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Fixed point theorems for α-β-ψ-contractive mappings in partially ordered sets

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Abstract

In this paper, we introduce a new concept of α-β-ψ-contractive type mappings and construct some fixed point theorems for such mappings in metric spaces endowed with partial order. Moreover, we use fixed point theorems to find a solution for the first-order boundary value differential equation.

Keywords: Fixed point, α-β-ψ-contractive mappings, partially ordered sets, lower and upper solutions.
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1. Introduction and Preliminaries

The existence of fixed point in partially ordered sets has been considered in [1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 15, 16, 19]. Furthermore, some applications to periodic boundary value problems and matrix equations were given in [13, 14, 17]. Recently, Samet et al. [18] introduced α-ψ-contractive type mappings in complete metric space and established some fixed point theorems as well as their applications to a second-order ordinary differential equation. In this paper, we introduce a new concept of α-β-ψ-contractive type mappings and establish some fixed point theorems in a metric space endowed with partial order. The presented theorems extend, generalize and improve many existing results in the literature, in particular the results of Ran and Reurings [17], Nieto and Rodríguez-López [12, 13] and Harjani and Sadarangani [7]. In the literature, we can find results on existence of solution for ordinary differential equations in presence of both lower and upper solutions. In this paper, we assume the existence of just one of them for the periodic boundary value problem

$$\begin{cases} u'(t) = h(t, u(t)), & t \in I = [0, T], \\ u(0) = u(T), \end{cases} \tag{1.1}$$

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where $T > 0$, and $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. A solution to (1.1) is a function $u \in C^1(I, \mathbb{R})$ satisfying conditions in (1.1). A lower solution for (1.1) is a function $u \in C^1(I, \mathbb{R})$ such that

$$\begin{cases} u'(t) \leq h(t, u(t)), & t \in I = [0, T], \\ u(0) \leq u(T). \end{cases}$$

An upper solution for (1.1) satisfies the reversed inequalities. It is well known [10] that the existence of a lower solution u and an upper solution v with $u \leq v$ implies the existence of a solution of (1.1) between u and v. In this paper, the existence of a unique solution for problem (1.1) was obtained under suitable conditions. Let's start by a few definitions and lemmas.

Definition 1.1. Let $(X, \leq)$ be a partially ordered set. We say that $f : X \to X$ is monotone nondecreasing if for all $x, y \in X$,

$$x \leq y \Longrightarrow f(x) \leq f(y).$$

Definition 1.2 ([18])**.** Let Ψ be a family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi$ and $t > 0$, $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, where ψ^n is the n-th iterate of ψ.

Lemma 1.3 ([18])**.** *Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function. If for each $t > 0$, $\lim_{t\to\infty} \psi^n(t) = 0$ then $\psi(t) < t$.*

Definition 1.4. Let $(X, \leq)$ be a partially ordered space with complete metric d. We say that $f : X \to X$ is a α-β-ψ-contractive mapping if there exist three functions $\alpha, \beta : X \times X \to [0, \infty)$, $\psi \in \Psi$ such that for all $x, y \in X$ with $x \geq y$,

$$\alpha(x, y)d(f(x), f(y)) \leq \beta(x, y)\psi(d(x, y)). \tag{1.2}$$

Example 1.5. If $f : X \to X$ satisfies the Banach contraction principle, then f is an α-β-ψ-contractive mapping, where $\alpha(x, y) = \beta(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = ct$ for all $t \geq 0$ and some $c \in [0, 1)$.

Definition 1.6. Let $f : X \to X$, $\alpha, \beta : X \times X \to [0, \infty)$ and $C_\alpha > 0$, $C_\beta \geq 0$. We say that f is an α-β-admissible mapping if for all $x, y \in X$ with $x \geq y$,

(i) $\alpha(x, y) \geq C_\alpha$ implies $\alpha(f(x), f(y)) \geq C_\alpha$;
(ii) $\beta(x, y) \leq C_\beta$ implies $\beta(f(x), f(y)) \leq C_\beta$;
(iii) $0 \leq C_\beta/C_\alpha \leq 1$.

