Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 63–69
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Various symmetries in matrix theory with application to modeling dynamic systems

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Communicated by S. M. Vaezpour

Abstract

In this paper, we recall centrally symmetric matrices and introduce some new kinds of symmetric matrices such as row-wise symmetric matrices, column-wise symmetric matrices, and plus symmetric matrices. The relations between these kinds of matrices are also presented. Furthermore, a useful result is obtained about the types of the eigenvectors of centrally symmetric matrices leading to a limit-wise relation between centrally symmetric matrices and plus symmetric matrices which can be applied to mathematical modeling of dynamical systems in engineering applications.

Keywords: Special symmetry, symmetric matrices, mathematical modeling.
2010 MSC: 47H10, 54H25.

1. Introduction

Mathematics can be used as a partial translation of the nature to analyze various characteristics of the real-world systems as well as to design appropriate mechanisms to control such systems. Regardless of the mathematical models of the systems, some systems naturally represent some kinds of symmetric behavior around their equilibrium states. In such cases, having more specialized mathematical tools leads to more useful specialized mathematical models and simpler behavioral analysis.

Besides, matrix representation is now a very well-known tool for modeling the real-world systems. For example: the position and/or orientation of a rigid body in the space can be effectively presented by

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Received 2012-3-5

an appropriate matrix; the parameters of an electrical circuit can be gathered in a specific matrix which represents the relations between the main variables of the circuit; the parameters of an economical system can be gathered in a specific matrix which describes the behavior of the system; and etc. To be more general, from a system theoretic view, every dynamic system can be modeled by state space equations which can be represented by appropriate matrices when the system is linear. This is also true when the system is nonlinear but can be approximated by a linear transformation. Moreover, the behavior of a completely nonlinear system can be represented by a matrix in the framework of fuzzy relational modeling, where a fuzzy relational composition by a fuzzy relational matrix is used to construct the required mapping.

This paper tends to introduce some useful kinds of symmetric matrices and the relations between them. The results may be used in various applications including the fuzzy relational modeling scheme. In this regard, in the next section, different kinds of symmetric matrices are defined first and then some of their characteristics and relations are studied.

Throughout this paper, it is assumed that $\mathbb{F}$ is an arbitrary field. Let $m, n \in \mathbb{N}$, the ring of all $n \times n$ matrices and the set of all $m \times n$ matrices over $\mathbb{F}$ be denoted by $M_n(\mathbb{F})$ and $M_{m\times n}(\mathbb{F})$ respectively, and for simplicity let $\mathbb{F}^n = M_{n\times 1}(\mathbb{F})$. The zero matrix is denoted by $\mathbf{0}$. Also the element of an $m \times n$ matrix, $\mathbf{R}$, on the ith row and jth column is denoted by $r(i, j)$. If $\mathbf{A} = (a_{ij}) \in M_n(\mathbb{F})$, then $[\mathbf{A}]_{ij} \in M_{(n-1)}(\mathbb{F})$ is a matrix obtained by omitting the ith row and the jth column of the matrix $\mathbf{A}$. Also $\mathrm{cof}(\mathbf{A}) \in M_n(\mathbb{F})$ is defined by $\mathrm{cof}(\mathbf{A}) = a_{ij}(-1)^{i+j}\det([\mathbf{A}]_{ij})$. For more details about the cofactor matrix and the other general algebraic materials see [5, 9]. For every i, j, $1 \leqslant i, j \leqslant n$, $\mathbf{E}_{ij}$ denotes an element in $M_n(\mathbb{F})$ whose entries are all 0 but the (i, j)th entry which is one. Also $\mathbf{0}_r$ and $\mathbf{I}_r$ denote the zero square matrix of size r and the identity square matrix of size r respectively. Finally, $[x]$ stands for the integer part of x, for every x in $\mathbb{R}$.

2. Main Results

First, we recall the definition of a centrally symmetric matrix from [8].

Definition 2.1. An $m \times n$ matrix $\mathbf{R}$ is called *centrally symmetric (CS)*, if

$$r(i,\, j) = r(m + 1 - i,\, n + 1 - j),$$

for every i, j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Centrally symmetric matrices has been introduced since many years ago and various topics about them have been studied in these years, e.g., calculating their determinants [8] and inverses [4]. Centrally symmetric matrices have been used in a variety of applications and are still studied by researchers; for example see [7] and [6] and the references therein.

