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Proof of one open inequality

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Abstract

In this paper, one conjecture presented in the paper [V. Cîrtoaje, Proofs of Three Open Inequalities With Power-Exponential Functions, J. Nonlinear Sci. Appl. 4 (2011) no.2, 130-137, http://www.tjnsa.com] is proved.

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1. Introduction and Preliminaries

In the paper [1], V. Cîrtoaje has posted 3 conjectures on some inequalities with power-exponential functions. In this paper, we prove the Conjecture 1.1:

Conjecture 1.1. *If $a; b \in (0; 1]$ and $r \in (0; e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}. \tag{1.1}$$

In the particular case $r = 2$, we get the elegant inequality

$$2a^a b^b \geq a^{2b} + b^{2a}. \tag{1.2}$$

which is also an open problem.

We show, that $r = e$ is the greatest possible value of a positive real number r such that the inequality (1.1) holds for all positive real numbers $a; b \leq 1$:

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2. Proof of the Conjecture 1.1

Theorem 2.1. *Let $a; b \in (0; 1]$ and $r \in (0; e]$, then*

$$2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}.$$

Proof. Inequality (1.1) is equivalent to

$$F(a, b, r) = 2e^{a\ln a + b\ln b - 2a\ln b} - e^{r(b\ln a = a\ln b)} - 1 \geq 0. \tag{2.1}$$

Without loss of generality, assume that $0 < a < b \leq 1$. Let b be fixed. Since $F(b, b, r) = 0$ it suffices to show that

$$\frac{\partial F(a, b, r)}{\partial a} \leq 0 \quad \text{for} \quad 0 < a < b \quad \text{and} \quad 0 < r \leq e. \tag{2.2}$$

Inequality (2.2) can be written as

$$e^{\frac{r}{2}(a\ln a + b\ln b - 2b\ln a}(\ln a + 1 - 2\ln b) - \frac{b}{a} + \ln b < 0. \tag{2.3}$$

From $a\ln a + b\ln b - 2b\ln a > 0$ for $0 < a < b \leq 1$ it suffices to prove inequality (2.4) only for a, b such that $b^2/e < a < b \leq 1$ $(\ln a + 1 - 2\ln b > 0)$.

$$H(a, b) = \frac{e}{2}(a\ln a + b\ln b - 2b\ln a) + \ln(\ln a + 1 - 2\ln b) - \ln\left(\frac{b}{a} - \ln b\right) < 0. \tag{2.4}$$

Since $H(b, b) = 0$ we show that

$$\frac{\partial H(a, b)}{\partial a} \geq 0 \quad \text{for} \quad b^2/e < a < b \leq 1. \tag{2.5}$$

Elementary calculation gives

$$\frac{\partial H(a, b)}{\partial a} = \frac{e}{2}\left(\ln a + 1 - \frac{2b}{a}\right) + \frac{1}{(\ln a + 1 - 2\ln b)a} + \frac{\ln b}{b - a\ln b} + \frac{1}{a}. \tag{2.6}$$

Inequality (2.5) is equivalent to

$$\begin{aligned} g(a, b) =& 2b(2 - eb + 2(eb - 1)\ln b) + a(eb - 4eb\ln^2 b - 2\ln b) + ea^2\ln b(2\ln b - 1) + 2b(1 - eb)\ln a + \\ & 2eab\ln a + 2ea^2\ln a\ln b(\ln b - 1) + eab\ln^2 a - ea^2\ln^2 a\ln b > 0. \end{aligned} \tag{2.7}$$

From the following formulas

$$\ln a = \ln b - \sum_{n=1}^{\infty}\frac{(b - a)^n}{nb^n}, \tag{2.8}$$

$$\ln^2 a = \ln^2 b - 2\ln b\sum_{n=1}^{\infty}\frac{(b - a)^n}{nb^n} + 2\sum_{n=2}^{\infty}\frac{(b - a)^n}{nb^n}\left(\sum_{n=1}^{i-1}\frac{1}{i}\right) \tag{2.9}$$

for $0 < a < b \leq 1$ we get

$$\ln a > \ln b - \frac{(b - a)}{b} - \frac{(b - a)^2}{2ab}, \tag{2.10}$$

$$\ln^2 a > \ln^2 b - \frac{2\ln b}{b}(b - a) + (1 - \ln b)\frac{(b - a)^2}{b^2}. \tag{2.11}$$

Using (2.10),(2.11) in (2.7) we find $g(a,b) > \Phi(a,b)$, where

$$\begin{aligned}\Phi(a,b) =& b(4-3eb+4(eb-1)\ln b) + a\left(\left(2e-\frac{1}{2}\right)b-(eb+2)\ln b - 3eb\ln^2 b\right) + 2b(1-eb)\ln a + \\ & a^2(-e+5e\ln^2 b+(2e+1)\ln b - e\ln^3 b) + \frac{a^3}{b}\left(\frac{1}{2}+e+e\ln b - 4e\ln^2 b\right) + \frac{a^4}{b^2}\ln b(\ln b - 1).\end{aligned} \tag{2.12}$$

We show that $\Phi(a,b) > 0$,

There are five cases for b.

