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Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem

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Abstract

Based on the nonlinearization technique, a binary Bargmann symmetry constraint associated with a new discrete 3×3 matrix eigenvalue problem, which implies that there exist infinitely many common commuting symmetries and infinitely many common commuting conserved functionals, is proposed. A new symplectic map of the Bargmann type is obtained through binary nonlinearization of the discrete eigenvalue problem and its adjoint one. The generating function of integrals of motion is obtained, by which the symplectic map is further proved to be completely integrable in the Liouville sense.

Keywords: Discrete Hamiltonian structure, binary Bargmann symmetry constraint, finite-dimensional integrable system .
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1. Introduction

Recently in the past decade, an unusual way of using the nonlinearization technique arose in the theory of soliton equations. In general, one considers the complicated nonlinear problems to be solved in such a way to break nonlinear problems into several linear or smaller ones and then to solve these resulting problems. It is following this idea that one has introduced the method of Lax pair to study nonlinear soliton equations. The Lax pairs are always linear with respect to their eigenfunctions. Nevertheless, the nonlinearization technique puts this original object, the Lax pair, into a nonlinear and more complicated object, the nonlinearized Lax system. The main reason why the nonlinearization technique takes effect is that kind of specific symmetry constraints expressed through the variational derivative of the potential.

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The study of symmetry constraints itself is an important part of the kernel of the mathematical theory of nonlinearization, which can manipulate both mono-nonlinearization [3] and binary nonlinearization [12, 25].

However, all examples of application of the nonlinearization technique, discussed so far, are related to lower-order matrix spectral problems of soliton equations, most of which are only concerned with second-order traceless matrix spectral problems. On the other hand, there appears much difficulty in handling the Liouville integrability of the so-called constrained flows generated from spectral problems, in the case of the third and fourth-order matrix spectral problems[5, 10, 15, 16, 29, 34]. It is a challenging task to extend the theory of nonlinearization to the case of higher-order matrix spectral problems. In this article, we would like to establish a concrete example to apply the nonlinearization technique to the case of higher-order matrix spectral problems, by manipulating binary nonlinearization[1, 4, 7, 9, 11, 13, 14, 17, 18, 22, 23, 24, 30, 31, 32] for arbitrary-order matrix spectral problems associated with 3×3 discrete matrix eigenvalue problem. The resulting theory will show a direct way for generating sufficiently many integrals of motion for the Liouville integrability of the constrained flows resulting from higher-order matrix spectral problems.

This article is organized as follows. In Section 2, a discrete 3×3 matrix spectral problem is introduced, and a hierarchy of lattice soliton equations is derived by the method of discrete zero curvature representation. A lattice system is proposed, it is a typical lattice system in resulting hierarchy. Infinitely many commuting symmetries and infinitely many commuting conserved functionals for the obtained hierarchy are given. In Section 3,we consider the Bargmann symmetry constraint for the proposed new Lax pairs and adjoint Lax pairs of the discrete soliton hierarchy. Finally in Section 4, conclusions and remarks are given.

2. A family of lattice soliton equations and its Liouville integrability

Let we define the shift operator E, the inverse of E by

$$\begin{aligned}&Ef_n = f_{n+1}, E^{-1}f_n = f_{n-1}, \Delta = E - E^{-1}, n \in Z,\\&(1-E)^{-1} = -(1+E^{-1})\Delta^{-1}, (1-E^{-1})^{-1} = (1+E)\Delta^{-1}.\end{aligned}$$

We introduce the new discrete 3×3 matrix spectral problem

$$E\psi_n = U_n(u_n, \lambda)\psi_n = \begin{pmatrix} p_n & 1 & q_n - \lambda \\ 0 & 0 & 1 \\ s_n & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \\ \psi_n^3 \end{pmatrix}, \tag{2.1}$$

where the potential vector $U_n = (p_n, q_n, s_n)^T, \lambda_t = 0$, and solve the stationary discrete zero curvature equation

$$(EV_n)U_n - U_nV_n = 0,\ V_n = (V_n^{ij})_{3\times3}, \tag{2.2}$$

where each entry $(V_n^{ij})_{3\times3} = V_n^{ij}(A_n(\lambda), B_n(\lambda), D_n(\lambda))$ of the 3×3 matrix V_n is a Laurent expansion of λ. When we choose $V_n^{12} = A_n(\lambda),\ V_n^{32} = B_n(\lambda),\ V_n^{22} = D_n(\lambda)$, we have

$$\begin{aligned}&V_n^{11} = E^{-1}p_nA_n(\lambda) - \lambda E^{-1}B_n(\lambda) + E^{-1}q_nB_n(\lambda) + E^{-1}D_n(\lambda),\\&V_n^{13} = q_nA_n(\lambda) - \lambda A_n(\lambda) + \tfrac{1}{s_n}EB_n(\lambda),\\&V_n^{21} = E^{-1}B_n(\lambda),\ V_n^{23} = E^{-1}\tfrac{1}{s_n}E^{-1}s_nA_n(\lambda) - E^{-1}\tfrac{p_n}{s_n}B_n(\lambda),\\&V_n^{31} = E^{-1}s_nA_n(\lambda),\ V_n^{33} = -\lambda B_n(\lambda) + q_nB_n(\lambda) + ED_n(\lambda).\end{aligned} \tag{2.3}$$