Example 1.7. Let $X = (0, +\infty)$. Define $f : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ by $f(x) = e^x$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} 3 & \text{if } x \geq y; \\ 0 & \text{otherwise,} \end{cases}, \quad \beta(x, y) = \begin{cases} 1/4 & \text{if } x \geq y; \\ 0 & \text{otherwise,} \end{cases}$$

let $C_\alpha = 2$ and $C_\beta = 1/2$ then f is α-β-admissible.

Example 1.8. Let $X = [0, +\infty)$. Define $f : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ by $f(x) = \sqrt[3]{x}$ for all $x \in X$ and

$$\alpha(x, y) = \begin{cases} xy^{-1} & \text{if } x \geq y; \\ 0 & \text{otherwise,} \end{cases}, \quad \beta(x, y) = \begin{cases} \frac{2^{y-x}}{3} & \text{if } x \geq y; \\ 0 & \text{otherwise,} \end{cases}$$

let $C_\alpha = 1/2$ and $C_\beta = 1/3$ then f is α-β-admissible.

2. Fixed point theorems

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a nondecreasing, α-β-ψ-contractive mapping satisfying the following conditions:*

(i) f is continuous;
(ii) f is α-β-admissible;
(iii) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$;
(iv) there exist $C_\alpha > 0$, $C_\beta \geq 0$ such that $\alpha(f(x_0), x_0) \geq C_\alpha$, $\beta(f(x_0), x_0) \leq C_\beta$.

Then, f has a fixed point.

Proof. If $f(x_0) = x_0$, then the proof is finished. Suppose that $f(x_0) \neq x_0$. Since $x_0 \leq f(x_0)$ and f is nondecreasing, we obtain by induction that

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \ldots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots, \tag{2.1}$$

also, since f is α-β-admissible by (iv), we get

$$\begin{cases} \alpha(f(x_0), x_0) \geq C_\alpha \to \alpha(f^2(x_0), f(x_0)) \geq C_\alpha \to \ldots \to \alpha(f^{n+1}(x_0), f^n(x_0)) \geq C_\alpha, \\ \beta(f(x_0), x_0) \leq C_\beta \to \beta(f^2(x_0), f(x_0)) \leq C_\beta \to \ldots \to \beta(f^{n+1}(x_0), f^n(x_0)) \leq C_\beta. \end{cases} \tag{2.2}$$

Now, by (1.2), (2.1) and (2.2), we obtain

$$\begin{aligned} C_\alpha d(f^2(x_0), f(x_0)) &\leq \alpha(f(x_0), x_0) d(f^2(x_0), f(x_0)) \\ &\leq \beta(f(x_0), x_0)\psi(d(f(x_0), x_0)) \\ &\leq C_\beta \psi(d(f(x_0), x_0)). \end{aligned}$$

Hence,

$$d(f^2(x_0), f(x_0)) \leq C_\beta / C_\alpha \psi(d(f(x_0), x_0)) \leq \psi(d(f(x_0), x_0)).$$

Continuing this process, we get

$$d(f^{n+1}(x_0), f^n(x_0)) \leq \psi^n(d(f(x_0), x_0)).$$

Now, as $n \to \infty$ then $d(f^{n+1}(x_0), f^n(x_0)) \to 0$. We show that $\{f^n(x_0)\}_{n=1}^\infty$ is a Cauchy sequence. Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(d(f(x_0), x_0)) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n(\epsilon)$, by triangular inequality,

$$\begin{aligned} d(f^n(x_0), f^m(x_0)) &\leq d(f^n(x_0), f^{n+1}(x_0)) + \ldots + d(f^{m-1}(x_0), f^m(x_0)) \\ &\leq \psi^n(d(f(x_0), x_0)) + \ldots + \psi^{m-1}(d(f(x_0), x_0)) \\ &= \sum_{k=n}^{m-1} \psi^k(d(f(x_0), x_0)) \\ &\leq \sum_{n \geq n(\epsilon)} \psi^n(d(f(x_0), x_0)) < \epsilon. \end{aligned}$$

