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Degenerate q-Changhee polynomials

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Abstract

In this paper, we consider the degenerate q-Changhee numbers and polynomials. From the definition of degenerate of q-Changhee polynomials, we derive some new interesting identities.

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1. Introduction

Recently, Changhee polynomials are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [9][10][12]).} \tag{1.1}$$

In [10], the degenerate Changhee polynomials are defined by Kwon-Kim-Seo to be

$$\frac{2\lambda}{2\lambda+\log(1+\lambda t)}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.2}$$

From (1.2), we note that

$$\lim_{\lambda\to 0} Ch_{n,\lambda}(x) = Ch_n(x), \quad (\text{for } n \geq 0).$$

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When $x=0$, $Ch_{n,\lambda}=Ch_{n,\lambda}(0)$ are called the degenerate Changhee numbers. In particular, $\lim_{\lambda\to 0} Ch_{n,\lambda}=Ch_n$, for $n\geq 0$, are called Changhee numbers, (see [1]–[12]). Throughout this paper, we denote the ring of p-adic integer, the field of p-adic number and the completion of algebraic closure of $\mathbb{Q}_p$ by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p=\frac{1}{p}$. Let q be an indeterminate with $|1-q|_p<p^{-\frac{1}{p-1}}$ and let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=\lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)q^x(-1)^x, \tag{1.3}$$

where $[x]_{-q}=\frac{1-(-q)^x}{1+q}$, (see [7][12]).

From (1.3), we note that

$$q^n I_{-q}(f_n)+(-1)^{n-1}I_{-q}(f)=[2]_q\sum_{i=0}^{n-1}(-1)^{n-1-i}q^i f(i), \tag{1.4}$$

where $[x]_q=\frac{1-q^x}{1-q}$ and $n\in\mathbb{N}$, $f_n(x)=f(x+n)$. Recently, Kim-Mansour-Rim-Seo considered the q-Changhee polynomials, $Ch_{n,q}(x)$, which are given by the generating function to be

$$\frac{1+q}{q(1+t)+1}(1+t)^x=\sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!},\ \text{(see [9])}. \tag{1.5}$$

In this paper, we consider the degenerate q-Changhee polynomials and we derive some new and interesting properties related to these polynomials and numbers.

2. Degenerate q-Changhee polynomials

Let us assume that $\lambda, t\in\mathbb{C}_p$ with $|\,\lambda t\,|_p<p^{-\frac{1}{p-1}}$. From (1.3), we have

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x)+\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x)=[2]_q f(0). \tag{2.1}$$

By (2.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x d\mu_{-q}(x)&=\frac{1+q}{q(\log(1+\lambda t)^{\frac{1}{\lambda}}+1)+1}\\&=\frac{q\lambda+\lambda}{q\log(1+\lambda t)+q\lambda+\lambda}=\sum_{n=0}^{\infty}Ch_{n,\lambda,q}\frac{t^n}{n!},\end{aligned} \tag{2.2}$$

where $Ch_{n,\lambda,q}$ are called degenerate q-Changhee numbers.

From (2.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^x d\mu_{-q}(x)&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x)_n d\mu_{-q}(x)\frac{1}{n!}(\log(1+\lambda t)^{\frac{1}{\lambda}})^n\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x)_n d\mu_{-q}(x)\lambda^{-n}\sum_{m=n}^{\infty}S_1(m,n)\frac{\lambda^m}{m!}t^m\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\lambda^{m-n}S_1(m,n)\int_{\mathbb{Z}_p}(x)_n d\mu_{-q}(x)\right)\frac{t^m}{m!},\end{aligned} \tag{2.3}$$

where $S_1(m,n)$ is the Stirling number of the first kind. It is known that

$$\int_{\mathbb{Z}_p}(1+t)^x d\mu_{-q}(x)=\frac{1+q}{q(1+t)+1}=\sum_{n=0}^{\infty}Ch_{n,q}\frac{t^n}{n!}. \tag{2.4}$$

Thus, by (2.4), we get

$$\int_{\mathbb{Z}_p}(x)_n d\mu_{-q}(x)=Ch_{n,q},\,(n\geq 0). \tag{2.5}$$

Therefore, by (2.2), (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $m\geq 0$, we have*

$$Ch_{m,\lambda,q}=\sum_{n=0}^{m}\lambda^{m-n}S_1(m,n)Ch_{n,q}.$$

We observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x+y}d\mu_{-q}(y)&=\frac{q\lambda+\lambda}{q\log(1+\lambda t)+q\lambda+\lambda}(1+\log(1+\lambda t)^{\frac{1}{\lambda}})^{x}\\&=\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

where $Ch_{n,\lambda,q}(x)$ are called degenerate q-Changhee polynomials.