$$\begin{array}{ll} I. & 0 < b < 0.41 \\ II. & 0.41 \le b \le b_0 \\ III. & b_0 < b \le b_1 \\ IV. & b_1 < b \le 0.766 \\ V. & 0.766 < b \le 1, \end{array}$$

where b_0 is the root of

$$r(b) = \frac{1}{2} + e\ln b + e - 4e\ln^2 b, \tag{2.13}$$

in $(0,1)$. b_1 is the root of

$$kp3(b) = \frac{1}{2} + e + \left(1+\frac{1}{2e}\right)b - \frac{1}{b} + \left(1+3e-\frac{b^2}{e}\right)\ln b + \left(e-3b+\frac{b^2}{e}\right)\ln^2 b - e\ln^3 b \tag{2.14}$$

in $(0.64,1)$. We note, that the existence of b_0 follows from $r'(b) > 0$ for $b \in (0,1)$, $\lim_{b\to 0^+} r(b) = -\infty$, $r(1) = 1/2 + e$. From $r(0.64) = -0.1605$, $r(0.65) = 0.0295$ we have $0.64 < b_0 < 0.65$. Similarly, we show that $kp3'(b) > 0$ for $b \in (0.64, 1]$. Then from $kp3(0.64) = -1.1743$, $kp3(1) = 3.4020$ we obtain there is only one b_1 such that $0.64 < b_1 < 1$ and $kp3(b_1) = 0$. From $kp3(0.726) = -0.0044$, $kp3(0.727) = 0.0091$, we have $0.726 < b_1 < 0.727$. We show that $kp3'(b) > 0$. After some computation we obtain

$$kp3'(b) = \frac{2e+(2e+6e^2)b+(2e+1)b^2-2b^3}{2eb^2} + \frac{2e-6b}{b} + \frac{-3e^2-3eb+2b^2}{eb}\ln^2 b. \tag{2.15}$$

To prove $kp3'(b) > 0$ it suffices to show

$$2e+(2e+6e^2)b+(2e+1)b^2-2b^3+4eb(e-3b)\ln b + \frac{(1-b)^2}{b}(-6e^2b-6eb^2+4b^3) > 0. \tag{2.16}$$

We used (25). From $b > 0.64$ we obtain that

$$\begin{aligned} kp3'(b) >& 2e(1-3e) + (18e^2-4e+4e(e-1.92)\ln 0.64)b + (14e+5-6e^2)b^2 - (10+6e)b^3 + 4b^4 \\ & > -38.9 + 118.2b - 1.3b^2 - 26.4b^3 + 4b^4 > 0 \end{aligned} \tag{2.17}$$

which can be easily showed.

The proof of the case I.

We show that $\Phi(b^2/e, b) > 0$ and $\frac{\partial \Phi(a,b)}{\partial a} > 0$ for fixed b such that $0 < b < 0.41$ and $b^2/e < a < b$. Some computations yields

$$\Phi(b^2/e, b) = k_1(b) + k_2(b)\ln b + k_3(b)\ln^2 b - \frac{b^4}{e}\ln^3 b. \tag{2.18}$$

where

$$k_1(b) = \left(2 - eb + \frac{4e-1}{2e}b^2 - \frac{1}{e}b^3 + \frac{2e+1}{2e^3}b^4\right)b, \tag{2.19}$$

$$k_2(b) = \left(\frac{-2}{e} - b + \frac{2e+1}{e^2}b^2 + \frac{1}{e^2}b^3 - \frac{1}{e^4}b^4\right)b^2, \tag{2.20}$$

$$k_3(b) = \left(-3 + \frac{5}{e}b - \frac{4}{e^2}b^2 + \frac{1}{e^3}b^3\right)b^3. \tag{2.21}$$

It is easy to show that $k_1(b) > 0$ $k_2(b) < 0$ $k_3(b) < 0$ for $b \in (0, 0.41)$. From (2.8), (2.9) and from

$$\frac{2}{n+1}\sum_{i=1}^{n}\frac{1}{i} \leq \frac{2}{n+1}\left(\frac{1}{2n} + \ln n + C\right) \leq 1 \quad \text{for} \quad n > 5 \tag{2.22}$$

we estimate

$$\ln b < 2\frac{b-1}{b+1} \quad \text{and} \quad \ln^2 b < 2\frac{(b-1)^2}{b} \quad \text{for} \quad b \in (0,1). \tag{2.23}$$

Using (2.23) we obtain that it suffices to prove that

$$\frac{1}{b}\left(k_1(b) + k_2(b)\frac{2(b-1)}{b+1} + k_3(b)\frac{(1-b)^2}{b}\right) > 0. \tag{2.24}$$

(2.24) can be rewriting as

$$p(b) = 2e^3 - e^2(e^2+e-4)b + e^2\left(\frac{1}{2} + 7e - e^2\right)b^2 - e\left(\frac{21e}{2} + 6 - 3e^2\right)b^3 - \left(-\frac{3}{2} - 5e + 2e^2 + 3e^3\right)b^4 + \left(\frac{3}{2} + 7e + 5e^2\right)b^5 - (3+4e)b^6 + b^7 > 0.$$

Simple computation gives

$$p(b) > 40 - 46 \times 0.41 - 34 \times 0.41^3 - 60 \times 0.41^4 - 14 \times 0.41^6 = 17.0347$$

We prove $\frac{\partial\Phi(a,b)}{\partial a} > 0$ for fixed b such that $0 < b < 0.41$ and $b^2/e < a < b$. From $\frac{\partial\Phi(a,b)}{\partial a} = \frac{1}{a}\Phi^*(a,b)$ it suffices to show that $\Phi^*(a,b) > 0$ where

$$\Phi^*(a,b) = \left(\left(2e - \frac{1}{2}\right)b - (eb+2)\ln b - 3eb\ln^2 b\right)a + 2(5e\ln^2 b - e + (2e+1)\ln b - e\ln^3 b)a^2 + \frac{3}{b}\left(\frac{1}{2} + e\ln b + e - 4e\ln^2 b\right)a^3 + \frac{4}{b^2}e\ln b(e\ln b - 1)a^4 + 2b(1-eb). \tag{2.25}$$