Since (X, d) is a complete metric space, then there exists $x \in X$ such that $\lim_{n\to\infty} f^n(x_0) = x$. Now, we show that x is a fixed point of $f(x)$. Suppose $\epsilon > 0$ is given. Since f is a continuous function, then there exists

$\delta > 0$ such that, for each $z \in X$, $d(z,x) < \delta$ implies that $d(f(z), f(x)) < \frac{\epsilon}{2}$. Given $\eta = min\{\frac{\epsilon}{2}, \delta\}$, now by convergence of $\{f^n(x_0)\}_{n=1}^{\infty}$ to x, there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n \geq n_0$,

$$d(f^n(x_0), x) < \eta.$$

Taking $n \in \mathbb{N}$, $n \geq n_0$, we get

$$\begin{aligned} d(f(x), x) &\leq d(f(f^n(x_0), f(x)) + d(f^{n+1}(x_0), x) \\ &< \frac{\epsilon}{2} + \eta \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

therefore, $d(f(x), x) = 0$. Consequently, $f(x) = x$. □

In the next theorem, the continuity hypothesis of f has been removed.

Theorem 2.2. *Let $(X, \leq)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a nondecreasing, α-β-ψ-contractive mapping satisfying the following conditions:*

(i) f is α-β-admissible;
(ii) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$;
(iii) there exist $C_\alpha > 0$, $C_\beta \geq 0$ such that $\alpha(f(x_0), x_0) \geq C_\alpha$, $\beta(f(x_0), x_0) \leq C_\beta$;
(iv) if $\{x_n\}_{n=1}^{\infty}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x) \geq C_\alpha$, $\beta(x_n, x) \leq C_\beta$;
(v) if $\{x_n\}$ be a nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$, for all $n \in \mathbb{N}$.

Then, f has a fixed point.

Proof. Following the proof of Theorem 2.1, since $\{f^n(x_0)\}$ is a cauchy sequence, then there exists $x \in X$ such that $\lim_{n\to\infty} f^n(x_0) = x$. We will show that x is a fixed point of $f(x)$. Given $\epsilon > 0$, since $\{f^n(x_0)\}_{n=1}^{\infty}$ converges to x, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(f^n(x_0), x) < \frac{\epsilon}{2}.$$

Moreover, since $\{f^n(x_0)\}$ is a nondecreasing sequence, from (v), we have

$$f^n(x_0) \leq x. \tag{2.3}$$

From (1.2), (2.2), (2.3) and (iv), we get

$$\begin{aligned} C_\alpha d(x, f(x)) &\leq C_\alpha d(f(f^n(x_0), f(x))) + C_\alpha d(f^{n+1}(x_0), x) \\ &\leq \alpha(f^n(x_0), x) d(f(f^n(x_0), f(x))) + C_\alpha d(f^{n+1}(x_0), x) \\ &\leq \beta(f^n(x_0), x)\psi(d(f^n(x_0), x)) + C_\alpha d(f^{n+1}(x_0), x) \\ &< C_\beta \psi(d(f^n(x_0), x)) + C_\alpha d(f^{n+1}(x_0), x), \end{aligned}$$

therefore,

$$d(x, f(x)) < C_\beta / C_\alpha \psi(d(f^n(x_0), x)) + d(f^{n+1}(x_0), x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $d(x, f(x)) = 0$, that is, $f(x) = x$. □

Example 2.3. Let $(\mathbb{R}, \leq)$ and $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta : X \times X \to [0, +\infty)$, by

$$f(x) = \begin{cases} \frac{x}{15} & \text{if } x \geq 0; \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x, y \geq 0; \\ 0 & \text{otherwise,} \end{cases} \quad , \quad \beta(x,y) = \begin{cases} 1/3 & \text{if } x, y \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for each $t > 0$. Clearly, f is an α-β-ψ-contractive mapping. Moreover, f is nondecreasing and continuous. We show that f is α-β-admissible. For all $x, y \in [0, +\infty)$ with $x \geq y$. Let $C_\alpha = 3/2$ and $C_\beta = 1/2$, we have

$$\alpha(x,y) \geq C_\alpha \quad \to \quad \alpha(f(x), f(y)) = \alpha(\frac{x}{15}, \frac{y}{15}) \geq C_\alpha,$$

also

$$\beta(x,y) \leq C_\beta \quad \to \quad \beta(f(x), f(y)) = \beta(\frac{x}{15}, \frac{y}{15}) \leq C_\beta.$$