Substituting the following expressions

$$A_n(\lambda) = \sum_{m=-1}^{\infty} A_n^{(m)}\lambda^{-m}, B_n(\lambda) = \sum_{m=-1}^{\infty} B_n^{(m)}\lambda^{-m}, D_n(\lambda) = \sum_{m=-1}^{\infty} D_n^{(m)}\lambda^{-m}. \tag{2.4}$$

The stationary discrete zero-curvature equation (2.2) is equivalent to the recursion relation:

$$
\begin{aligned}
&(s_nE - E^{-1}s_n)A_n^{(j)}(\lambda) + p_n(1-E^{-1})B_n^{(j)}(\lambda)\\
&= (p_n^2 - p_nE^{-1}p_n + s_nEq_n - q_nE^{-1}s_n)A_n^{(j-1)}(\lambda)\\
&\quad + (p_nq_n - p_nE^{-1}q_n + s_nE\tfrac{1}{s_n}E - E^{-1})B_n^{(j-1)}(\lambda) + p_n(1-E^{-1})D_n^{(j-1)}(\lambda),\\
&(1-E)D_n^{j}(\lambda)\\
&= (E - E^{-1}\tfrac{1}{s_n}E^{-1}s_n)A_n^{(j-1)}(\lambda) + (E^{-1}\tfrac{p_n}{s_n} - \tfrac{p_n}{s_n}E)B_n^{(j-1)}(\lambda) + q_n(1-E)D_n^{(j-1)}(\lambda),\\
&s_n(E-E^{-1})B_n^{(j)}(\lambda)\\
&= s_n(1-E^{-1})p_nA_n^{(j-1)}(\lambda) + s_n(E-E^{-1})q_nB_n^{(j-1)}(\lambda) + s_n(E^2-E^{-1})D_n^{(j-1)}(\lambda).
\end{aligned}
\tag{2.5}
$$

From the above recursion equations, we obtain the initial data

$$A_n^{(-1)} = 0, B_n^{(-1)} = 1,\ D_n^{(-1)} = 0,\ A_n^{(0)} = \frac{1}{s_n},\ B_n^{(0)} = q_n,\ D_n^{(0)} = \frac{p_{n-1}}{s_{n-1}}, \cdots$$

To obtain Lax integrable equations, we define F_n^j by the following relation:

$$D_n^{(j)}(\lambda) = -p_nA_n^{(j)}(\lambda) - (1+E^{-1})s_nF_n^{(j)}(\lambda). \tag{2.6}$$

It is easy to see that

$$
\begin{aligned}
&(s_nE - E^{-1}s_n)A_n^{(j)}(\lambda) + p_n(1-E^{-1})B_n^{(j)}(\lambda)\\
&= (s_nEq_n - q_nE^{-1}s_n)A_n^{(j-1)}(\lambda) + (p_nq_n - p_nE^{-1}q_n + s_nE\tfrac{1}{s_n}E - E^{-1})B_n^{(j-1)}(\lambda)\\
&\quad + p_n(E^{-2}-1)s_nF_n^{(j-1)}(\lambda),\\
&(E-1)p_nA_n^{(j)}(\lambda) + \Delta s_nF_n^{j}(\lambda)\\
&= (E - E^{-1}\tfrac{1}{s_n}E^{-1}s_n + q_nEp_n - p_nq_n)A_n^{(j-1)}(\lambda) + (E^{-1}\tfrac{p_n}{s_n} - \tfrac{p_n}{s_n}E)B_n^{(j-1)}(\lambda)\\
&\quad + q_n\Delta s_nF_n^{(j-1)}(\lambda),\\
&s_n\Delta B_n^{(j)}(\lambda) = s_n(1-E^2)p_nA_n^{(j-1)}(\lambda) + s_n\Delta q_nB_n^{(j-1)}(\lambda)\\
&\qquad\qquad + s_n(E^{-2} - E^2 + E^{-1} - E)s_nF_n^{(j-1)}(\lambda).
\end{aligned}
\tag{2.7}
$$

