod_{j=2,j\neq i}^{m} r(x_1+x_j)\right] \neq 0, \sum_{i=2}^{m} r(x_1)\left[\prod_{j=2,j\neq i}^{m} r(x_j)\right] + (m-1)\prod_{i=2}^{m} r(x_i) \neq 0$$

for all $x_i \in X$; $i = 1, 2, \dots, m; m \geq 3$ and $x_1 \neq -x_i$, for all $i; 2 \leq i \leq m; m \geq 3$.

2. General solution of functional equation (1.7)

Theorem 2.1. *A mapping $r : X \to \mathbb{R}$ satisfies the functional equation (1.7) for all $x_1, x_2, \dots, x_m \in X$ if and only if there exists a reciprocal mapping $r : X \to \mathbb{R}$ satisfying the reciprocal functional equation (1.1) for all $x, y \in X$.*

Proof. Let the mapping $r : X \to \mathbb{R}$ satisfy the functional equation (1.7). Replacing x_1 by x and x_i by y for $i = 2, 3 \dots, m$ in (1.7), we arrive (1.1).

Conversely, let the mapping $r : X \to \mathbb{R}$ satisfy the functional equation (1.1). Replacing (x, y) by $(x_1, x_2 + x_3)$ in (1.1), we obtain

$$\begin{aligned} r(x_1+x_2+x_3) &= \frac{r(x_1)r(x_2+x_3)}{r(x_1)+r(x_2+x_3)} \\ &= \frac{r(x_1)r(x_2)r(x_3)}{r(x_1)r(x_2)+r(x_1)r(x_3)+r(x_2)r(x_3)} \\ &= \frac{\prod_{i=1}^{3} r(x_i)}{\sum_{i=2}^{3} r(x_1)\left[\prod_{j=2,j\neq i}^{3} r(x_j)\right] + \prod_{i=2}^{3} r(x_i)} \end{aligned}$$

for all $x_1, x_2, x_3 \in X$. Using induction on a positive integer $m-1$, we have

$$r(x_1+x_2+\cdots+x_{m-1}) = \frac{\prod_{i=1}^{m-1} r(x_i)}{\sum_{i=2}^{m-1} r(x_1)\left[\prod_{j=2,j\neq i}^{m-1} r(x_j)\right] + \prod_{i=2}^{m-1} r(x_i)} \tag{2.1}$$

for all $x_1, x_2, \dots, x_{m-1} \in X$. Now, replacing x_i by x for $i = 1, 2, \dots, m-1$ in (2.1), we get $r((m-1)x) = \frac{1}{m-1} r(x)$, for all $x \in X$. Replacing x_i by x_{i+1} for $i = 1, 2, \dots, m-1$ in (2.1), we obtain

$$r(x_2+x_3+\cdots+x_m) = \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)\left[\prod_{j=3,j\neq i}^{m} r(x_j)\right] + \prod_{i=3}^{m} r(x_i)} \tag{2.2}$$

for all $x_2, x_3, \dots, x_m \in X$. Now, replacing x_i by $x + x_i$ for $i = 2, 3 \dots, m$ in (2.2), we get

$$\begin{aligned} &\frac{\prod_{i=2}^{m} r(x_1+x_i)}{\sum_{i=3}^{m} r(x_1+x_2)\left[\prod_{j=3,j\neq i}^{m} r(x_1+x_j)\right] + \prod_{i=3}^{m} r(x_1+x_j)} \\ &= r((m-1)x_1+x_2+\cdots+x_m) = \\ &= \frac{\frac{1}{m-1} r(x_1) \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)\left[\prod_{j=3,j\neq i}^{m} r(x_j)\right]+\prod_{i=3}^{m} r(x_i)}}{\frac{1}{m-1} r(x_1) + \frac{\prod_{i=2}^{m} r(x_i)}{\sum_{i=3}^{m} r(x_2)\left[\prod_{j=3,j\neq i}^{m} r(x_j)\right]+\prod_{i=3}^{m} r(x_i)}} \\ &= \frac{\prod_{i=1}^{m} r(x_i)}{\sum_{i=2}^{m} r(x_1)\left[\prod_{j=2,j\neq i}^{m} r(x_j)\right] + (m-1)\prod_{i=2}^{m} r(x_i)} \end{aligned} \tag{2.3}$$

for all $x_1, x_2, \dots, x_m \in X$. On further simplification of the above equation (2.3) yields the equation (1.7). This completes the proof of Theorem 2.1. □