In [2] some results are obtained about the eigenvectors and eigenvalues of a matrix which is both symmetric and centrally symmetric. In this paper, the results of [2] are extended so that the same results are achieved with less constraints, i.e., the same characteristics are obtained for the eigenvectors of a matrix which is only centrally symmetric and not symmetric. These characteristics help finding a limit-wise relation for centrally symmetric matrices with another kind of symmetric matrices which is introduced in this paper.

In this regard, the new kinds of symmetric matrices are defined as follows.

Definition 2.2. An $m \times n$ matrix $\mathbf{R}$ is called *row-wise symmetric (column-wise symmetric) (RWS/CWS)*, if

$$r(i,\, j) = r(m + 1 - i,\, j) \qquad (r(i,\, j) = r(i,\, n + 1 - j)),$$

for every i, j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Definition 2.3. An $m \times n$ matrix $\mathbf{R}$ is called *row-wise (column-wise) skew symmetric (RWSS/CWSS)*, if

$$r(i,\, j) = -r(m + 1 - i,\, j) \qquad (r(i,\, j) = -r(i,\, n + 1 - j)),$$

for every i, j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Definition 2.4. An $m \times n$ matrix $\mathbf{R}$ is called *plus symmetric (PS)*, if it is both RWS matrix and CWS matrix, or equivalently

$$r(i,\, j) = r(m+1-i,\, j) = r(i,\, n+1-j) = r(m+1-i,\, n+1-j),$$

for every i, j, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.

Theorem 2.5. *Let $n \in \mathbb{N}$, and $\mathbf{R} \in M_n(\mathbb{R})$ be a CS matrix which has n distinct eigenvalues. Then each of its eigenvectors is either RWS vector or RWSS vector. Furthermore, the number of RWS eigenvectors is $[\frac{n+1}{2}]$ and the number of RWSS eigenvectors is $[\frac{n}{2}]$. (Note that $[\frac{n+1}{2}] + [\frac{n}{2}] = n$).*

Proof. Since $\mathbf{A}\mathrm{adj}(\mathbf{A}) = \mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I}$, for every $\mathbf{A} \in M_n(\mathbb{R})$, thus if λ is an eigenvalue of $\mathbf{R}$, then $(\mathbf{R}-\lambda\mathbf{I})\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I}) = \det(\mathbf{R}-\lambda\mathbf{I})\mathbf{I} = 0$. Therefore every column of $\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I})$ is an eigenvector $\mathbf{R}$ which is associated to λ.

Let us write the columns of $\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I})$ as $\begin{bmatrix} \underline{c}_1 & \cdots & \underline{c}_n \end{bmatrix}$. Now consider a pair of column vectors $\underline{\mathbf{c}}_j$ and $\underline{\mathbf{c}}_{n+1-j}$, for j, $1 \leqslant j \leqslant n$. Note that $\mathbf{R}$ has n distinct eigenvalues and hence $\mathrm{rank}(\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I})) = 1$, so $\underline{\mathbf{c}}_j$ is proportional to $\underline{\mathbf{c}}_{n+1-j}$. Since $\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I}) = \mathrm{cof}(\mathbf{R})^T$ and $\mathbf{R}$ is CS, then $\mathrm{adj}(\mathbf{R}-\lambda\mathbf{I})$ is CS. Therefore for any value of j, the two vectors $\underline{\mathbf{c}}_j$ and $\underline{\mathbf{c}}_{n+1-j}$ are reverse of each other, i.e., one can be obtained by making the other upside down. Hence the vector $\underline{\mathbf{c}}_j$ (and $\underline{\mathbf{c}}_{n+1-j}$ too) is proportional to its reverse and this may not happen except in two ways:

1. When the vector $\underline{\mathbf{c}}_j$ (and also $\underline{\mathbf{c}}_{n+1-j}$) is RWS.
2. When the vector $\underline{\mathbf{c}}_j$ (and also $\underline{\mathbf{c}}_{n+1-j}$) is RWSS.

Therefore each of the eigenvectors of $\mathbf{R}$ is either RWS vector or RWSS vector.