We prove that $\Phi^*(b^2/e, b) > 0$ and $\frac{\partial\Phi^*(a,b)}{\partial a} > 0$. $\Phi^*(b^2/e, b) > 0$ is equivalent to

$$k_1^*(b)b + k_2^*(b)b^2\ln b + k_3^*(b)b^3\ln^2 b > 0. \tag{2.26}$$

where

$$k_1^*(b) = 2 - 2eb + \left(2 - \frac{1}{2e}\right)b^2 - \frac{2}{e}b^3 + \left(\frac{3}{2e^3} + \frac{3}{e^2}\right)b^4, \tag{2.27}$$

$$k_2^*(b) = -\frac{2}{e} - b + \left(\frac{4}{e} + \frac{2}{e^2}\right)b^2 + \frac{3}{e^2}b^3 - \frac{4}{e^3}b^4, \tag{2.28}$$

$$k_3^*(b) = -3 + \frac{10}{e}b - \frac{12}{e^2}b^2 + \frac{4}{e^3}b^3. \tag{2.29}$$

It is easy to show that $k_1^*(b) > 0$, $k_2^*(b) < 0$ $k_3^*(b) < 0$ for $b \in (0, 0.41)$. Using (2.23) we have that it suffices to show

$$k_1^*(b) + \frac{2k_2^*(b)b(b-1)}{b+1} + k_3^*(b)b(1-b)^2 > 0. \tag{2.30}$$

(2.30) is equivalent to

$$\begin{aligned} g_1(b) =& 2e^3 + (-e^3 - 2e^4 + 4e^2)b + (-2e^4 + 7e^3 + 5.5e^2)b^2 + (3e^3 - 20.5e^2 - 16e)b^3 - (12 + 12e)b^6 + 4b^7 \\ &+ (5.5 + 21e + 10e^2)b^5 + (-4e^2 + 5.5 + 13e - 3e^3)b^4 > 0 \end{aligned} \tag{2.31}$$

and

$$g_1(b) > 40 - 94 \times 0.41 + b^2(66 - 135 \times 0.41 - 50 \times 0.41^2) + b^5(136 - 45 \times 0.41 + 4b^2) > 0.$$

for $b \in (0, 0.41]$.

We show that $\frac{\partial \Phi^*(a,b)}{\partial a} > 0$ for $0 < b < 0.41$ and $b^2/e < a < b$. Simple computation gives

$$\frac{\partial \Phi^*(a,b)}{\partial a} = a_0(b) + a_1(b)a + a_2(b)a^2 + a_3(b)a^3, \tag{2.32}$$

where

$$a_0(b) = \left(2e - \frac{1}{2}\right) b - (2 + eb)\ln b - 3eb\ln^2 b, \tag{2.33}$$

$$a_1(b) = 4(-e + (2e + 1)\ln b + 5e\ln^2 b - e\ln^3 b), \tag{2.34}$$

$$a_2(b) = \frac{9}{b}\left(\frac{1}{2} + e\ln b + e - 4e\ln^2 b\right), \tag{2.35}$$

$$a_3(b) = \frac{16}{b^2} e\ln b(\ln b - 1). \tag{2.36}$$

It is evident that $a_3(b) > 0$ for $b \in (0, 0.41)$.

We prove $a_0(b) > 0$, $a_1(b) > 0$, $a_2(b) < 0$ for $b \in (0, 0.41)$. We show

$$\frac{3}{2} e\ln^2 b + \frac{e}{2} e\ln b - e + \frac{1}{4} < -5 + \frac{5}{2b}, \tag{2.37}$$

and

$$-5 + \frac{5}{2b} < -\frac{\ln b}{b}. \tag{2.38}$$

It implies $a_0(b) > 0$. Denote

$$s_1(b) = \frac{3}{2} e\ln^2 b + \frac{e}{2}\ln b - e + \frac{21}{4} - \frac{5}{2b},$$

$$s_2(b) = -5b + \frac{5}{2} + \ln b.$$

If we show $s_1(b) < 0$, $s_2(b) < 0$ the inequality $a_0(b) > 0$ will be proved. We show $s_1'(b) > 0$ for $b \in (0, 0.25)$ and from $s_1(0.25) = -1.5164$ we obtain $s_1(b) < 0$ for $b \in (0, 0.25]$. Denote

$$hh(b) = 3e\ln b + \frac{e}{2} + \frac{5}{2b}.$$

It is evident that $s_1'(b) = hh(b)/b$. From $hh(0) = +\infty$, $hh'(b) = \frac{-2.5+3eb}{b^2} < 0$ for $b \in (0, 0.25)$ and from $hh(0.25) = 0.0541$ we have $s_1'(b) > 0$ for $b \in (0, 0.25)$. If $b \in (0.25, 0.41)$ by using (2.23) we get $s_1(b) < \frac{1}{b}h(b)$ where

$$h(b) = \frac{3}{2} e(1 - b)^2 + \frac{e}{2} b\ln b - eb + \frac{21}{4} b - \frac{5}{2}.$$

From $h''(b) = 3e + e/(2b) > 0$ and $h'(0.41) = -2.1323$ we obtain $h'(b) < 0$ and from $h(0.25) = -0.1783$ we get $h(b) < 0$. From $s_2(0) = -\infty$, $s_2'(b) = -5 + 1/b$, $s_2(0.2) = -0.1094$ we obtain $s_2(b) < 0$ for $b \in (0, 0.41)$.