3. Generalized Hyers-Ulam stability of equation (1.7)

Theorem 3.1. *Let $\varphi : X^m \to \mathbb{R}$ be a function satisfying*

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi\left(\frac{x_1}{2^{i+1}}, \frac{x_2}{2^{i+1}}, \dots, \frac{x_m}{2^{i+1}}\right) < +\infty \tag{3.1}$$

for all $x_1, x_2, \dots, x_m \in X$. If a function $f : X \to \mathbb{R}$ satisfies the functional inequality

$$|D_m f(x_1, x_2, \dots, x_m)| \le \varphi(x_1, x_2, \dots, x_m) \tag{3.2}$$

for all $x_1, x_2, \dots, x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (1.7) and the inequality

$$|r(x) - f(x)| \le 2(m-1) \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \dots, \frac{x}{2^{i+1}}\right) \tag{3.3}$$

for all $x \in X$.

Proof. Replacing x_i by $\frac{x}{2}$ for $i = 1, 2, \dots, m$ in (3.2) and multiplying by $(m-1)$, we get

$$\left| f(x) - \frac{1}{2} f\left(\frac{x}{2}\right) \right| \le (m-1) \varphi\left(\frac{x}{2}, \frac{x}{2}, \dots, \frac{x}{2}\right) \tag{3.4}$$

for all $x \in X$. Now, replacing x by $\frac{x}{2}$ in (3.4), dividing by 2 and summing the resulting inequality with (3.4), we obtain

$$\left| f(x) - \frac{1}{2^2} f\left(\frac{x}{2^2}\right) \right| \le 2(m-1) \sum_{i=0}^{1} \frac{1}{2^{i+1}} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \dots, \frac{x}{2^{i+1}}\right)$$

for all $x \in X$. Proceeding further and using induction arguments on a positive integer n, we arrive

$$\left| f(x) - \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right| \le 2(m-1) \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \dots, \frac{x}{2^{i+1}}\right) \tag{3.5}$$

for all $x \in X$. For any positive integer i and $x \in X$, we have

$$\begin{aligned} \left| \frac{1}{2^{i+1}} f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i} f\left(\frac{x}{2^i}\right) \right| &= \frac{1}{2^i} \left| f\left(\frac{x}{2^i}\right) - \frac{1}{2} f\left(\frac{x}{2^{i+1}}\right) \right| \\ &\le 2(m-1) \frac{1}{2^{i+1}} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \dots, \frac{x}{2^{i+1}}\right). \end{aligned}$$

Hence for any integers l, k with $l > k > 0$, we obtain by using the triangular inequality

$$\begin{aligned} &\left| \frac{1}{2^l} f\left(\frac{x}{2^l}\right) - \frac{1}{2^k} f\left(\frac{x}{2^k}\right) \right| \\ &\quad = \left| \frac{1}{2^l} f\left(\frac{x}{2^l}\right) - \frac{1}{2^{l-1}} f\left(\frac{x}{2^{l-1}}\right) + \frac{1}{2^{l-1}} f\left(\frac{x}{2^{l-1}}\right) - \dots + \frac{1}{2^{k+1}} f\left(\frac{x}{2^{k+1}}\right) \right| \\ &\quad \le 2(m-1) \frac{1}{2^l} \varphi\left(\frac{x}{2^l}, \frac{x}{2^l}, \dots, \frac{x}{2^l}\right) + \dots + 2(m-1) \frac{1}{2^{k+1}} \varphi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, \dots, \frac{x}{2^{k+1}}\right) \\ &\quad \le 2(m-1) \sum_{i=k+1}^{l} \frac{1}{2^i} \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}, \dots, \frac{x}{2^i}\right) \\ &\quad \le 2(m-1) \sum_{i=k}^{l-1} \frac{1}{2^{i+1}} \varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \dots, \frac{x}{2^{i+1}}\right) \end{aligned} \tag{3.6}$$

for all $x \in X$. Taking the limit as $k \to +\infty$ in (3.6) and considering (3.1), it follows that the sequence $\{\frac{1}{2^n} f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for each $x \in X$. Since $\mathbb{R}$ is complete, we can define $r : X \to \mathbb{R}$ by $r(x) = \lim_{n\to\infty} \frac{1}{2^n} f\left(\frac{x}{2^n}\right)$. To show that r satisfies (1.7), replacing $(x_1, x_2, \ldots, x_m)$ by $(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_m)$ in (3.2) and dividing by 2^n, we obtain