In addition, there exists $x_0 = 0 \in \mathbb{R}$ such that $\alpha(f(x_0), x_0) \geq C_\alpha$ and $\beta(f(x_0), x_0) \leq C_\beta$. Further, since $0 \leq f(0) = 0$ then $x_0 \leq f(x_0)$. Now, all the hypotheses of Theorem 2.1 are satisfied consequently, f has a fixed point. Here, 0 is a fixed point of f.

In the following example, the continuity of f has been removed.

Example 2.4. Let $(\mathbb{R}, \leq)$ and $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta : X \times X \to [0, +\infty)$, by

$$f(x) = \begin{cases} 2x - \frac{1}{2} & \text{if } x \geq \frac{1}{2}; \\ \frac{x}{10} & \text{if } 0 \leq x < \frac{1}{2}; \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{1}{2}); \\ 0 & \text{otherwise,} \end{cases} \quad , \quad \beta(x,y) = \begin{cases} 1/3 & \text{if } x, y \in [0, \frac{1}{2}); \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is nondecreasing and discontinuous. Let $\psi(t) = \frac{t}{3}$ for each $t > 0$. Obviously, if $x, y \in \mathbb{R} - [0, 1/2)$, then f is an α-β-ψ-contractive mapping. Suppose that $x, y \in [0, 1/2)$ with $x \geq y$, let $C_\alpha = 1/2$ and $C_\beta = 1/3$ then $\alpha(x,y) \geq C_\alpha$ and $\beta(x,y) \leq C_\beta$. Hence,

$$\alpha(x,y)d(f(x), f(y)) = |f(x) - f(y)| = |\frac{x}{10} - \frac{y}{10}| = \frac{|x-y|}{10}$$

and

$$\beta(x,y)\psi(d(x,y)) = \frac{d(x,y)}{9} = \frac{|x-y|}{9},$$

therefore,

$$\frac{|x-y|}{10} \leq \frac{|x-y|}{9}.$$

In other words,

$$\alpha(x,y)d(f(x), f(y)) \leq \beta(x,y)\psi(d(x,y)).$$

So, for all $x, y \in \mathbb{R}$, f is an α-β-ψ-contractive mapping. Moreover, there exists $x_0 \in \mathbb{R}$ such that $\alpha(f(x_0), x_0) \geq C_\alpha$ and $\beta(f(x_0), x_0) \leq C_\beta$. Let $x_0 = 0$ then

$$\alpha(f(x_0), x_0) = \alpha(f(0), 0) = \alpha(0, 0) = 1 \geq C_\alpha = 1/2,$$

and

$$\beta(f(x_0), x_0) = \beta(f(0), 0) = \beta(0, 0) = 1/3 \leq C_\beta = 1/3.$$

Since $0 = x_0 \leq 0 = f(x_0)$ then $x_0 \leq f(x_0)$. Clearly, f is α-β-admissible. Finally, if $\{x_n\}$ be a nondecreasing sequence in $\mathbb{R}$ such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$ and $\beta(x_n, x_{n+1}) \leq C_\beta$ for all $n \in \mathbb{N}$ and $x_n \to x$ then, by definitions of α and β, $x_n \in [0, \frac{1}{2})$. Consequently, $x \in [0, \frac{1}{2})$. In addition, $\{x_n\}$ is nondecreasing hence $x_n \leq x$. Therefore, all the required hypotheses of Theorem 2.2 are satisfied, then f has a fixed point. Here, 0 and $\frac{1}{2}$ are two fixed points of f.

Regarding to the above examples , it is seen that f may have more than one fixed point. In the following, additional condition is applied to the hypotheses of Theorems 2.1 and 2.2 to obtain the singularity of the fixed point.