We prove $a_1(b) > 0$. Making the substitution $t = 1/\ln b$ the inequality $a_1(b) > 0$ is equivalent to

$$-et^3 + (2e + 1)t^2 + 5et - e < 0 \quad \text{for} \quad t \in (1/\ln 0.41, 0)$$

which can be easily showed.

We prove $a_2(b) < 0$. Putting $t = \ln b$ we get $\frac{1}{2} + et + e - 4et^2 < 0$ for $-\infty < t < \ln(0.41)$ which implies $a_2(b) < 0$. If we show that

$$4a_1(b)a_3(b) - a_2^2(b) > 0 \tag{2.39}$$

which is

$$-\frac{81}{4} - 81e - 81e^2 + (-81e + 94e^2)\ln b + (68e - 201e^2)\ln^2 b + (-120e^2 + 256e)\ln^3 b + 240e^2 \ln^4 b - 256e^2 \ln^5 b > 0. \tag{2.40}$$

Making the substitution $t = 1/\ln b$ it suffices to prove that

$$\begin{aligned} ff(t) = &-\left(\frac{81}{4} + 81e + 81e^2\right)t^5 + (-81e + 94e^2)t^4 + (68e - 201e^2)t^3 + (-120e^2 + 256e)t^2 + 240e^2 t \\ &- 256e^2 < 0 \quad \text{for} \quad 1/\ln 0.41 < t < 0. \end{aligned} \tag{2.41}$$

According to $t > 1/\ln 0.41$ we have

$$\begin{aligned} ff(t) < &\frac{\left(-\frac{81}{4} - 13e - 81e^2 - 201e^2 + \frac{94e^2 - 81e}{\ln 0.41}\right)t^3}{\ln^2 0.41} + (-120e^2 + 256e)t^2 + 240e^2 t - 256e^2 \\ < &3050t^2 + 1774t - 1981 < 0. \end{aligned}$$

This completes the proof of the case I.

The case II. Let $b \in (0.41, b_0)$. Using

$$a^3 r(b) > a^2\left(\frac{1}{2} + e\ln b + e - 4e\ln^2 b\right)$$

and

$$\frac{a^4}{b^2} e \ln b(\ln b - 1) > \frac{a^2 b^2}{e} \ln b(\ln b - 1),$$

$$\ln a < \ln b - \frac{b-a}{b} - \frac{(b-a)^2}{2b^2}$$

we find that $\Phi(a,b) > \Phi_3(a,b)$ where

$$\Phi_3(a,b) = l_1(b) + a l_2(b) + a^2 l_3(b). \tag{2.42}$$

where

$$l_1(b) = b(1 + 2(eb - 1)\ln b), \tag{2.43}$$

$$l_2(b) = 4 - \left(2e + \frac{1}{2}\right)b - (eb + 2)\ln b - 3eb\ln^2 b, \tag{2.44}$$

$$l_3(b) = \frac{1}{2} + e - \frac{1}{b} + \left(3e + 1 - \frac{b^2}{e}\right)\ln b + \left(e^2 + \frac{b^2}{e}\right)\ln^2 b - e\ln^3 b. \tag{2.45}$$

We show that $l_1(b) > 0$, $l_2(b) > 0$ for $b \in (0.41, b_0]$. Using

$$\ln b > b - 1 - \frac{(b-1)^2}{2b}, \tag{2.46}$$

in $l_1(b) > 0$ it suffices to show that

$$1 + (1-e)b - b^2 + eb^3 > 0$$

which can be easily showed. Using (2.23) in $l_2(b) > 0$ we have that

$$l_2(b) > j(b) = 4 - 3e + \left(4e - \frac{1}{2}\right) b - 3eb^2 - (2 + eb) \ln b > 0.$$

From $\ln b < \ln 0.65$ we have

$$j(b) > 4 - 3e - 2 \ln 0.65 + (4e - 0.5 - e \ln 0.65)b - 3eb^2 > 0$$

which can be easily showed. We show that $\Phi_3(a, b) > 0$. It suffices to show that

$$l_1(b) + \frac{b^2}{e} l_2(b) + \frac{b^4}{e^2} l_3(b) > 0 \tag{2.47}$$

and

$$l_2(b) + bl_3(b) > 0. \tag{2.48}$$

Indeed, (2.48) implies

$$l_2(b) + \frac{b^2}{e} l_3(b) > 0. \tag{2.49}$$

(where $l_3(b) < 0$ we have $(b^2/e)l_3(b) > l_3(b)b$). (2.48), (2.49) imply

$$l_2(b) + al_3(b) > 0 \quad \text{for} \quad b \in (0.41, b_0], \quad \frac{b^2}{e} < a < b. \tag{2.50}$$

From this we obtain $\Phi_3(a, b) > 0$. We show (2.47), (2.48). (2.47) is equivalent to

$$\begin{aligned} v(b) =& b\left(1 + \frac{4}{e} b - \left(2 + \frac{1}{2e} + \frac{1}{e^2}\right) b^2 + \left(\frac{1}{2e^2} + \frac{1}{e}\right) b^3\right) + \left(-2 + \left(2e - \frac{2}{e}\right) b - b^2 + \left(\frac{1}{e^2} + \frac{3}{e}\right) b^3 - \frac{b^5}{e^3}\right) + \\ & b \ln b + b^3 \ln^2 b \left(-3 + b + \frac{b^3}{e^3}\right) - \frac{b^4}{e} \ln^3 b > 0. \end{aligned} \tag{2.51}$$