$$|2^{-n} D_m f(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_m)| \le 2^{-n}\varphi(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_m) \tag{3.7}$$

for all $x_1, x_2, \ldots, x_m \in X$ and for all positive integer n. Using (3.1) and (3.5) in (3.7), we see that r satisfies (1.7), for all $x_1, x_2, \ldots, x_m \in X$. Taking limit $n \to \infty$ in (3.5), we arrive (3.3). Now, it remains to show that r is uniquely defined. Let $r_1 : X \to \mathbb{R}$ be another reciprocal mapping which satisfies (1.7) and the inequality (3.3). Clearly, $r_1(2^{-n}x) = 2^n r_1(x)$, $r(2^{-n}x) = 2^n r(x)$ and using (3.3), we arrive

$$\begin{aligned}
|r_1(x) - r(x)| &= 2^{-n}\left|r_1(2^{-n}x) - r(2^{-n}x)\right| \\
&\le 4(m-1)\sum_{i=0}^{\infty} \frac{1}{2^{n+i+1}}\varphi\left(\frac{x}{2^{n+i+1}}, \frac{x}{2^{n+i+1}}, \ldots, \frac{x}{2^{n+i+1}}\right) \\
&\le 4(m-1)\sum_{i=n}^{\infty} \frac{1}{2^{i+1}}\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, \ldots, \frac{x}{2^{i+1}}\right)
\end{aligned} \tag{3.8}$$

for all $x \in X$. Allowing $n \to \infty$ in (3.8), we find that r is unique. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $\varphi : X^m \to \mathbb{R}$ be a function satisfying*

$$\sum_{i=0}^{\infty} 2^i \varphi(2^i x_1, 2^i x_2, \ldots, 2^i x_m) < +\infty \tag{3.9}$$

for all $x_1, x_2, \ldots, x_m \in X$. If a function $f : X \to \mathbb{R}$ satisfies the functional inequality

$$|D_m f(x_1, x_2, \ldots, x_m)| \le \varphi(x_1, x_2, \ldots, x_m) \tag{3.10}$$

for all $x_1, x_2, \ldots, x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ which satisfies (1.7) and the inequality

$$|r(x) - f(x)| \le 2(m-1)\sum_{i=0}^{\infty} 2^i \varphi(2^i x, 2^i x, \ldots, 2^i x) \tag{3.11}$$

for all $x \in X$.

Proof. The proof is obtained by replacing x_i by x for $i = 1, 2, \ldots, m$ in (3.10) and proceeding further by similar arguments as in Theorem 3.1. □

Corollary 3.3. *For any fixed $c_1 \ge 0$ and $p > -1$ or $p < -1$, if $f : X \to \mathbb{R}$ satisfies*

$$|D_m f(x_1, x_2, \ldots, x_m)| \le c_1\left(\sum_{i=1}^{m} |x_i|^p\right)$$

for all $x_1, x_2, \ldots, x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ such that

$$|r(x) - f(x)| \le \begin{cases} \frac{2m(m-1)c_1}{2^{p+1}-1}|x|^p & \text{for } p > -1 \\ \frac{2m(m-1)c_1}{1-2^{p+1}}|x|^p & \text{for } p < -1 \end{cases}$$

for all $x \in X$.

Proof. If we choose $\varphi(x_1, x_2, \ldots, x_m) = c_1 \left(\sum_{i=1}^m |x_i|^p\right)$, for all $x_1, x_2, \ldots, x_m \in X$, then by Theorem 3.1, we arrive

$$|r(x) - f(x)| \leq \frac{2m(m-1)c_1}{2^{p+1} - 1}|x|^p, \text{ for all } x \in X \text{ and } p > -1$$

and using Theorem 3.2, we arrive

$$|r(x) - f(x)| \leq \frac{2m(m-1)c_1}{1 - 2^{p+1}}|x|^p, \text{ for all } x \in X \text{ and } p < -1.$$

□

Corollary 3.4. *Let $f : X \to \mathbb{R}$ be a mapping and there exists p such that $p > -1$ or $p < -1$. If there exists $c_2 \geq 0$ such that*

$$|D_m f(x_1, x_2, \ldots, x_m)| \leq c_2 \left(\prod_{i=1}^m |x_i|^{\frac{p}{m}}\right)$$

for all $x_1, x_2, \ldots, x_m \in X$, then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying the functional equation (1.7) and

$$|r(x) - f(x)| \leq \begin{cases} \frac{2(m-1)c_2}{2^{p+1}-1}|x|^p & \text{for } p > -1 \\ \frac{2(m-1)c_2}{1-2^{p+1}}|x|^p & \text{for } p < -1 \end{cases}$$

for all $x \in X$.