Theorem 2.5. *Suppose all the hypotheses of Theorems 2.1 and 2.2 are satisfied. If there exists $z \in X$ such that for all $x, y \in X$ with $x \geq z$, $y \geq z$,*

$$\begin{cases} \alpha(x, z) \geq C_\alpha \quad \text{and} \quad \beta(x, z) \leq C_\beta, \\ \alpha(y, z) \geq C_\beta \quad \text{and} \quad \beta(y, z) \leq C_\beta. \end{cases} \tag{2.4}$$

Then, f has a unique fixed point.

Proof. Suppose $x^\star$ and $y^\star$ are two fixed points of f, then, $f(x^\star) = x^\star$ and $f(y^\star) = y^\star$. By the first part of (2.4), there exists $z \in X$ such that

$$\alpha(x^\star, z) \geq C_\alpha \quad \text{and} \quad \beta(x^\star, z) \leq C_\beta, \quad x^\star \geq z. \tag{2.5}$$

Since f is α-β-admissible, we get

$$\alpha(f(x^\star), f(z)) \geq C_\alpha \quad \text{and} \quad \beta(f(x^\star), f(z)) \leq C_\beta, \quad f(x^\star) \geq f(z),$$

therefore,

$$\alpha(x^\star, f(z)) \geq C_\alpha \quad \text{and} \quad \beta(x^\star, f(z)) \leq C_\beta, \quad x^\star \geq f(z).$$

Continuing this process, we have

$$\alpha(x^\star, f^n(z)) \geq C_\alpha \quad \text{and} \quad \beta(x^\star, f^n(z)) \geq C_\beta, \quad x^\star \geq f^n(z), \tag{2.6}$$

for all $n \in \mathbb{N}$. Since f is α-β-ψ-contractive mapping, then we get

$$\begin{aligned} C_\alpha d(x^\star, f^n(z)) &= C_\alpha d(f(x^\star), f(f^{n-1}(z))) \\ &\leq \alpha(x^\star, f^{n-1}(z)) d(f(x^\star), f(f^{n-1}(z))) \\ &\leq \beta(x^\star, f^{n-1}(z)) \psi(d(x^\star, f^{n-1}(z))) \\ &\leq C_\beta \psi(d(x^\star, f^{n-1}(z))), \end{aligned}$$

so,

$$\begin{aligned} d(x^\star, f^n(z)) &\leq C_\beta / C_\alpha \psi(d(x^\star, f^{n-1}(z))) \\ &\leq \psi(d(x^\star, f^{n-1}(z))) \\ &\leq \psi(\psi(d(x^\star, f^{n-2}(z)))) \\ &\vdots \\ &\leq \psi^n(d(x^\star, z)), \end{aligned}$$

which implies that,

$$d(x^\star, f^n(z)) \le \psi^n(d(x^\star, z)).$$

For all $n \in \mathbb{N}$. Now, as $n \to \infty$ then $f^n(z) \to x^\star$. Similarly for the second part of (2.4), $f^n(z) \to y^\star$. Therefore, $x^\star = y^\star$. That means f has a unique fixed point. □

Theorem 2.6. *Let $(X, \le)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a nondecreasing, α-β-ψ-contractive mapping satisfying the following conditions:*

(i) f is continuous;
(ii) f is α-β-admissible;
(iii) there exists $x_0 \in X$ such that $x_0 \ge f(x_0)$;
(iv) there exist $C_\alpha > 0$, $C_\beta \ge 0$ such that $\alpha(x_0, f(x_0)) \ge C_\alpha$, $\beta(x_0, f(x_0)) \le C_\beta$.

Then, f has a fixed point.

Theorem 2.7. *Let $(X, \le)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a nondecreasing, α-β-ψ-contractive mapping satisfying the following conditions:*

(i) f is α-β-admissible;
(ii) there exists $x_0 \in X$ such that $x_0 \ge f(x_0)$;
(iii) there exist $C_\alpha > 0$, $C_\beta \ge 0$ such that $\alpha(x_0, f(x_0)) \ge C_\alpha$, $\beta(x_0, f(x_0)) \le C_\beta$;
(iv) if $\{x_n\}_{n=1}^\infty$ be a sequence in X such that $\alpha(x_{n+1}, x_n) \ge C_\alpha$, $\beta(x_{n+1}, x_n) \le C_\beta$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x, x_n) \ge C_\alpha$, $\beta(x, x_n) \le C_\beta$;
(v) if $\{x_n\}$ be a nonincreasing sequence in X such that $x_n \to x$ then $x \le x_n$, for all $n \in \mathbb{N}$.