For the number of each type of the eigenvectors the proof is as follows. Assume that $\mathbf{R} = [r_{ij}]$. First, let $n = 2m+1$, for some $m \in \mathbb{N}$. Suppose that

$$\mathbf{x} = [x_1\ x_2\ \cdots\ x_m\ y\ x_m\ x_{m-1}\ \cdots\ x_1]^T$$

is RWS eigenvector of $\mathbf{R}$. Consider the equation $\mathbf{Rx} = \lambda\mathbf{x}$ which is equivalent to the following, since $\mathbf{R}$ is CS matrix.

$$\begin{bmatrix} r_{11} & \cdots & r_{1(m+1)} & r_{1(m+2)} & r_{1(m+3)} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{(m+1)1} & \ddots & r_{(m+1)(m+1)} & r_{(m+1)m} & r_{(m+1)(m-1)} & \cdots & r_{(m+1)1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{1n} & \cdots & r_{1(m+1)} & r_{1m} & r_{1(m-1)} & \cdots & r_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ y \\ x_m \\ \vdots \\ x_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ y \\ x_m \\ \vdots \\ x_1 \end{bmatrix}$$

Thus

$$\begin{cases} r_{11}x_1 + r_{12}x_2 + \cdots + r_{1(m+1)}y + r_{1(m+2)}x_m + \cdots + r_{1n}x_1 = \lambda x_1 \\ \vdots \\ r_{m1}x_1 + r_{m2}x_2 + \cdots + r_{m(m+1)}y + r_{m(m+2)}x_m + \cdots + r_{mn}x_1 = \lambda x_m \\ r_{(m+1)1}x_1 + r_{(m+1)2}x_2 + \cdots + r_{(m+1)m}x_m + r_{(m+1)(m+1)}y + r_{(m+1)m}x_m + \cdots + r_{(m+1)1}x_1 = \lambda y \\ r_{mn}x_1 + r_{m(n-1)}x_2 + \cdots + r_{m(m+2)}x_m + r_{m(m+1)}y + r_{mm}x_m + \cdots + r_{m1}x_1 = \lambda x_m \\ \vdots \\ r_{1n}x_1 + r_{1(n-1)}x_2 + \cdots + r_{1(m+2)}x_m + r_{1(m+1)}y + r_{1m}x_m + \cdots + r_{11}x_1 = \lambda x_1 \end{cases}$$

An straightforward manipulation shows that the first m equations and the last m equations are the same. Therefore we have the following set of equations

$$\begin{cases} (r_{11}+r_{1n}-\lambda)x_1+(r_{12}+r_{1(2m)})x_2+\cdots+(r_{1m}+r_{1(m+1)})x_m+r_{1(m+1)}y=0 \\ \vdots \\ (r_{m1}+r_{mn})x_1+(r_{m2}+r_{m(2m)})x_2+\cdots+(r_{mm}+r_{m(m+2)}-\lambda)x_m+r_{m(m+1)}y=0 \\ 2r_{(m+1)1}x_1+2r_{(m+1)2}x_2+\cdots+2r_{(m+1)m}x_m+(r_{(m+1)(m+1)}-\lambda)y=0 \end{cases}$$

Since a set of homogenous equations has a non-zero solution if and only if the determinant of the matrix of its coefficients is equal to zero and the determinant of the matrix of coefficients of the above set of equations is a polynomial of degree $m+1$ of λ and it has at most $m+1$ solutions, therefore there are at most $m+1$ eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWS.

Now, Suppose that

$$\mathbf{x}=[x_1\ x_2\ \cdots\ x_m\ y\ -x_m\ -x_{m-1}\ \cdots\ -x_1]^T$$

is a RWSS eigenvector of $\mathbf{R}$. Consider the equation $\mathbf{Rx}=\lambda\mathbf{x}$ which is equivalent to the following, since $\mathbf{R}$ is CS matrix.

$$\begin{bmatrix} r_{11} & \cdots & r_{1(m+1)} & r_{1(m+2)} & r_{1(m+3)} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{(m+1)1} & \ddots & r_{(m+1)(m+1)} & r_{(m+1)m} & r_{(m+1)(m-1)} & \cdots & r_{(m+1)1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{1n} & \cdots & r_{1(m+1)} & r_{1m} & r_{1(m-1)} & \cdots & r_{11} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ y \\ -x_m \\ \vdots \\ -x_1 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ -x_m \\ \vdots \\ -x_1 \end{bmatrix}$$