Using the matrix notation, the above expressions (2.3) can be written as

$$KG_n^{j-1} = JG_n^j,\ G_n^j = (A_n^{(j)}, B_n^{(j)}, F_n^{(j)})^T,\ j \geq 0, \tag{2.8}$$

where so-called Lenards operator pair J and K are two skew-symmetric operators

$$J = \begin{pmatrix} s_nE - E^{-1}s_n & p_n(1-E^{-1}) & 0 \\ (E-1)p_n & 0 & \Delta s_n \\ 0 & s_n\Delta & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} s_nEq_n - q_nE^{-1}s_n & p_nq_n - p_nE^{-1}q_n + s_nE\frac{1}{s_n}E - E^{-1} & p_n(E^{-2}-1)s_n \\ E - E^{-1}\frac{1}{s_n}E^{-1}s_n + q_nEp_n - p_nq_n & E^{-1}\frac{p_n}{s_n} - \frac{p_n}{s_n}E & q_n\Delta s_n \\ s_n(1-E^2)p_n & s_n\Delta q_n & s_n(E^{-2} - E^2 + E^{-1} - E)s_n \end{pmatrix}.$$

From (2.8), we have

$$
\begin{aligned}
&G_n^{-1} = (A_n^{(-1)}, B_n^{(-1)}, F_n^{(-1)})^T = (0,1,0)^T,\ G_n^0 = (A_n^{(0)}, B_n^{(0)}, F_n^{(0)})^T = (\frac{1}{s_n}, q_n, 0)^T,\\
&G_n^1 = (A_n^{(1)}, B_n^{(1)}, F_n^{(1)})^T = (\frac{q_n + q_{n+1}}{s_n}, q^2 + \frac{p_n}{s_n} + \frac{p_{n-1}}{s_{n-1}}, \frac{p_n(q_n + q_{n+1}) - 1}{s_n}), \cdots
\end{aligned}
$$

Let $\psi_n(\lambda)$ satisfy (2.1) and its auxiliary problem

$$\frac{\partial \psi_n(\lambda)}{\partial t_n} = V_n^{(m)} \psi_n(\lambda), \tag{2.9}$$

where

$$V_n^{(m)} = (V_n^{(ijm)})_{3\times 3},\ V_n^{(ijm)} = V_n^{(ij)}(A_n^{(m)}(\lambda), B_n^{(m)}(\lambda), D_n^{(m)}(\lambda))$$

and

$$A_n^{(m)}(\lambda) = \sum_{i=-1}^{\infty} A_n^{(m)} \lambda^{m-i},\ B_n^{(m)}(\lambda) = \sum_{i=-1}^{\infty} B_n^{(m)} \lambda^{m-i},\ D_n^{(m)}(\lambda) = \sum_{i=-1}^{\infty} D_n^{(m)} \lambda^{m-i}.$$

Then the compatibility conditions of (2.1) and (2.9) are

$$\frac{\partial U_n}{\partial t_{n_m}} = (EV_n^{(m)})U_n - U_n V_n^{(m)},\ m \geq -1, \tag{2.10}$$

which implies the lattice solition equations

$$\frac{\partial U_n}{\partial t_{n_m}} = X_n^{m+1},\ U_n = (p_n, q_n, s_n)^T,\ m \geq -1,$$

and

$$X_n^j = JG_n^j = KG_n^{j-1},\ j \geq 0$$

which give rise to the following hierarchy of lattice soliton equations

$$\left\{\begin{array}{l} \dfrac{\partial p_n}{\partial t_{n_m}} = (s_n E - E^{-1} s_n) A_n^{(j)}(\lambda) + p_n(1-E^{-1}) B_n^{(j)}(\lambda), \\ \dfrac{\partial q_n}{\partial t_{n_m}} = (E-1) p_n A_n^{(j)}(\lambda) + \Delta s_n F_n^j(\lambda), \qquad\qquad j \geq -1. \\ \dfrac{\partial s_n}{\partial t_{n_m}} = s_n \Delta B_n^{(j)}(\lambda). \end{array}\right. \tag{2.11}$$

So the (2.10) are discrete zero curvature representation of (2.11), the discrete spectral problem (2.3) and (2.9) constitute the Lax pair of (2.11), and (2.11) is a hierarchy of Lax integrable nonlinear lattice equations. It is easy to verify that the new first Liouville integrable differential-difference equation in (2.11), when $m = 0$, is

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t_n} p_n = p_n(q_n - q_{n-1}) + \dfrac{s_n - s_{n+1}}{s_{n+1}}, \\ \dfrac{\partial}{\partial t_n} q_n = \dfrac{p_{n+1}}{s_{n+1}} - \dfrac{p_n}{s_n}, \\ \dfrac{\partial}{\partial t_n} s_n = s_n(q_{n+1} - q_{n-1}). \end{array}\right. \tag{2.12}$$