From (2.6), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x)\frac{t^n}{n!}&=\left(\sum_{m=0}^{\infty}Ch_{m,\lambda,q}\frac{t^m}{m!}\right)\sum_{l=0}^{\infty}(x)_l\lambda^{-l}\frac{(\log(1+\lambda t))^l}{l!}\\&=\left(\sum_{m=0}^{\infty}Ch_{m,\lambda,q}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x)_l\lambda^{-l}\sum_{k=l}^{\infty}S_1(k,l)\frac{\lambda^k t^k}{k!}\right)\\&=\left(\sum_{m=0}^{\infty}Ch_{m,\lambda,q}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}(x)_l\lambda^{k-l}S_1(k,l)\right)\frac{t^k}{k!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}(x)_l\lambda^{k-l}S_1(k,l)Ch_{n-l,\lambda,q}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem:

Theorem 2.2. *For $n\geq 0$, we have*

$$Ch_{n,\lambda,q}(x)=\sum_{k=0}^{n}\sum_{l=0}^{k}(x)_l\lambda^{k-l}S_1(k,l)Ch_{n-l,\lambda,q}.$$

By (2.6), we easily get

$$\begin{aligned}\int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x+y}d\mu_{-q}(y)&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)_m d\mu_{-q}(y)\frac{\lambda^{-m}}{m!}(\log(1+\lambda t))^m\\&=\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(x+y)_m d\mu_{-q}(y)\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}(x+y)_m d\mu_{-q}(y)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

It is not difficult to show that

$$\int_{\mathbb{Z}_p}(1+t)^{x+y}d\mu_{-q}(y)=\frac{1+q}{q(1+t)+1}(1+x)=\sum_{n=0}^{\infty}Ch_{n,q}(x)\frac{t^n}{n!}. \tag{2.9}$$

Thus, by (2.9), we get

$$\int_{\mathbb{Z}_p}(x+y)_n d\mu_{-q}(y)=Ch_{n,q}(x),(\text{for } n\geq 0). \tag{2.10}$$

Therefore, by (2.8) and (2.10), we obtain the following theorem:

Theorem 2.3. *For $n \geq 0$, we have*

$$Ch_{n,\lambda,q}(x)=\sum_{m=0}^{n}Ch_{m,q}(x)\lambda^{n-m}S_1(n,m).$$

From (2.6), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x)\lambda^{-n}\frac{1}{n!}(e^t-1)^n&=\int_{\mathbb{Z}_p}\left(1+\frac{1}{\lambda}t\right)^{x+y}d\mu_{-q}(y)\\&=\sum_{m=0}^{\infty}\lambda^{-m}Ch_{m,q}(x)\frac{t^m}{m!},\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x)\lambda^{-n}\frac{1}{n!}(e^t-1)^n&=\sum_{n=0}^{\infty}Ch_{n,\lambda,q}(x)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}Ch_{n,\lambda,q}(x)\lambda^{-n}S_2(m,n)\right)\frac{t^m}{m!},\end{aligned} \tag{2.12}$$

where $S_2(m,n)$ is the Stirling number of the second kind. Therefore, by (2.11) and (2.12), we obtain the following theorem:

Theorem 2.4. *For $m \geq 0$, we have*

$$Ch_{m,q}(x)=\sum_{n=0}^{m}\lambda^{m-n}Ch_{n,\lambda,q}(x)S_2(m,n).$$

We observe that

$$\int_{\mathbb{Z}_p}(x+y)_n d\mu_{-q}(y)=\sum_{l=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}(x+y)^l d\mu_{-q}(y), \tag{2.13}$$

where $n\in\mathbb{N}\cup\{0\}$.

The q-analogue of Euler polynomials are given by the generating function to be

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{(x+y)t}d\mu_{-q}(y)&=\frac{q+1}{qe^t+1}e^{xt}\\&=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

Thus, by (2.14), we get

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)=E_{n,q}(x),(n\geq 0). \tag{2.15}$$

Therefore, by (2.6), (2.8) and (2.15), we obtain the following theorem:

Theorem 2.5. *For $n \geq 0$, we have*

$$Ch_{n,\lambda,q}(x) = \sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)E_{l,q}(x)\lambda^{n-m}S_1(n,m).$$

By (2.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(qCh_{n,\lambda,q}(x+1) + Ch_{n,\lambda,q}(x)\right)\frac{t^n}{n!}\\
&= \frac{q\lambda+\lambda}{q\log(1+\lambda t)+q\lambda+\lambda}\left(q(1+\log(1+\lambda t)^{\frac{1}{\lambda}})+1\right)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x\\
&= \frac{1}{\lambda}(q\lambda+\lambda)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x = (q+1)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x\\
&= [2]_q\sum_{m=0}^{\infty}(x)_m\frac{\lambda^{-m}}{m!}(\log(1+\lambda t))^m\\
&= [2]_q\sum_{m=0}^{\infty}(x)_m\lambda^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{\lambda^n t^n}{n!}\\
&= [2]_q\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem:

Theorem 2.6. *For $n \geq 0$, we have*

$$qCh_{n,\lambda,q}(x+1) + Ch_{n,\lambda,q}(x) = [2]_q\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m).$$

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