Thus

$$\begin{cases} r_{11}x_1+r_{12}x_2+\cdots+r_{1m}x_m+0-r_{1(m+2)}x_m-\cdots-r_{1n}x_1=\lambda x_1 \\ \vdots \\ r_{m1}x_1+r_{m2}x_2+\cdots+r_{mm}x_m+0-r_{m(m+2)}x_m-\cdots-r_{mn}x_1=\lambda x_m \\ r_{(m+1)1}x_1+r_{(m+1)2}x_2+\cdots+r_{(m+1)m}x_m+0-r_{(m+1)m}x_m-\cdots-r_{(m+1)1}x_1=0 \\ r_{mn}x_1+r_{m(n-1)}x_2+\cdots+r_{m(m+2)}x_m+0-r_{mm}x_m-\cdots-r_{m1}x_1=-\lambda x_m \\ \vdots \\ r_{1n}x_1+r_{1(n-1)}x_2+\cdots+r_{1(m+2)}x_m+0-r_{1m}x_m-\cdots-r_{11}x_1=-\lambda x_1 \end{cases}$$

An straightforward manipulation shows that the first m equations and the last m equations are the same and the $(m+1)$-th equation is trivial. Therefore we have the following set of equations

$$\begin{cases} (r_{11}-r_{1n}-\lambda)x_1+(r_{12}-r_{1(2m)})x_2+\cdots+(r_{1m}-r_{1(m+1)})x_m=0 \\ \vdots \\ (r_{m1}-r_{mn})x_1+(r_{m2}-r_{m(2m)})x_2+\cdots+(r_{mm}-r_{m(m+2)}-\lambda)x_m=0 \end{cases}$$

Since a set of homogenous equations has a non-zero solution if and only if the determinant of the matrix of its coefficients is equal to zero and the determinant of the matrix of coefficients of the above set of equations is a polynomial of degree m of λ and it has at most m solutions, therefore there are at most m eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWSS. Hence the number of RWS eigenvectors is $m+1=[\frac{n+1}{2}]$ and the number of RWSS eigenvectors is $m=[\frac{n}{2}]$, when $n=2m+1$, for some $m\in\mathbb{N}$.

If $n=2m$, for some $m\in\mathbb{N}$, a similar method follows to obtain the result. The proof is completed. □

Let $\mathbf{R}\in M_n(\mathbb{R})$ be a CS matrix. In the next theorem $\mathbf{R}^k$ is considered, when $k\longrightarrow\infty$. The $\lim_{k\to\infty}\mathbf{R}^k$ does not necessarily exist but it is interesting that $\mathbf{R}^k$ tends towards a PS matrix as $k\longrightarrow\infty$.

Theorem 2.6. *Let $n \in \mathbb{N}$, and $\boldsymbol{R} \in M_n(\mathbb{R})$ be a CS matrix which has n distinct eigenvalues. Then $\boldsymbol{R}^k$ tends towards a PS matrix if all of the eigenvalues associated with RWSS eigenvectors are located in the unit circle.*

Proof. Let $\mathbf{M}$ be the modal matrix of $\mathbf{R}$, i.e., the columns of $\mathbf{M}$ are eigenvectors of $\mathbf{R}$. The form of $\mathbf{M}$ is determined according to Theorem 2.5, i.e., all the eigenvectors of $\mathbf{R}$ are either RWS or RWSS matrices. Besides, the relational matrix $\mathbf{R}$ can be decomposed as $\mathbf{MJM}^{-1}$, in which $\mathbf{J}$ is a diagonal matrix containing all the eigenvalues of $\mathbf{R}$. Here just the form of these matrices are important to us. Let's show $\mathbf{M}^{-1}$ with $\mathbf{N}$ for convenience. The form of $\mathbf{N}$ is like that of the transpose of $\mathbf{M}$.