Then, f has a fixed point.

Theorem 2.8. *Suppose all the hypotheses of Theorems 2.6 and 2.7 are satisfied. If there exists $z \in X$ such that for all $x, y \in X$ with $z \ge x$, $z \ge y$,*

$$\begin{cases} \alpha(z,x) \ge C_\alpha \quad \text{and} \quad \beta(z,x) \le C_\beta, \\ \alpha(z,y) \ge C_\beta \quad \text{and} \quad \beta(z,y) \le C_\beta. \end{cases} \tag{2.7}$$

Then, f has a unique fixed point.

3. Application to ordinary differential equations

In this section, we prove the existence of the unique solution of problem (1.1) in the presence of it's lower solution with α-β-ψ-contractive mappings. This problem is solved by Nieto and Rodríguez-López [13] in the presence of a lower solution, as follow.

Theorem 3.1. *Consider problem* (1.1) *with $h : I \times \mathbb{R} \to \mathbb{R}$ continuous. Suppose that there exist $\lambda > 0$ and $\mu > 0$ with $\mu < \lambda$ such that for all $x, y \in \mathbb{R}$, with $y \ge x$,*

$$0 \le h(t,y) + \lambda y - h(t,x) - \lambda x \le \mu(y - x),$$

then, the existence of a lower solution for (1.1)*, provides the existence of a unique solution of* (1.1).

Also, Harjani and Sadarangani [7] have established following theorem:

Theorem 3.2. *Consider problem* (1.1) *with $h : I \times \mathbb{R} \to \mathbb{R}$ continuous. Suppose that there exists $\lambda > 0$ such that for all $x, y \in \mathbb{R}$, with $y \ge x$,*

$$0 \le h(t,y) + \lambda y - h(t,x) - \lambda x \le \lambda\psi(y - x),$$

where $\psi : [0,\infty) \to [0,\infty)$ can be written by $\psi(x) = x - \phi(x)$ with $\phi : [0,\infty) \to [0,\infty)$ continuous, increasing, positive in $(0,\infty)$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Then the existence of a lower solution of (1.1) *provides the existence of a unique solution of* (1.1).

Now, we are ready to solve problem (1.1) according to our presented theorems.

Remark 3.3. For each $\lambda > 0$, problem (1.1) is written as

$$\begin{cases} u'(t) + \lambda u(t) = h(t, u(t)) + \lambda u(t), & t \in I = [0, T]; \\ u(0) = u(T), \end{cases}$$

this differential equation is equivalent to the integral equation:

$$u(t) = \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds,$$

where,

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T; \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

In the theory of differential equations, $G(t,s)$ is called Green function.

Theorem 3.4. *Consider differential equation* (1.1) *with continuous* $h : I \times \mathbb{R} \to \mathbb{R}$ *by following conditions:*

(i) there exists $\lambda > 0$ *such that for all* $x, y \in \mathbb{R}$, *with* $y \geq x$, *and* $\psi \in \Psi$,

$$0 \leq h(t,y) + \lambda y - h(t,x) - \lambda x \leq \lambda \psi(y-x);$$

(ii) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ *such that for all* $t \in I$, *for all* $a, b \in \mathbb{R}$ *with* $\xi(a,b) \geq 0$,

$$\xi\left(\int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds, \gamma(t)\right) \geq 0,$$

where $\gamma \in C(I,\mathbb{R})$ *be a lower solution of* (1.1)*;*

(iii) for all $t \in I$ *and all* $x, y \in C(I,\mathbb{R})$, $\xi(x(t), y(t)) \geq 0$ *implies,*

$$\xi\left(\int_0^T G(t,s)[h(s,x(s)) + \lambda u(s)]ds, \int_0^T G(t,s)[h(s,y(s)) + \lambda u(s)]ds\right) \geq 0;$$

(iv) if $x_n \to x \in C(I,\mathbb{R})$ *and* $\xi(x_n, x_{n+1}) \geq 0$ *then* $\xi(x_n, x) \geq 0$ *for all* $n \in \mathbb{N}$.