The variational derivative, the Gateaux derivative and the inner product are defined, respectively, by

$$\frac{\delta H_n}{\delta u_n} = \sum_{m\in Z} E^{-m}\left(\frac{\partial H_n}{\partial u_{n+m}}\right), J'(u_n)[v_n] = \frac{\partial}{\partial \varepsilon} J(u_n + \varepsilon v_n)|_{\varepsilon=0}, \langle f_n, g_n\rangle = \sum_{n\in Z} (f_n, g_n)_{R^2}, \tag{2.13}$$

where f_n, g_n are required to be rapidly vanished at the infinity, and $(f_n, g_n)_{R^2}$ denotes the standard inner product of f_n and g_n in the Euclidean space R_2. Operator J^* is defined by $\langle f, J^* g\rangle = \langle J f_n, g_n\rangle$; it is called adjoint operator of J with respect to (2.8). If an operator J has the property $J^* = -J$, then J is called to be a skew-symmetric. A linear operator J is called a Hamiltonian operator, if J is a skew-symmetric operator and satisfies the Jacobi identity, i.e., it satisfies that

$$\langle f, Jg\rangle = -\langle Jf, g\rangle, \quad \langle J'(u_n)[Jf]g, h\rangle + Cycle(f, g, h) = 0 \tag{2.14}$$

based on a given Hamiltonian operator J, we can define a corresponding Poisson bracket

$$\{f,g\}_J = \langle \frac{\delta f}{\delta u_n}, J\frac{\delta g}{\delta u_n}\rangle = \sum_{n\in Z}(\frac{\delta f}{\delta u_n}, J\frac{\delta g}{\delta u_n}). \tag{2.15}$$

To establish the Hamiltonian structures for (2.11), we define

$$R_n = V_n U_n^{-1} = \begin{pmatrix} V_n^{12} & (\lambda - q_n)V_n^{12} + V_n^{13} & \frac{V_n^{11} - p_n V_n^{12}}{s_n} \\ V_n^{22} & (\lambda - q_n)V_n^{22} + V_n^{23} & \frac{V_n^{21} - p_n V_n^{22}}{s_n} \\ V_n^{32} & (\lambda - q_n)V_n^{32} + V_n^{33} & \frac{V_n^{31} - p_n V_n^{32}}{s_n} \end{pmatrix}$$

and $\langle A, B\rangle = Tr(AB)$, where A and B are the some order square matrices. We have

$$\frac{\partial U_n}{\partial \lambda} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{\partial U_n}{\partial p_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{\partial U_n}{\partial q_n} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{\partial U_n}{\partial s_n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} &\langle R_n, \frac{\partial U_n}{\partial \lambda}\rangle = -V_n^{32} = -B_n(\lambda),\ \langle R_n, \frac{\partial U_n}{\partial p_n}\rangle = V_n^{12} = A_n(\lambda),\ \langle R_n, \frac{\partial U_n}{\partial q_n}\rangle = V_n^{32} = B_n(\lambda), \\ &\langle R_n, \frac{\partial U_n}{\partial s_n}\rangle = \frac{V_n^{11} - p_n V_n^{12}}{s_n} = \tfrac{1}{s_n}[(E^{-1}-1)p_n A_n(\lambda) - E^{-1}(\lambda - q_n)B_n(\lambda) + E^{-1}D_n(\lambda)]. \end{aligned} \tag{2.16}$$

By virtue of the discrete trace identity

$$\frac{\delta}{\delta u}\sum_{n\in Z}\langle R_n, \frac{\partial U_n}{\partial \lambda}\rangle = \left(\lambda^{-\varepsilon}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\varepsilon}\right)\langle R_n, \frac{\partial U_n}{\partial u_n^i}\rangle, \quad i = 1,2,3. \tag{2.17}$$

The substitution of (2.4) into (2.17), and comparing the coefficients of λ^{-m-1} in (2.17), we get

$$(\frac{\delta}{\delta p_n}, \frac{\delta}{\delta q_n}, \frac{\delta}{\delta s_n})\left(B_n^{(m+1)}\right) = (\varepsilon - m)\begin{pmatrix} A_n^{(m)} \\ B_n^{(m)} \\ \frac{1}{s_n}(E^{-1}-1)A_n^{(m)} - E^{-1}B_n^{(m+1)} + E^{-1}(q_n B_n^{(m)} + D_n^{(m)}) \end{pmatrix}. \tag{2.18}$$

When $m = 0$ in the (2.18), through a direct calculation, we find that $\varepsilon = 0$. So we have

$$(\frac{\delta}{\delta s_n}, \frac{\delta}{\delta w_n}, \frac{\delta}{\delta p_n})\left(-\frac{B_n^{(m+1)}}{m+1}\right) = \begin{pmatrix} A_n^{(m)} \\ B_n^{(m)} \\ \frac{1}{s_n}(E^{-1}-1)A_n^{(m)} - E^{-1}B_n^{(m+1)} + E^{-1}(q_n B_n^{(m)} + D_n^{(m)}) \end{pmatrix}, m \geq -1.$$

Now we can rewrite the (2.11) in the following Hamiltonian forms

$$\frac{\partial U_n}{\partial t_{n_m}} = X_n^{m+1} = J(\frac{\delta}{\delta p_n}, \frac{\delta}{\delta q_n}, \frac{\delta}{\delta s_n})H_n^{m+1} = JL(\frac{\delta}{\delta p_n}, \frac{\delta}{\delta q_n}, \frac{\delta}{\delta s_n})H_n^m, \quad m \geq -1. \tag{2.19}$$