After some computation and using (2.23) we have

$$\frac{v(b)}{b} > ch_1(b) + ch_2(b) \ln b \tag{2.52}$$

where

$$ch_1(b) = 1 + \left(\frac{4}{e} - 3\right) b + \left(5 - \frac{1}{2e} - \frac{1}{e^2}\right) b^2 - \frac{2}{e^3} b^5 + \left(-5 + \frac{1}{2e^2} + \frac{1}{e}\right) b^3 + \left(1 + \frac{1}{e^3}\right) b^4 + \frac{1}{e^3} b^6, \tag{2.53}$$

$$ch_2(b) = -2 + \left(2e - \frac{2}{e}\right) b - b^2 + \left(\frac{1}{e^2} + \frac{3}{e}\right) b^3 - \frac{1}{e^3} b^5. \tag{2.54}$$

It is easy to show that $ch_1(b) > 0$ for $b \in (0.41, b_0]$. If $ch_2(b) \leq 0$ then (2.51) is evident. If $ch_2(b) > 0$ then from $\ln b > b - 1 - \frac{(b-1)^2}{2b}$ it suffices to prove that

$$ch_1(b) + \left(b - 1 - \frac{(b-1)^2}{2b}\right) ch_2(b) > 0 \tag{2.55}$$

for $b \in (0.41, b_0] \cap \{b; ch_2(b) > 0\}$. (2.55) is equivalent to

$$\begin{aligned} &\frac{1}{b} + 1 - e + \frac{1}{e} + \left(\frac{4}{e} - \frac{7}{2}\right) b + \left(5 + e - \frac{3}{e} - \frac{3}{2e^2}\right) b^2 + \left(-\frac{11}{2} + \frac{1}{e} + \frac{1}{2e^2}\right) b^3 + \left(1 + \frac{3}{2e} + \frac{1}{2e^2} + \frac{3}{2e^3}\right) b^4 - \\ &\frac{2}{e^3} b^5 + \frac{1}{2e^3} b^6 > 0 \end{aligned} \tag{2.56}$$

which can be easily showed for $b \in (0.41, b_0]$. We prove (2.48). Inequality (2.48) can be rewriting as

$$3 - eb + \left(-2 + (2e+1)b - \frac{1}{e}b^3\right)\ln b + b\left(e^2 - 3e + \frac{1}{e}b^2\right)\ln^2 b - eb\ln^3 b > 0. \tag{2.57}$$

Using (2.46) and $\ln^2 b < (1-b)^2/b$ omitting $-eb\ln^3 b$ we obtain that it suffices to show

$$\frac{1}{b} + \frac{5}{2} - 4e + e^2 + (-1 + 5e - 2e^2)b + \left(\frac{1}{2} - 2e + \frac{3}{2e} + e^2\right)b^2 - \frac{2}{e}b^3 + \frac{1}{2e}b^4 > 0 \tag{2.58}$$

which can be easily showed. Thus, the proof of the case II. is completed.

Cases III, IV, V ($b \in (b_0, 1]$).

According to $a > b^2/e$ we get

$$\Phi(a,b) > b(4 - 3eb) + 4b(eb-1)\ln b + a((2e - 0.5)b - (eb+2)\ln b - 3eb\ln^2 b) + +2b(1-eb)\ln a +$$
$$a^2(-e + (2e+1)\ln b + 5e\ln^2 b - e\ln^3 b) + \frac{a^3}{b}\left(\frac{1}{2} + e + e\ln b - 4e\ln^2 b\right) + a^3\ln b(\ln b - 1). \tag{2.59}$$

Using $\ln a < \ln b - (b-a)/b - (b-a)^2/(2b^2)$ we estimate

$$\Phi(a,b) > \Phi_4(a,b) = u_1(b) + au_2(b) + a^2u_3(b) + a^3u_4(b). \tag{2.60}$$

where

$$u_1(b) = l_1(b) = b(1 + 2(eb-1)\ln b), \tag{2.61}$$

$$u_2(b) = l_2(b) = 4 - \left(2e + \frac{1}{2}\right)b - (eb+2)\ln b - 3eb\ln^2 b, \tag{2.62}$$

$$u_3(b) = 5e\ln^2 b + (2e+1)\ln b - e\ln^3 b - \frac{1}{b}. \tag{2.63}$$

$$u_4(b) = \left(e + \frac{1}{2}\right)\frac{1}{b} + \left(\frac{e}{b} - 1\right)\ln b + \left(1 - \frac{4e}{b}\right)\ln^2 b. \tag{2.64}$$

We show that $u_1(b) > 0$, $u_3(b) < 0$,$u_4(b) > 0$, for $b \in (0.64, 1]$. Using (2.46) we obtain

$$u_1(b) > 1 + (1-e)b - b^2 + eb^3 > 0$$

which can be easily showed. Simple calculation gives $u_3''(b) = 1/b^3 uu(b)$ where

$$uu(b) = -2 + (8e-1)b - 16eb\ln b + 3eb\ln^2 b > 11.$$

It implies $u_3(b)$ is a convex function. From $u_3(0.64) = -1.9864$, $u_3(1) = -1$ we get $u_3(b) < 0$. Elementary calculation gives $u_4(b) = 1/b vv(b)$ where

$$vv(b) = 0.5 + e + (e-b)\ln b + (b - 4e)\ln^2 b.$$

Simple calculation gives $vv_3''(b) = (-9e + b + 8e\ln b)/b^2 < 0$ So $vv(b)$ is a concave function. From $vv(0.64) = 0.2526$, $vv(1) = 3.2183$ we obtain $u_4(b) > 0$ for $b \in (0.64, 1]$.