Proof. Considering $\varphi(x_1, x_2, \ldots, x_m) = c_2 \left(\prod_{i=1}^m |x_i|^{\frac{p}{m}}\right)$, for all $x_1, x_2, \ldots, x_m \in X$, then by Theorem 3.1, we arrive

$$|r(x) - f(x)| \leq \frac{2(m-1)c_2}{2^{p+1} - 1}|x|^p, \text{ for all } x \in X \text{ and } p > -1$$

and using Theorem 3.2, we arrive

$$|r(x) - f(x)| \leq \frac{2(m-1)c_2}{1 - 2^{p+1}}|x|^p, \text{ for all } x \in X \text{ and } p < -1.$$

□

Corollary 3.5. *Let $c_3 > 0$ and $\alpha > -\frac{1}{m}$ or $\alpha < -\frac{1}{m}$ be real numbers, and $f : X \to \mathbb{R}$ be a mapping satisfying the functional inequality*

$$|D_m f(x_1, x_2, \ldots, x_m)| \leq c_3 \left\{\sum_{i=1}^m |x_i|^{m\alpha} + \left(\prod_{i=1}^m |x_i|^{\alpha}\right)\right\}$$

for all $x_1, x_2, \ldots, x_m \in X$. Then there exists a unique reciprocal mapping $r : X \to \mathbb{R}$ satisfying the functional equation (1.7) and

$$|r(x) - f(x)| \leq \begin{cases} \frac{2(m-1)(m+1)c_3}{2^{m\alpha+1}-1}|x|^{m\alpha} & \text{for } \alpha > -\frac{1}{m} \\ \frac{2(m-1)(m+1)c_3}{1-2^{m\alpha+1}}|x|^{m\alpha} & \text{for } \alpha < -\frac{1}{m} \end{cases}$$

for all $x \in X$.

Proof. Choosing $\varphi(x_1, x_2, \ldots, x_m) = c_3 \left\{\sum_{i=1}^m |x_i|^{m\alpha} + \left(\prod_{i=1}^m |x_i|^{\alpha}\right)\right\}$, for all $x_1, x_2, \ldots, x_m \in X$, then by Theorem 3.1, we arrive

$$|r(x) - f(x)| \leq \frac{2(m-1)(m+1)c_3}{2^{m\alpha+1} - 1}|x|^{m\alpha}, \text{ for all } x \in X \text{ and } \alpha > -\frac{1}{m}$$

and using Theorem 3.2, we arrive

$$|r(x) - f(x)| \leq \frac{2(m-1)(m+1)c_3}{1 - 2^{m\alpha+1}}|x|^{m\alpha}, \text{ for all } x \in X \text{ and } \alpha < -\frac{1}{m}.$$

□

4. Counter-examples

The following example illustrates the fact that the functional equation (1.7) is not stable for $p = -1$ in Corollary 3.3. We present the following counter-example modified by the well-known counter-example of Z. Gajda [2].

Example 4.1. Let $\varphi : X \to \mathbb{R}$ be a mapping defined by

$$\varphi(x) = \begin{cases} \frac{\mu}{x} & \text{for } x \in (1, \infty) \\ \mu & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a mapping $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\varphi(2^{-n}x)}{2^n}, \text{ for all } x \in X.$$

Then the mapping f satisfies the inequality

$$|D_m f(x_1, x_2, \dots, x_m)| \leq \frac{6\mu}{m-1} \left(\sum_{i=1}^{m} |x_i|^{-1} \right) \tag{4.1}$$

for all $x_1, x_2, \dots, x_m \in X$. Therefore there do not exist a reciprocal mapping $r : X \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - r(x)| \leq \beta |x|^{-1} \tag{4.2}$$

for all $x \in X$.