Therefore, the existence of a lower solution for (1.1) *provides a unique solution of* (1.1).

Proof. Regarding to the Remark 3.3, we define $\mathcal{A} : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$[\mathcal{A}u](t) = \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds, \quad t \in I.$$

Note that if $u \in C(I,\mathbb{R})$ is a fixed point of $\mathcal{A}$, then $u \in C^1(I,\mathbb{R})$ is a solution of (1.1). Let $X = C(I,\mathbb{R})$. By the following order relation, X is a partially ordered set.

$$x, y \in X, \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \qquad t \in I.$$

If we choose

$$d(x,y) = \sup_{t \in I} |x(t) - y(t)|, \qquad x, y \in X$$

then (X,d) is a complete metric space. Assume a monotone nondecreasing sequence $\{x_n\} \subseteq C(I,\mathbb{R})$ converging to $x \in C(I,\mathbb{R})$, then for each $t \in I$,

$$x_1(t) \leq x_2(t) \leq x_3(t) \leq \cdots \leq x_n(t) \leq \cdots.$$

The convergence of this sequence to $x(t)$ implies that $x_n(t) \leq x(t)$, for all $t \in I$, all $n \in \mathbb{N}$. Therefore, $x_n \leq x$ for all $n \in \mathbb{N}$. Moreover, $\mathcal{A}$ is a nondecreasing mapping, since for all $u, v \in X$ with $u \geq v$,

$$h(t,u) + \lambda u \geq h(t,v) + \lambda v$$

and also $G(t,s) > 0$ for all $(t,s) \in I \times I$, then

$$\begin{aligned} [\mathcal{A}u]\,(t) &= \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds \\ &\geq \int_0^T G(t,s)[h(s,v(s)) + \lambda v(s)]ds = [\mathcal{A}v]\,(t). \end{aligned}$$

In addition, for $u \geq v$ by (i) and the definition of $G(t,s)$, we obtain

$$\begin{aligned} d(\mathcal{A}u, \mathcal{A}v) &= \sup_{t\in I} |\mathcal{A}u(t) - \mathcal{A}v(t)| \\ &\leq \sup_{t\in I} \int_0^T G(t,s)|h(s,u(s)) + \lambda u(s) - h(s,v(s)) - \lambda v(s)|ds \\ &\leq \sup_{t\in I} \int_0^T G(t,s)|\lambda\psi(u(s) - v(s))|ds \\ &\leq \sup_{t\in I} \int_0^T G(t,s)\lambda\psi(|u(s) - v(s)|)ds \\ &\leq \lambda\psi(d(u,v)) \sup_{t\in I} \int_0^T G(t,s)ds \\ &= \lambda\psi(d(u,v)) \sup_{t\in I} \frac{1}{e^{\lambda T} - 1} \left(\frac{1}{\lambda}\, e^{\lambda(T+s-t)}\Big|_0^t + \frac{1}{\lambda}\, e^{\lambda(s-t)}\Big|_t^T \right) \\ &= \lambda\psi(d(u,v)) \times \frac{1}{\lambda} = \psi(d(u,v)), \end{aligned}$$

then

$$(\mathcal{A}u, \mathcal{A}v) \leq \psi(d(u,v)).$$

Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } \xi(u(t), v(t)) \geq 0, \ t \in I; \\ 0 & \text{otherwise,} \end{cases}$$

and $\beta : X \times X \to [0,\infty)$ by

$$\beta(u,v) = \begin{cases} 1 & \text{if } \xi(u(t), v(t)) \geq 0, \ t \in I; \\ 0 & \text{otherwise,} \end{cases}$$