Let

$$L = \begin{pmatrix} L_{11} & \frac{1}{s_n}\Delta^{-1} & L_{13} \\ \Delta^{-1}\frac{1}{s_n} & 0 & 0 \\ L_{31} & 0 & (Es_n - s_n E^{-1})^{-1} \end{pmatrix}, \tag{2.20}$$

where

$$L_{11} = -\frac{1}{s_n}\Delta^{-1}p_n(Es_n - s_nE^{-1})^{-1}p_n(E-1)\Delta^{-1}\frac{1}{s_n},$$
$$L_{13} = -\frac{1}{s_n}\Delta^{-1}(1-E^{-1})p_n(Es_n - s_nE^{-1})^{-1},$$
$$L_{31} = -(Es_n - s_nE^{-1})^{-1}p_n(E-1)\Delta^{-1}\frac{1}{s_n}.$$

It is easy to verify that K is a skew-symmetric operator in this way and the positive hierarchy (2.10) is derived. It is easy to verify that the positive hierarchy has the discrete zero-curvature representation (2.9). And, every soliton equation in (2.11) or the discrete Hamiltonian system (2.19) is a discrete Liouville integrable system.

3. A binary Symmetry constraint by binary nonlinearization

In order to impose the Bargmann symmetry constraint by binary nonlinearization, we consider the adjoint spectral problem of spectral problem (2.1)

$$E^{-1}\psi_n = (E^{-1}\tilde{U_n}^T(u_n,\lambda)\psi_n),\ \psi_n = (\psi_n^{1j}, \psi_n^{2j}, \psi_n^{3j})^T \tag{3.1}$$

and temporal spectral problem

$$\psi_{nt_m} = -(\tilde{V}_n^m(u_n,\lambda))^T\psi_n. \tag{3.2}$$

From the compatibility condition$(E^{-1}\psi_n)_{t_m} = E^{-1}\psi_{nt_m}$, we know that

$$E^{-1}\tilde{U_n}^T_{t_m} = (E^{-1}\tilde{U_n}^T)(\tilde{V}_n^m)^T - (E^{-1}(\tilde{V}_n^m)^T)(E^{-1}\tilde{U_n}^T) \tag{3.3}$$

Let $\lambda_1, \lambda_2, ..., \lambda_N$ be N distinct eigenvalues of spectral problem (1) and $\lambda_j \neq 0, j = 1, 2, ..., N$, we have

$$\begin{cases} (E\varphi_n^{1j}, E\varphi_n^{2j}, E\varphi_n^{3j}) = (\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})U_n^T(u_n,\lambda), \\ (\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})_{t_m} = (\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})V_n^T(u_n,\lambda), \\ (E\psi_n^{1j}, E\psi_n^{2j}, E\psi_n^{3j}) = (\psi_n^{1j}, \psi_n^{2j}, \psi_n^{3j})(U_n(u_n,\lambda))^{-1}, \\ (\psi_n^{1j}, \psi_n^{2j}, \psi_n^{3j})_{t_m} = (\psi_n^{1j}, \psi_n^{2j}, \psi_n^{3j})(-V_n(u_n,\lambda)). \end{cases} \tag{3.4}$$

We can compute the variational derivative of the spectral parameter λ with respect to the potential u

$$\frac{\delta\lambda_j}{\delta u_n} = \alpha_j(E\psi_n^{1j}, E\psi_n^{2j}, E\psi_n^{3j})\frac{\partial U_n(u_n,\lambda_j)}{\partial u_n}(\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})^T. \tag{3.5}$$

Namely

$$\nabla\lambda_j = \begin{pmatrix} \frac{\delta\lambda_j}{\delta p_n} \\ \frac{\delta\lambda_j}{\delta q_n} \\ \frac{\delta\lambda_j}{\delta s_n} \end{pmatrix} = \alpha_j \begin{pmatrix} \varphi_n^{2j}\psi_n^{3j} \\ \varphi_n^{2j}\psi_n^{1j} \\ s_n^{-1}\varphi_n^{4j}\psi_n^{4j} \end{pmatrix}, \tag{3.6}$$

where$\frac{\delta\lambda_j}{\delta u_n}$ is a variational derivative for eigenvalue λ_j, α_j is a constant and $\varphi_n^i, \psi_n^i, i = 1, 2, 3, 4$ are required to be rapidly vanishing at the infinity, and we denote the inner product in R^N by $< ., . >$ and use the following notations

$$\Phi_n^i = (\varphi_n^{i1}, \varphi_n^{i2}, ..., \varphi_n^{iN}),\ \Psi_n^i = (\psi_n^{i1}, \psi_n^{i2}, ..., \psi_n^{iN}), i = 1, 2, 3,\ \wedge = diag(\lambda_1, \lambda_2, ..., \lambda_N).$$