To prove $\Phi_4(a,b) > 0$ we show that

$$\alpha(a,b) = u_2(b) + au_3(b) + a^2u_4(b) \geq c(b)a + d(b). \tag{2.65}$$

$$c(b)a + d(b) > -\frac{u_1(b)}{a} \quad \text{for} \quad b \in (0.64, 1], \quad \frac{b^2}{e} < a < b. \tag{2.66}$$

where

$$c(b) = u_3(b) + u_4(b)\frac{eb + b^2}{e}, \tag{2.67}$$

$$d(b) = u_2(b) - u_4(b)\frac{(eb+b^2)^3}{4e^2}. \tag{2.68}$$

The inequality (2.65) is equivalent to

$$\frac{(eb+b^2)^3}{4e^2} - a\frac{(eb+b^2)}{e} + a^2 \geq 0. \tag{2.69}$$

which is evidently fulfilled. We prove (2.66). After some computation we have that (2.65) can be rewriting as

$$kp_1(b) + kp_2(b)a + kp_3(b)a^2 > 0, \tag{2.70}$$

where

$$kp_1(b) = u_1(b) = l_1(b) = b(1 + 2(eb-1)\ln b), \tag{2.71}$$

$$\begin{aligned} kp_2(b) =& 4 - \left(\frac{9e}{4} + \frac{5}{8}\right)b - \left(\frac{1}{2} + \frac{1}{4e}\right)b^2 + \left(-2 - \frac{5e}{4}b - \frac{1}{4}b^2 + \frac{1}{4e}b^3 + \frac{1}{4e^2}b^4\right)\ln b + \\ & \left(-2eb + \frac{7}{4}b^2 + \frac{1}{2e}b^3 - \frac{1}{4e^2}b^4\right)\ln^2 b - \left(\frac{1}{4e} + \frac{1}{8e^2}\right)b^3, \end{aligned} \tag{2.72}$$

$$kp_3(b) = -\frac{1}{b} + \frac{1}{2} + e + \left(1 + \frac{1}{2e}\right)b + \left(3e + 1 - \frac{1}{e}b^2\right)\ln b + \left(e - 3b + \frac{1}{e}b^2\right)\ln^2 b - e\ln^3 b. \tag{2.73}$$

From $kp_1(b) = u_1(b)$ we have that $kp_1(b) > 0$. If we show that (2.70) is true for $b \in (0.64, 1]$ the proof will be completed.

The case III. ($b_0 < b \leq b_1$).

We have $kp_3(b) \leq 0$. From $a < b$ we obtain

$$kp_1(b) + kp_2(b)a + kp_3(b)a^2 > kp_1(b) + kp_2(b)a + kp_3(b)ab.$$

If we show that

$$\Delta(b) = \frac{b^2}{e}(kp_3(b)b + kp_2(b)) + kp_1(b) > 0 \tag{2.74}$$

and

$$\Theta(b) = b(kp_3(b)b + kp_2(b)) + kp_1(b) > 0 \tag{2.75}$$

then

$$a(kp_3(b)b + kp_2(b)) + kp_1(b) > 0 \quad \text{for} \quad \frac{b^2}{e} < a < b \tag{2.76}$$

and the proof of the case III will be completed. We prove (2.74). $\Delta(b) > 0$ is equivalent to

$$\begin{aligned} & b + \frac{3}{e}b^2 - \left(\frac{5}{4} + \frac{1}{8e}\right)b^2 + \left(\frac{1}{2e} + \frac{1}{4e^2}\right)b^4 - b^3\ln^3 b - \left(\frac{1}{4e^2} + \frac{1}{8e^3}\right)b^5 - 2b\ln b + \\ & \left(\left(2e - \frac{2}{e}\right)b^2 + \left(\frac{7}{4} + \frac{1}{e}\right)b^3 - \frac{1}{4e}b^4 - \frac{3}{4e^2}b^5 + \frac{1}{4e^3}b^6\right)\ln b + \left(-b^3 - \frac{5}{4e}b^4 + \frac{3}{2e^2}b^5 - \frac{1}{4e^3}b^6\right)\ln^2 b > 0. \end{aligned} \tag{2.77}$$

Using (2.23), (2.46), simple computation and omitting $-b^3\ln^3 b$ we obtain

$$\Delta(b) > 1 - 1.36b + 3.92b^2 + 1.64b^3 + 1.45b^4 - 0.971b^5 + 0.177b^6 - 0.0063b^7 > 0.$$

which can be easily showed.