Proof. $|f(x)| \leq \sum_{n=0}^{\infty} \frac{|\varphi(2^{-n}x)|}{|2^n|} \leq \sum_{n=0}^{\infty} \frac{\mu}{2^n} = \mu \left(1 - \frac{1}{2}\right)^{-1} = 2\mu$. Hence f is bounded by 2μ. If

$$\left(\sum_{i=1}^{m} |x_i|^{-1} \right) \geq 1,$$

then the left hand side of (4.1) is less than $\frac{6\mu}{m-1}$. Now, suppose that $0 < \left(\sum_{i=1}^{m} |x_i|^{-1}\right) < 1$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \leq \sum_{i=1}^{m} |x_i|^{-1} < \frac{1}{2^k}. \tag{4.3}$$

Hence $\sum_{i=1}^{m} |x_i|^{-1} < \frac{1}{2^k}$ implies

$$\begin{aligned} & 2^k \sum_{i=1}^{m} |x_i|^{-1} < 1 \\ \text{or} \quad & \frac{x_i}{2^k} > 1 \text{ for } i = 1, 2, \dots, m \\ \text{or} \quad & \frac{x_i}{2^k} > 1 > \frac{1}{2} \text{ for } i = 1, 2, \dots, m \\ \text{or} \quad & \frac{x_i}{2^{k-1}} > 2 > 1 \text{ for } i = 1, 2, \dots, m \end{aligned}$$

and consequently

$$\frac{1}{2^{k-1}}(x_1), \frac{1}{2^{k-1}}(x_i), \frac{1}{2^{k-1}}(x_1 + x_i) > 1 \text{ for } i = 2, 3 \dots, m.$$

Therefore, for each value of $n = 0, 1, 2, \dots, k-1$, we obtain

$$\frac{1}{2^n}(x_1), \frac{1}{2^n}(x_i), \frac{1}{2^n}(x_1 + x_i) > 1 \text{ for } i = 2, 3, \dots, m$$

and $D_m\varphi(2^{-n}x_1, 2^{-n}x_2, \ldots, 2^{-n}x_m) = 0$ for $n = 0, 1, 2, \ldots, k-1$. Using (4.3) and the definition of f, we obtain

$$\begin{aligned} |D_m f(x_1, x_2, \ldots, x_m)| &= \Bigg| \frac{\prod_{i=2}^m f(x_1 + x_i)}{\sum_{i=2}^m \left[\prod_{j=2, j\neq i}^m f(x_1 + x_j)\right]} \\ &\quad - \frac{\prod_{i=1}^m f(x_i)}{\sum_{i=2}^m f(x_1) \left[\prod_{j=2, j\neq i}^m f(x_j)\right] + (m-1) \prod_{i=2}^m f(x_i)} \Bigg| \\ &\leq \frac{\prod_{i=2}^m \sum_{n=k}^\infty \frac{\mu}{2^n}}{(m-1) \prod_{i=2}^{m-1} \left(\sum_{n=k}^\infty \frac{\mu}{2^n}\right)} + \frac{\prod_{i=1}^m \sum_{n=k}^\infty \frac{\mu}{2^n}}{2(m-1) \prod_{i=2}^m \left(\sum_{n=k}^\infty \frac{\mu}{2^n}\right)} \\ &\leq \frac{3}{2(m-1)} \frac{\mu}{2^k} \left(1 - \frac{1}{2}\right)^{-1} \\ &\leq \frac{6\mu}{m-1} \frac{1}{2^{k+1}} \\ &\leq \frac{6\mu}{m-1} \left(\sum_{i=1}^m |x_i|^{-1}\right) \end{aligned}$$

for all $x_1, x_2, \ldots, x_m \in X$. Therefore the inequality (4.1) holds true.

We claim that the reciprocal functional equation (1.7) is not stable for $p = -1$ in Corollary 3.3.