First, suppose that $n = 2m$, for some m, $m \in \mathbb{N}$. An straightforward manipulation shows that

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{PA} & \textbf{-PB} \end{bmatrix}, \text{ and } \mathbf{N} = \begin{bmatrix} \mathbf{C} & \mathbf{CP} \\ \mathbf{D} & \textbf{-DP} \end{bmatrix},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{P} \in M_m(\mathbb{R})$, and

$$\mathbf{P} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in M_m(\mathbb{R}).$$

By Theorem 2.5, it can be assumed that $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWS and $\lambda_{m+1}, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWSS. Let $k \in \mathbb{N}$ and

$$\mathbf{S} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} \text{ and } \mathbf{T} = \begin{bmatrix} \lambda_{m+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Thus $\mathbf{J}^k = \begin{bmatrix} \mathbf{S}^k & 0 \\ 0 & \mathbf{T}^k \end{bmatrix}$. Therefore

$$\begin{aligned} \mathbf{R}^k = \mathbf{MJ}^k\mathbf{N} &= \begin{bmatrix} \mathbf{AS}^k\mathbf{C} + \mathbf{BT}^k\mathbf{D} & \mathbf{AS}^k\mathbf{CP} - \mathbf{BT}^k\mathbf{DP} \\ \mathbf{PAS}^k\mathbf{C} - \mathbf{PBT}^k\mathbf{D} & \mathbf{PAS}^k\mathbf{CP} + \mathbf{PBT}^k\mathbf{DP} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{AS}^k\mathbf{C} & \mathbf{AS}^k\mathbf{CP} \\ \mathbf{PAS}^k\mathbf{C} & \mathbf{PAS}^k\mathbf{CP} \end{bmatrix} + \begin{bmatrix} \mathbf{BT}^k\mathbf{D} & -\mathbf{BT}^k\mathbf{DP} \\ -\mathbf{PBT}^k\mathbf{D} & \mathbf{PBT}^k\mathbf{DP} \end{bmatrix}. \end{aligned}$$

Clearly, $\begin{bmatrix} \mathbf{AS}^k\mathbf{C} & \mathbf{AS}^k\mathbf{CP} \\ \mathbf{PAS}^k\mathbf{C} & \mathbf{PAS}^k\mathbf{CP} \end{bmatrix}$ is both RWS matrix and CWS matrix, and so it is PS matrix. Also by an straightforward manipulation it can be seen that every element of $\begin{bmatrix} \mathbf{BT}^k\mathbf{D} & -\mathbf{BT}^k\mathbf{DP} \\ -\mathbf{PBT}^k\mathbf{D} & \mathbf{PBT}^k\mathbf{DP} \end{bmatrix}$ is a polynomial of $\lambda_{m+1}^k, \ldots, \lambda_n^k$, which by hypothesis tend towards 0 as $k \longrightarrow \infty$. Thus $\mathbf{R}^k$ tends towards a PS matrix.

Next, suppose that $n = 2m + 1$, for some $m \in \mathbb{N}$. In a similar way, it is shown that

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & 0 \\ \mathbf{PA} & \textbf{-PB} \end{bmatrix}, \text{ and } \mathbf{N} = \begin{bmatrix} \mathbf{D} & \mathbf{F} & \mathbf{DP} \\ \mathbf{E} & 0 & \textbf{-EP} \end{bmatrix},$$

where $\mathbf{A}$, $\mathbf{D}^T \in M_{m\times(m+1)}(\mathbb{R})$, $\mathbf{B}$, $\mathbf{E} \in M_m(\mathbb{R})$, and $\mathbf{C}$, $\mathbf{F}^T \in M_{1\times(m+1)}(\mathbb{R})$, and

$$\mathbf{P} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in M_m(\mathbb{R}).$$

By Theorem 2.5, we may assume that $\lambda_1, \ldots, \lambda_{m+1}$ are the eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWS and $\lambda_{m+2}, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{R}$ such that their associated eigenvectors are RWSS. Let $k \in \mathbb{N}$ and

$$\mathbf{S} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{m+1} \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} \lambda_{m+2} & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Thus $\mathbf{J}^k = \begin{bmatrix} \mathbf{S}^k & 0 \\ 0 & \mathbf{T}^k \end{bmatrix}$. Therefore