for all $u, v \in X$ with $u \geq v$. Then,

$$\alpha(u,v)d(\mathcal{A}u, \mathcal{A}v) \leq \beta(u,v)\psi(d(u,v)),$$

which implies that $\mathcal{A}$ is an α-β-ψ-contractive mapping. Let $C_\alpha = C_\beta = 1$. From (iii), for all $u, v \in X$ with $u \geq v$, we get

$$\alpha(u,v) \geq 1 = C_\alpha \Longrightarrow \xi(u(t), v(t)) \geq 0 \Longrightarrow \xi(\mathcal{A}u(t), \mathcal{A}v(t)) \geq 0 \Longrightarrow \alpha(\mathcal{A}u, \mathcal{A}v) \geq 1 = C_\alpha,$$

also,

$$\beta(u,v) \leq 1 = C_\beta \Longrightarrow \xi(u(t), v(t)) \geq 0 \Longrightarrow \xi(\mathcal{A}u(t), \mathcal{A}v(t)) \geq 0 \Longrightarrow \beta(\mathcal{A}u, \mathcal{A}v) \leq 1 = C_\beta.$$

Therefore, $\mathcal{A}$ is α-β-admissible. Let η be a lower solution of (1.1), from (ii),

$$\xi((\mathcal{A}\eta)(t),\eta(t)) \geq 0 \Longrightarrow \begin{cases} \alpha(\mathcal{A}\eta,\eta) \geq C_\alpha; \\ \beta(\mathcal{A}\eta,\eta) \leq C_\beta. \end{cases}$$

Now, we show that $\mathcal{A}\eta \geq \eta$. From the definition of lower solution, we have

$$\begin{cases} \eta'(t) \leq h(t,\eta(t)), \quad t \in I = [0,T]; \\ \eta(0) \leq \eta(T). \end{cases}$$

For all $t \in I$ and $\lambda > 0$, hence

$$\eta'(t) + \lambda\eta(t) \leq h(t,\eta(t)) + \lambda\eta(t),$$

multiplying by $e^{\lambda t}$, we get

$$(\eta(t)e^{\lambda t})' \leq (h(t,\eta(t)) + \lambda\eta(t))e^{\lambda t},$$

by integration, we obtain

$$\eta(t)e^{\lambda t} \leq \eta(0) + \int_0^t [h(s,\eta(s)) + \lambda\eta(s)]e^{\lambda s}ds, \tag{3.1}$$

which implies that

$$\eta(0)e^{\lambda T} \leq \eta(T)e^{\lambda T} \leq \eta(0) + \int_0^T [h(s,\eta(s)) + \lambda\eta(s)]e^{\lambda s}ds,$$

and so

$$\eta(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds. \tag{3.2}$$

From (3.1) and (3.2),

$$\begin{aligned} \eta(t)e^{\lambda t} &\leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds + \int_0^t [h(s,\eta(s)) + \lambda\eta(s)]e^{\lambda s}ds \\ &\leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds, \end{aligned}$$

dividing by $e^{\lambda t}$, we obtain

$$\eta(t) \leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[h(s,\eta(s)) + \lambda\eta(s)]ds.$$

Then, by the definition of $G(t,s)$, we have

$$\eta(t) \leq \int_0^T G(t,s)[h(s,\eta(s)) + \lambda\eta(s)]ds = [\mathcal{A}\eta](t),$$

for all $t \in I$, then, $\mathcal{A}\eta \geq \eta$. Finally, from (iv) if $x_n \to x \in X$ for all n, we get

$$\xi(x_n, x_{n+1}) \geq 0 \Longrightarrow \xi(x_n, x) \geq 0,$$

therefore

$$\alpha(x_n, x_{n+1}) \geq C_\alpha \Longrightarrow \alpha(x_n, x) \geq C_\alpha,$$

also

$$\beta(x_n, x_{n+1}) \leq C_\beta \Longrightarrow \beta(x_n, x) \leq C_\beta.$$

Then, all the hypotheses of Theorem 2.2 are satisfied. Consequently, $\mathcal{A}$ has a fixed point and so equation (1.1) has a solution. The uniqueness of the solution comes from Theorem 2.5. □

Theorem 3.5. *If we replace the existence of lower solution to* (1.1) *by upper solution, Theorem 3.4 still holds.*

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