Such a gradient satisfies the following equation

$$K\nabla\lambda_j = \lambda_j J\nabla\lambda_j. \tag{3.7}$$

Consider the discrete symmetry constraint

$$G_{-1} = \sum_{j=1}^{N} \nabla \lambda_j. \tag{3.8}$$

That is

$$\frac{\delta H_n^{(0)}}{\delta u_n} = \begin{pmatrix} \frac{1}{s_n} \\ q_n \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_n^{1j}\psi_n^{2j} \\ \varphi_n^{3j}\psi_n^{2j} \\ s_n^{-1}(\varphi_n^{1j}\psi_n^{1j} - p_n\varphi_n^{1j}\psi_n^{2j}) \end{pmatrix}. \tag{3.9}$$

Note that the explicit constraints of potential functions and eigenvalue functions can not be obtained with the express above. Under the constraint (3.8), we obtain a discrete binary constrained flows

$$\begin{array}{l} E\varphi_n^{1j} = p_n\varphi_n^{1j} + \varphi_n^{2j} + (q_n - \lambda)\varphi_n^{3j}, \quad 1 \le j \le N, \\ E\varphi_n^{2j} = \varphi_n^{3j}, \quad 1 \le j \le N, \\ E\varphi_n^{3j} = s_n\varphi_n^{1j}, \quad 1 \le j \le N, \\ E\psi_n^{1j} = \psi_n^{2j}, \quad 1 \le j \le N, \\ E\psi_n^{2j} = (\lambda - q_n)\psi_n^{2j} + \psi_n^{3j}, \quad 1 \le j \le N, \\ E\psi_n^{3j} = s_n^{-1}(\psi_n^{1j} - p_n\psi_n^{2j}), \quad 1 \le j \le N, \end{array} \tag{3.10}$$

Here, $< .,. >$ is the standard inner product of R^N. The symmetry constraint (3.8) yields explicit expressions from (3.9):

$$\begin{cases} p_n =< \Phi_n^1, \Psi_n^1 >< \Phi_n^1, \Psi_n^2 >^{-1}, \\ q_n =< \Phi_n^3, \Psi_n^2 >, \\ s_n =< \Phi_n^1, \Psi_n^2 >^{-1} . \end{cases} \tag{3.11}$$

So the discrete symmetry constraint (3.8) is a Bargmann constraint. Setting

$$P_n = (\varphi_n^{11}, \varphi_n^{12}, \cdots, \varphi_n^{1N}, \cdots, \varphi_n^{31}, \varphi_n^{32}, \cdots, \varphi_n^{3N})^T, Q_n = (\psi_n^{11}, \psi_n^{12}, \cdots, \psi_n^{1N}, \cdots, \psi_n^{31}, \psi_n^{32}, \cdots, \psi_n^{3N})^T$$

and

$$\begin{array}{l} \frac{\partial}{\partial P_n} = (\frac{\partial}{\partial \varphi_n^{11}}, \frac{\partial}{\partial \varphi_n^{12}}, \cdots, \frac{\partial}{\partial \varphi_n^{1N}}, \cdots, \frac{\partial}{\partial \varphi_n^{31}}, \frac{\partial}{\partial \varphi_n^{32}}, \cdots \frac{\partial}{\partial \varphi_n^{3N}},)^T, \\ \frac{\partial}{\partial Q_n} = (\frac{\partial}{\partial \psi_n^{11}}, \frac{\partial}{\partial \psi_n^{12}}, \cdots, \frac{\partial}{\partial \psi_n^{1N}}, \cdots, \frac{\partial}{\partial \psi_n^{31}}, \frac{\partial}{\partial \varphi_n^{32}}, \cdots, \frac{\partial}{\partial \psi_n^{3N}},)^T, \end{array}$$

the Poisson bracket of between two arbitrary function of α, β in symplectic apace R^{6N} is defined by

$$\{\alpha, \beta\} = \langle \frac{\partial \alpha}{\partial P}, \frac{\partial \beta}{\partial Q} \rangle - \langle \frac{\partial \beta}{\partial P}, \frac{\partial \alpha}{\partial Q} \rangle = (\frac{\partial \alpha}{\partial P})^T (\frac{\partial \beta}{\partial Q}) - (\frac{\partial \beta}{\partial P})^T (\frac{\partial \alpha}{\partial Q}).$$

This is skew-symmetric, bilinear, and satisfies the Jacobi identity. In particular, any two of α, β is called involutive if $\{\alpha, \beta\} = 0$.