$$\begin{aligned} \mathbf{R}^k = \mathbf{M}\mathbf{J}^k\mathbf{N} &= \begin{bmatrix} \mathbf{AS}^k\mathbf{D} + \mathbf{BT}^k\mathbf{E} & \mathbf{AS}^k\mathbf{F} & \mathbf{AS}^k\mathbf{DP} - \mathbf{BT}^k\mathbf{EP} \\ \mathbf{CS}^k\mathbf{D} & \mathbf{CS}^k\mathbf{F} & \mathbf{CS}^k\mathbf{DP} \\ \mathbf{PAS}^k\mathbf{D} - \mathbf{PBT}^k\mathbf{E} & \mathbf{PAS}^k\mathbf{F} & \mathbf{PAS}^k\mathbf{DP} + \mathbf{PBT}^k\mathbf{EP} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{AS}^k\mathbf{D} & \mathbf{AS}^k\mathbf{F} & \mathbf{AS}^k\mathbf{DP} \\ \mathbf{CS}^k\mathbf{D} & \mathbf{CS}^k\mathbf{F} & \mathbf{CS}^k\mathbf{DP} \\ \mathbf{PAS}^k\mathbf{D} & \mathbf{PAS}^k\mathbf{F} & \mathbf{PAS}^k\mathbf{DP} \end{bmatrix} + \begin{bmatrix} \mathbf{BT}^k\mathbf{E} & 0 & -\mathbf{BT}^k\mathbf{EP} \\ 0 & 0 & 0 \\ -\mathbf{PBT}^k\mathbf{E} & 0 & \mathbf{PBT}^k\mathbf{EP} \end{bmatrix}. \end{aligned}$$

Clearly, $\begin{bmatrix} \mathbf{AS}^k\mathbf{D} & \mathbf{AS}^k\mathbf{F} & \mathbf{AS}^k\mathbf{DP} \\ \mathbf{CS}^k\mathbf{D} & \mathbf{CS}^k\mathbf{F} & \mathbf{CS}^k\mathbf{DP} \\ \mathbf{PAS}^k\mathbf{D} & \mathbf{PAS}^k\mathbf{F} & \mathbf{PAS}^k\mathbf{DP} \end{bmatrix}$ is both RWS matrix and CWS matrix, and so it is PS matrix. Also by an straightforward manipulation it can be seen that every element of $\begin{bmatrix} \mathbf{BT}^k\mathbf{E} & 0 & -\mathbf{BT}^k\mathbf{EP} \\ 0 & 0 & 0 \\ -\mathbf{PBT}^k\mathbf{E} & 0 & \mathbf{PBT}^k\mathbf{EP} \end{bmatrix}$ is a polynomial of $\lambda_{m+2}^k, \ldots, \lambda_n^k$, which by hypothesis tend towards 0 as $k \longrightarrow \infty$. Thus $\mathbf{R}^k$ tends towards a PS matrix. This competes the proof. $\square$

Each kind of the symmetric matrices introduced in this paper, may be seen in many types of mathematical models, where a matrix representation is used to store the parameters of the model. The inputs of such models are usually processed through compositions with matrices under a matrix equation. Therefore, solving a matrix equation for a special kind of symmetric matrix becomes important as the applications grow. See [11] and [10] for example, where some methods are proposed respectively for analytical and numerical solution of algebraic matrix equations for centrally symmetric matrices. Algebraic matrix equations may be used as state equations to construct discrete-time state-space models leading to a very important modeling framework in system theory which can be used for modeling linear dynamical systems or linear approximations of nonlinear dynamical systems [3]. As another important example, the proposed symmetric matrices may be used as the fuzzy relational matrix in the fuzzy relational modeling framework, where the model is ruled by a fuzzy relational equation [1]. State-space models and fuzzy relational models, both can be used to model discrete-time dynamical systems. Theorem 2.6 may be used to study the convergence of the output of a model in which the involving matrix is centrally symmetric as time goes by.

It is worth mentioning that the dynamic behavior of the system might be exactly or approximately symmetric about an operating point which is very common in practical cases specially in a local view. A well-known benchmark example in control theory is the behavior of an inverted pendulum around its upward or downward position. Likewise, consider a dynamic system which has a symmetric behavior about its unique equilibrium point. In such cases, the system may be modeled by a fuzzy relational model with a centrally symmetric relational matrix in which the center of the membership functions are distributed symmetrically about the origin and so the convergence of the output to the origin can be achieved.

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