Assume that there exists a reciprocal mapping $r : X \to \mathbb{R}$ satisfying (4.2). Therefore, we have

$$|f(x)| \leq (\beta + 1)|x|^{-1}. \tag{4.4}$$

However, we can choose a positive integer m with $m\mu > \beta + 1$. If $x \in \left(1, 2^{m-1}\right)$ then $2^{-n}x \in (1, \infty)$ for all $n = 0, 1, 2, \ldots, m-1$ and therefore

$$|f(x)| = \sum_{n=0}^\infty \frac{\varphi(2^{-n}x)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\frac{2^n\mu}{x}}{2^n} = \frac{m\mu}{x} > (\beta + 1)x^{-1}$$

which contradicts (4.4). Therefore, the reciprocal type functional equation (1.7) is not stable for $p = -1$ in Corollary 3.3. □

The following example illustrates the fact that the functional equation (1.7) is not stable for $\alpha = -\frac{1}{m}$ in Corollary 3.5.

Example 4.2. Let $\phi : X \to \mathbb{R}$ be a mapping defined by

$$\phi(x) = \begin{cases} \frac{\delta}{x} & \text{for } x \in (1, \infty) \\ \delta & \text{otherwise} \end{cases}$$

where $\delta > 0$ is a constant, and define a mapping $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^\infty \frac{\phi(2^{-n}x)}{2^n}, \text{ for all } x \in X.$$

Then the mapping f satisfies the inequality

$$|D_m f(x_1, x_2, \ldots, x_m)| \leq \frac{6\delta}{m-1} \left\{ \sum_{i=1}^m |x_i|^{-1} + \left(\prod_{i=1}^m |x_i|^{-\frac{1}{m}} \right) \right\} \tag{4.5}$$

for all $x_1, x_2, \ldots, x_m \in X$. Therefore there do not exist a reciprocal mapping $r : X \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x) - r(x)| \leq \beta|x|^{-1} \tag{4.6}$$

for all $x \in X$.

Proof. It is easy to show that f is bounded by 2δ, by similar arguments as in Example 4.1. If

$$\left\{\sum_{i=1}^{m}|x_i|^{-1}+\left(\prod_{i=1}^{m}|x_i|^{-\frac{1}{m}}\right)\right\}\geq 1,$$

then the left hand side of (4.5) is less than $\frac{6\delta}{m-1}$. Now, suppose that

$$0<\left\{\sum_{i=1}^{m}|x_i|^{-1}+\left(\prod_{i=1}^{m}|x_i|^{-\frac{1}{m}}\right)\right\}<1.$$

Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}}\leq\left\{\sum_{i=1}^{m}|x_i|^{-1}+\left(\prod_{i=1}^{m}|x_i|^{-\frac{1}{m}}\right)\right\}<\frac{1}{2^k}. \tag{4.7}$$

Hence $\left\{\sum_{i=1}^{m}|x_i|^{-1}+\left(\prod_{i=1}^{m}|x_i|^{-\frac{1}{m}}\right)\right\}<\frac{1}{2^k}$ implies

$$\begin{array}{ll} & \left\{2^k\sum_{i=1}^{m}|x_i|^{-1}+2^k\left(\prod_{i=1}^{m}|x_i|^{-\frac{1}{m}}\right)\right\}<1 \\ \text{or} & 2^kx_i^{-1}<1 \text{ for } i=1,2,\ldots,m \\ \text{or} & \dfrac{x_i}{2^k}>1 \text{ for } i=1,2,\ldots,m \\ \text{or} & \dfrac{x_i}{2^k}>1>\dfrac{1}{2} \text{ for } i=1,2,\ldots,m \\ \text{or} & \dfrac{x_i}{2^{k-1}}>2>1 \text{ for } i=1,2,\ldots,m \end{array}$$

and consequently

$$\frac{1}{2^{k-1}}(x_1),\frac{1}{2^{k-1}}(x_i),\frac{1}{2^{k-1}}(x_1+x_i)>1 \text{ for } i=2,3,\ldots,m.$$

Therefore, for each value of $n=0,1,2,\ldots,k-1$, we obtain

$$\frac{1}{2^n}(x_1),\frac{1}{2^n}(x_i),\frac{1}{2^n}(x_1+x_i)>1 \text{ for } i=2,3,\ldots,m$$

and $D_m\phi(2^{-n}x_1,2^{-n}x_2,\ldots,2^{-n}x_m)=0$ for $n=0,1,2,\ldots,k-1$. Using (4.7), the definition of f and similar arguments as in Example 4.1, we obtain the inequality (4.5). The remaining part of the proof is obtained by similar arguments as in Example 4.1. Hence, the reciprocal type functional equation (1.7) is not stable for $\alpha=-\frac{1}{m}$ in Corollary 3.5. □

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