We prove (2.75). After some calculation we have $\Theta(b) > 0$ is equivalent to

$$4b - \left(\frac{1}{8} + \frac{5e}{4}\right) b^2 + \left(\frac{1}{2} + \frac{1}{4e}\right) b^3 - \left(\frac{1}{4e} + \frac{1}{8e^2}\right) b^4 + \left(-eb^2 - \frac{5}{4}b^3 + \frac{3}{2e}b^4 - \frac{1}{4e^2}b^5\right) \ln^2 b +$$
$$\left(-4b + \left(1 + \frac{15e}{4}\right) b^2 - \frac{1}{4}b^3 - \frac{3}{4e}b^4 + \frac{1}{4e^2}b^5\right) \ln b - eb^2 \ln^3 b > 0. \tag{2.78}$$

Using (2.23), (2.46), simple computation and omitting $-eb^2 \ln^3 b$ we obtain

$$2 + \left(\frac{7}{2} - \frac{23e}{8}\right) b + \left(-\frac{13}{4} + \frac{3e}{4}\right) b^2 + \left(\frac{7}{2} + \frac{7e}{8} + \frac{17}{8e}\right) b^3 - \left(\frac{1}{2e^2} + \frac{13}{4e} + \frac{11}{8}\right) b^4 + \left(\frac{9}{8e} + \frac{1}{2e^2}\right) b^5 - \frac{1}{8e^2}b^6 > 0$$

which can be easily proved.

The case IV ($b_1 < b \leq 0,766$).

We show that

$$kp_3(b)b + kp_2(b) < 0 \tag{2.79}$$

and

$$\Delta_1(b) = \frac{b^2}{e}kp_3(b)b + kp_2(b)b + kp_1(b) > 0 \tag{2.80}$$

for $b_1 < b \leq 0.766$.

Since $kp_3(b) > 0$ we have

$$kp_3(b)a + kp_2(b) < 0 \quad \text{for} \quad \frac{b^2}{e} < a < b.$$

It implies

$$v_1(a, b) = a(kp_3(b)a + kp_2(b)) + kp_1(b) > b(kp_3(b)a + kp_2(b)) + kp_1(b). \tag{2.81}$$

From (2.80) we obtain that $v_1(a, b) > 0$ and the proof of the case IV. will be completed.

We prove (2.79), (2.80). (2.79) is equivalent to

$$3 - \left(\frac{1}{8} + \frac{5e}{4}\right) b + \left(\frac{1}{2} + \frac{1}{4e}\right) b^2 - \left(\frac{1}{4e} + \frac{1}{8e^2}\right) b^3 + \left(-eb - \frac{5}{4}b^2 + \frac{3}{2e}b^3 - \frac{b^4}{4e^2}\right) \ln^2 b - eb \ln^3 b +$$
$$\left(\left(1 + 3e - \frac{5e}{4}\right) b - 2 - \frac{b^2}{4} + \left(-\frac{1}{e} + \frac{1}{4e}\right) b^3 + \frac{b^4}{4e^2}\right) \ln b < 0. \tag{2.82}$$

Using

$$\ln^2 b > (1 - b)^2 + (1 - b)^3.$$
$$\ln b < 2\frac{b - 1}{1 + b}.$$

Simple computation and omitting $-eb \ln^3 b$ we obtain

$$2e + 7 - \left(\frac{25}{8} + \frac{51e}{4}\right) b + \left(\frac{3}{8} + \frac{45e}{4} + \frac{1}{4e}\right) b^2 + \left(\frac{15}{4} - e + \frac{18}{4e} - \frac{1}{8e^2}\right) b^3 + \left(\frac{5}{4} - 3e - \frac{25}{4r} - \frac{9}{8e^2}\right) b^4 +$$
$$\left(-\frac{15}{4} + e - \frac{3}{2e} + \frac{5}{4e^2}\right) b^5 + \left(\frac{5}{4} + \frac{9}{2e} + \frac{1}{4e^2}\right) b^6 - \left(\frac{3}{2e} + \frac{3}{4e^2}\right) b^7 + \frac{1}{4e^2}b^8 < 0$$

which can be easily showed.

We prove (2.80). After some calculation we have $\Delta_1(b) > 0$ is equivalent to

$$5 - \left(\frac{5}{8} + \frac{9e}{4} + \frac{1}{e}\right) b + \left(\frac{1}{2} + \frac{1}{4e}\right) b^2 + \left(\frac{3}{4e} + \frac{3}{8e^2}\right) b^3 + \left(-4 + \frac{3e}{4}b + \left(\frac{11}{4} + \frac{1}{e}\right) b^2 - \frac{1}{4e}b^3 - \frac{3}{4e^2}b^4\right) \ln b +$$
$$\left(-2eb + \frac{11}{4}b^2 - \frac{5}{2e}b^3 + \frac{3b^4}{4e^2}\right) \ln^2 b - b^3 \ln^3 b > 0. \tag{2.83}$$

Using

$$\ln^2 b < (1-b)^2 + (1-b)^3 + \frac{11}{12}(1-b)^4 + \frac{(1-b)^5}{b},$$

$$\ln b < b - 1 - \frac{(1-b)^2}{2} - \frac{2(1-b)^3}{3(1+b)},$$

simple computation and omitting $-b^3 \ln^3 b$ we obtain

$$\Delta_1(b) > 13.6667 - 32.3829b + 33.1054b^2 + 61.9658b^3 - 18.8435b^4 + 32.5485b^5 - 19.5906b^6 + 6.4433b^7 - 1.2237b^8 + 0.0930b^9 > 0$$

which can be easily showed.

The case V $(0.766 < b \leq 1)$.