The map H defined as

$$H(\varphi_n^1, \varphi_n^2, \varphi_n^3, \psi_n^1, \psi_n^2, \psi_n^3) = (E\varphi_n^1, E\varphi_n^2, E\varphi_n^3, E\psi_n^1, E\psi_n^2, E\psi_n^3) \tag{3.12}$$

is a symplectic map. Through laborious but direct computation, we get

$$\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0, \{\alpha_i, \beta_j\} = \delta_{ij}, 1 \le i, j \le N$$

and the γ_i, δ_j are of the same forms. Furthmore, we can deduce

$$d(EP_n) \wedge d(EQ_n) = dP_n \wedge dQ_n.$$

Therefore, (3.12) determine a symplectic map.
Now, we will solve recursion equations (2.7). When $m > 1$, we have

$$\begin{aligned} U_{n_{t_m}} = \begin{pmatrix} p_n \\ q_n \\ s_n \end{pmatrix}_{t_m} &= \begin{pmatrix} (s_nE - E^{-1}s_n)A_n^{(j)}(\lambda) + p_n(1-E^{-1})B_n^{(j)}(\lambda) \\ (E-1)p_nA_n^{(j)}(\lambda) + \Delta s_n F_n^j(\lambda) \\ s_n\Delta B_n^{(j)}(\lambda) \end{pmatrix} \\ &= J\frac{\delta H_n^m}{\delta u_n} = J\Phi_n^{m-1}\frac{\delta H_n^1}{\delta u_n} = J\sum_{j=1}^{N}\lambda_j^{m-1}\frac{\delta \lambda_j}{\delta u_n}. \end{aligned} \tag{3.13}$$

Using (3.8) and (3.10) and the constraint (3.11), we take the following restriction:

$$G_{j-1} = \sum_{k=1}^{N}\lambda_k^j\nabla\lambda_k. \tag{3.14}$$

That is to say,

$$\begin{pmatrix} (s_nE - E^{-1}s_n)A_n^{(j)}(\lambda) + p_n(1-E^{-1})B_n^{(j)}(\lambda) \\ (E-1)p_nA_n^{(j)}(\lambda) + \Delta s_n F_n^j(\lambda) \\ s_n\Delta B_n^{(j)}(\lambda) \end{pmatrix} = J\sum_{j=1}^{N}\lambda_j^m \begin{pmatrix} \varphi_n^{1j}\psi_n^{2j} \\ \varphi_n^{3j}\psi_n^{2j} \\ \frac{1}{s_n}(\varphi_n^{1j}\psi_n^{1j} - p_n\varphi_n^{1j}\psi_n^{2j}) \end{pmatrix}. \tag{3.15}$$

From (3.15), we can conclude

$$\begin{aligned} &A_n^j =< \wedge^j\Phi_n^1, \Psi_n^2 >,\ B_n^j =< \wedge^j\Phi_n^3, \Psi_n^2 >, \\ &F_n^j = s_n^{-1}(< \wedge^j\Phi_n^1, \Psi_n^1 > -p_n < \wedge^j\Phi_n^1, \Psi_n^2 >). \end{aligned} \tag{3.16}$$

Substituting (3.16) into the relation (2.6), we obtain a solution of D_n^j, that is

$$D_n^j =< \wedge^j\Phi_n^2, \Psi_n^2 > . \tag{3.17}$$

By using (3.16), (3.17) and (2.7), we have

$$\begin{aligned} &E^{-1}s_nA_n(\lambda) =< \wedge^j\Phi_n^3, \Psi_n^1 >,\ E^{-1}B_n(\lambda) =< \wedge^j\Phi_n^2, \Psi_n^1 >, \\ &E^{-1}\tfrac{1}{s_n}E^{-1}s_nA_n(\lambda) - E^{-1}\tfrac{p_n}{s_n}B_n(\lambda) =< \wedge^j\Phi_n^2, \Psi_n^3 >, \\ &E^{-1}p_nA_n(\lambda) - \lambda E^{-1}B_n(\lambda) + E^{-1}q_nB_n(\lambda) + E^{-1}D_n(\lambda) =< \wedge^j\Phi_n^1, \Psi_n^1 >, \\ &q_nA_n(\lambda) - \lambda A_n(\lambda) + \tfrac{1}{s_n}EB_n(\lambda) =< \wedge^j\Phi_n^1, \Psi_n^3 >, \\ &-\lambda B_n(\lambda) + q_nB_n(\lambda) + ED_n(\lambda) =< \wedge^j\Phi_n^3, \Psi_n^3 > . \end{aligned} \tag{3.18}$$

In the following, we would like to discuss the Louville integrability on the nonlinearized temporal parts of the Lax pairs and adjoint Lax pairs.

Under the control of (3.11), the temporal parts of the Lax pairs and the adjoint Lax pairs by substituting (3.18) into (3.4) become

$$\begin{aligned} &\tfrac{\partial}{\partial t}(\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})^T = V|_B(\varphi_n^{1j}, \varphi_n^{2j}, \varphi_n^{3j})^T, j = 1, 2, \cdots, N, \\ &\tfrac{\partial}{\partial t}(\psi_n^{1j}, \psi_n^{2j}, \psi_n^{3j})^T = -V^T|_B(\psi_n^{1j}, \psi_n^{2j}, \psi_{3j})^T, j = 1, 2, \cdots, N. \end{aligned} \tag{3.19}$$

We arrive at the finite-dimensional Hamiltonian systems. Here, the subscript B means substitution of (3.18) into the expression.

The temporal parts of the nonlinearized Lax pairs and the adjoint Lax pairs (3.19) may be rewritten as

$$\frac{\partial}{\partial t}\Phi_n^i = \frac{\partial F_n^m}{\partial \Psi_n^i}, \quad \frac{\partial}{\partial t}\Psi_n^i = -\frac{\partial F_n^m}{\partial \Phi_n^i}, \tag{3.20}$$

which is the finite-dimensional Hamiltonian systems:

$$\begin{aligned}&(\frac{\partial F_n^{-1}}{\partial \Psi_n^1},\frac{\partial F_n^{-1}}{\partial \Psi_n^2},\frac{\partial F_n^{-1}}{\partial \Psi_n^3})=(\Phi_n^1,\Phi_n^2,\Phi_n^3)V_{-1}^T,\\&(\frac{\partial F_n^{-1}}{\partial \Phi_n^1},\frac{\partial F_n^{-1}}{\partial \Phi_n^2},\frac{\partial F_n^{-1}}{\partial \Phi_n^3})=-(\Psi_n^1,\Psi_n^2,\Psi_n^3)V_{-1},\end{aligned}\tag{3.21}$$

where

$$V_{-1}=\begin{pmatrix}<\Phi_n^2,\Psi_n^1>-\Lambda & 0 & <\Phi_n^1,\Psi_n^2>\\ 1 & 0 & 0\\ 0 & 1 & <\Phi_n^3,\Psi_n^2>-\Lambda\end{pmatrix}.\tag{3.22}$$

The associated Hamiltonian functions are given as follows

$$\begin{aligned}F_n^{-1}=&<\Phi_n^1,\Psi_n^2>+<\Phi_n^2,\Psi_n^3>-<\Lambda\Phi_n^1,\Psi_n^1>-<\Lambda\Phi_n^3,\Psi_n^3>\\&+<\Phi_n^1,\Psi_n^1><\Phi_n^2,\Psi_n^1>+<\Phi_n^1,\Psi_n^2><\Phi_n^3,\Psi_n^1>+<\Phi_n^3,\Psi_n^2><\Phi_n^3,\Psi_n^3>,\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}&(\frac{\partial F_n^m}{\partial \Psi_n^1},\frac{\partial F_n^m}{\partial \Psi_n^2},\frac{\partial F_n^m}{\partial \Psi_n^3})=(\Phi_n^1,\Phi_n^2,\Phi_n^3)V_n^{(m)}(u,\Lambda)^T,\ m\geq 0,\\&(\frac{\partial F_n^m}{\partial \Phi_n^1},\frac{\partial F_n^m}{\partial \Phi_n^2},\frac{\partial F_n^m}{\partial \Phi_n^3})=-(\Psi_n^1,\Psi_n^2,\Psi_n^3)V_n^{(m)}(u,\Lambda)^{-1},\ m\geq 0.\end{aligned}\tag{3.24}$$

Let $\Phi_i(n,t_m),\Psi_i(n,t_m),i=1,2,3$ be a solution of the finite-dimensional completely integrable systems (3.24). Then, the solution of the discrete nonlinear equation (2.12)

$$\begin{cases}p(n,t_0)=<\Phi_1(n,t_0),\Psi_1(n,t_0)><\Phi_1(n,t_0),\Psi_2(n,t_0)>^{-1},\\ q(n,t_0)=<\Phi_3(n,t_0),\Psi_2(n,t_0)>,\\ s(n,t_0)=<\Phi_1(n,t_0),\Psi_2(n,t_0)>^{-1}\end{cases}\tag{3.25}$$

is a Bӑcklund transformation between the integrable symplectic map (3.12) and the finite-dimensional completely integrable systems (3.24).

4. Conclusions and Remarks

In this paper, we have proposed a interesting and meaningful hierarchy of differential-difference equations associated with a new s-order discrete matrix isospectral problem through the discrete zero curvature equation and then the Liouville integrability of the obtained the family of differential-difference equations is proved. Furthermore, under the binary Bargmann symmetry constraint between the potentials and the eigenfunctions, the binary nonlinearization of the Lax pairs and the adjoint Lax pairs of the obtained family is presented. This will provide us with a large number of examples of the related fields.

As we know that the r-matrix formula [2], Lax representation and separation of variables [25, 26] have a direct link between the classical integrable problem and the finite-dimensional integrable problem. In addition, bilinear Bӑcklund transformation [8, 19], Darboux transformation [28, 33], Bell polynomials [6, 20], Hirota bilinear solution [21, 27]are all the key areas for solitons which will motivated us do further research to improve the classical binary nonlinearization.

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