We show that

$$kp_3(b) > 12.1b - 8.8, \tag{2.84}$$

$$kp_2(b) > -12.5b + 9, \tag{2.85}$$

$$4(12.1b - 8.8))kp_1(b) - (9 - 12.5b)^2 > 0. \tag{2.86}$$

It implies

$$kp_3(b)a^2 + kp_2(b)a + kp_1(b) > (12.1b - 8.8)a^2 + (-12.5b + 9)a + kp_1(b) > 0 \text{ for } \frac{b^2}{e} < a < b,\ 0.766 < b \leq 1$$

which completes the proof of the case V.

We prove (2.84), (2.85), (2.86). The proof of (2.84). Using $\ln^2 b > (1-b)^2$, (2.46) we have

$$kp_3(b) - 12.1b + 8.8 > aa(b) = 9.3 + e + \left(-11.1 + \frac{1}{2e}\right) b - \frac{1}{b} + \left(3e + 1 - \frac{b^2}{e}\right)\left(b - 1 - \frac{(b-1)^2}{2b}\right) + \left(e - 3b + \frac{b^2}{e}\right)(1-b)^2.$$

After some computation we get that $aa(b) > 0$ is equivalent to

$$(9.3 + 2e)b + \left(-13.6 + \frac{1}{e} - \frac{e}{2}\right) b^2 + \left(6 + \frac{1}{e} + e\right) b^3 - \frac{3}{2}(1+e) - \left(3 + \frac{5}{2e}\right) b^4 + \frac{1}{e} b^5 > 0.$$

which can be easily showed.

The proof of (2.85). Using (2.23), $\ln b < 2\frac{b-1}{b+1}$ we have

$$kp_2(b) + 12.5b - 9 > bb(b) = -5 + \left(12.5 - \frac{9}{4}e - \frac{5}{8}\right) b - \left(\frac{1}{2} + \frac{1}{4e}\right) b^2 - \left(\frac{1}{4e} + \frac{1}{8e^2}\right) b^3 + \frac{b-1}{b+1}\left(-4 - \frac{5e}{2} b - \frac{1}{2} b^2 + \frac{1}{2e} b^3 + \frac{1}{2e^2} b^4\right) + (1-b)^2 \left(-2e + \frac{7}{4} b + \frac{1}{2e} b^2 - \frac{1}{4e^2} b^3\right).$$

It is easy to show that $bb(b) > 0$. It follows from $bb(b) > 0$ is equivalent to

$$-1 - 2e + \left(\frac{37}{8} + \frac{9e}{4}\right) b + \left(\frac{81}{8} + \frac{1}{4e} - \frac{11e}{4}\right) b^2 + \left(-\frac{11}{4} - 2e - \frac{6}{4e} - \frac{3}{8e^2}\right) b^3 + \left(\frac{7}{4} - \frac{1}{4e} b - \frac{3}{8e^2}\right) b^4 + \left(\frac{1}{2e} + \frac{3}{4e^2}\right) b^5 - \frac{1}{4e^2} b^6 > 0$$

The proof of (2.86).

(2.86) can be rewriting as

$$-81 + 192.5b - 107.85b^2 + (70.4 - (96.8 + 70.4e)b + 96.8eb^2)b \ln b > 0.$$

Using (2.46)) we obtain

$$-116.2 + (240.9 + 35.2e)b - (72.65 + 48.4e)b^2 - (48.4 + 35.2e)b^3 + 48.4eb^4 > 0. \tag{2.87}$$

which can be easily showed.

We note, that e is the greatest possible value of a positive real number r such that the inequality (1.1) holds for all positive real numbers a and b such that $a, b \leq 1$. It follows from the following lemma.

Lemma 2.2. *Let r be a real number such that $r > e$. Then there are numbers a, b such that $0 < a < b < 1$ and*

$$2\sqrt{a^{ra}b^{rb}} < a^{rb} + b^{ra}.$$

Proof. Put $a = 1/r$, $b = 1/e$ and define

$$gf(r) = 2\left(\frac{1}{r}\right)^{\frac{1}{2}}\left(\frac{1}{e}\right)^{\frac{r}{2e}} - \left(\frac{1}{r}\right)^{\frac{r}{e}} - \left(\frac{1}{e}\right).$$

If we show $gf(r) < 0$ then the proof will be completed. Put $s = e/r$ then $0 < s < 1$ and $gf(r) < 0$ is equivalent to

$$fg(s) = 2e^{\frac{1}{2}\ln s + \frac{1}{2} - \frac{1}{2s}} - e^{1 - \frac{1}{2} + \frac{\ln s}{s}} < 1.$$

From $\lim\limits_{s\to 1^-} fg(s) = 1$ it suffices to show that

$$fg'(s) = e^{\frac{1}{2}\ln s + \frac{1}{2} - \frac{1}{2s}}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{1 - \frac{1}{2} + \frac{\ln s}{s}}\left(\frac{2}{s^2} - \frac{\ln s}{s^2}\right) > 0.$$

It is equivalent to

$$\ln(1 + s) - \ln(2 - \ln s) - \frac{1}{2} + \frac{1}{2s} + \frac{s - 2}{2s}\ln s > 0. \tag{2.88}$$

From $\ln x > (x - 1)/x$ for $0 < x < 1$ we have

$$\ln\left(\frac{1 + s1}{2 - s}\right) > \frac{s + \ln s - 1}{1 + s}. \tag{2.89}$$

Using (2.89) in (2.88) we get that it suffices to show that

$$gg(s) = s - 1 + (s + 2)\ln s < 0. \tag{2.90}$$

To complete the proof, we only need to show that $gg(1) = 0$, $gg'(s) > 0$ which is evident.

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