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Sufficient conditions for pulse phenomena of nonlinear systems with state-dependent impulses

Guixia Sui[a,b], Xiaodi Li[a,*], Jinjun Fan[a], Donal O'Regan[c]

[a] *School of Mathematical Sciences, Shandong Normal University, Ji'nan, 250014, P. R. China.*
[b] *Primary Education, Jinan Preschool Education College, Ji'nan, 250307, P. R. China.*
[c] *Department of Mathematics, National University of Ireland, Galway, Ireland.*

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Abstract

This paper is concerned with the problem of pulse phenomena of nonlinear systems with state-dependent impulses. Some sufficient conditions which guarantee the absence or presence of pulse phenomena are derived using impulsive control theory. Those results are more general than that given in some earlier references. Two examples are given to illustrate the feasibility and advantage of the results.

Keywords: Pulse phenomena, nonlinear systems, state-dependent impulses, impulsive control theory.
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1. Introduction

Impulsive systems describe a kind of evolution processes that are subject to abrupt changes in their states, whose duration is negligible in comparison with the duration of entire evolution processes [2, 7, 10]. This type of system was recognized as an excellent model to simulate processes and phenomena in many fields such as control theory, engineering, population dynamics and economics ect. For example, in biological neural networks [8], when a stimuli from the body or the external environment is received by receptors, the electrical impulses will be conveyed to the neural net and impulsive effects arise naturally in the net. In a nanoscale electronic circuit consisting of single-electron tunneling junctions (SETJ) [16], the electron tunneling effects can cause impulsive changes of charge in SETJ junction capacitors. Since the charge is the state variable of a SETJ capacitor, the quantum mechanical effects can be modeled by impulsive differential

*Corresponding author
Email address: sodymath@163.com (Xiaodi Li)

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equations. In some cases, moreover, an impulsive system is more efficient than a continuous system. For example, a government can not change savings rates of its central bank everyday. A deep-space spacecraft can not leave its engine on continuously if it has only limited fuel supply. In the last decade, a large number of results on impulsive systems have appeared in the literature, see [5, 11, 12, 14, 15, 17] and reference therein.

As is known [7, 10], the solutions of impulsive differential systems may experience pulse phenomena, namely the solutions may hit certain surfaces finite or infinite number of times causing rhythmical beating. This situation presents difficulties in the investigation of properties of solutions of such systems. Consequently, it is desirable to find conditions that guarantee the absence or presence of pulse phenomena. In fact, pulse phenomena can be regarded as one of the main differences between impulsive differential equations at fixed times and at variable times. Refs. [4, 13] dealt with impulsive differential equations with no pulse phenomena and Refs. [1, 3, 6, 9] dealt with pulse phenomena. In particular, Ref. [9] proposed some conditions that guarantee the absence or presence of pulse phenomena. Ref. [3] considered pulse accumulation in impulsive differential equations with variable times and established some necessary and sufficient conditions to assure pulse accumulation. Ref. [1] presented some sufficient conditions for the absence of beating phenomenon of impulsive differential equations. Ref. [6] proposed some new results on pulse phenomena for impulsive differential systems with variable moments. In previous studies, however, such as those in [1, 3, 6, 9], the authors assume the surfaces functions in impulsive perturbations are independent of time t, so the results can only be applied to some special cases.

Our goal in this paper is to improve the results in [1, 3, 6, 9] and establish some sufficient conditions ensuring the absence or presence of pulse phenomena of nonlinear systems with state-dependent impulses. We consider the case when surfaces functions in impulsive perturbations are time-dependent. The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we present the main results. Two examples are given in Section 4 to illustrate the feasibility and advantage of the results, and conclusions follow in Section 5.

2. Preliminaries

Notations. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{R}^n$ the n-dimensional real space and $\mathbb{Z}_+$ the set of positive integer numbers. For any set $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^k (1 \le k \le n), C[\mathcal{A}, \mathcal{B}] = \{\varphi : \mathcal{A} \to \mathcal{B}\}$ is continuous and $C^1[\mathcal{A}, \mathcal{B}] = \{\varphi : \mathcal{A} \to \mathcal{B}\}$ is continuously differentiable. Let Ω be an open set in $\mathbb{R}^n$ and define $\mathbb{D} = \mathbb{R}_+ \times \Omega$.

Consider the impulsive differential systems

$$\begin{cases} \dot{x}(t) = f(t,x), & t \neq \tau_k(t,x),\ t \ge t_0, \\ x(t_0^+) = x_0, & t_0 \ge 0, \\ \Delta x = I_k(x), & t = \tau_k(t,x), \end{cases} \tag{2.1}$$

where $f \in C[\mathbb{D}, \mathbb{R}^n]$, $I_k \in C[\Omega, \mathbb{R}^n]$, $\tau_k \in C^1[\mathbb{D}, \mathbb{R}_+]$, and $S_k : t = \tau_k(t,x)$ denotes the impulsive surface for every $k \in \mathbb{Z}_+$.

Under the following assumptions:

(A_1) $\tau_k < \tau_{k+1}$ for every $k \in \mathbb{Z}_+$, and $\lim_{k\to\infty} \tau_k(t,x) = \infty$ for any given $(t,x) \in \mathbb{D}$;

(A_2) for $(t,x) \in \mathbb{D}$ and every $k \in \mathbb{Z}_+$,

$$\frac{\partial \tau_k(t,x)}{\partial t} + \frac{\partial \tau_k(t,x)}{\partial x} \cdot f(t,x) \le \mu < 1,$$

where $\mu \in (0,1)$ is a given constant;

(A_3) there exist function sequences $\alpha_k(t)$ and $\beta_k(t) \in C[\mathbb{R}_+, \mathbb{R}_+]$ such that

$$0 < \alpha_k(t) \le \tau_{k+1}(t,x) - \tau_k(t,x) \le \beta_k(t), \quad k \ge 1,\ t \ge t_0.$$

Definition 2.1 ([10]). A function $x : [t_0, t_0 + a) \to R^n, t_0 \geq 0, a > 0$ is said to be a solution of (2.1) if

(i) $x(t_0^+) = x_0$ and $(t, x(t)) \in \mathbb{D}$ for $t \in [t_0, t_0 + a)$;

(ii) $x(t)$ is continuously differentiable and satisfies $\dot{x}(t) = f(t, x)$ for $t \in [t_0, t_0 + a)$ and $t \neq \tau_k(t, x)$;

(iii) if $t \in [t_0, t_0 + a)$ and $t = \tau_k(t, x)$, then $x(t^+) = x(t) + I_k(x(t))$ and at such $t^{'}s$ we always assume that $x(t)$ is right continuous and $s \neq \tau_j(s, x(s))$ for any $j \in \mathbb{Z}_+$, $t < s < \delta$ for any small $\delta > 0$.

3. Main results

In this section, we shall establish some sufficient conditions which will control the absence/presence of pulse phenomena of system (2.1).

Theorem 3.1. *Assume that assumptions $(A_1) - (A_3)$ hold. Suppose further that $x + I_k(x) \in \Omega$ for $x \in \Omega$ and for any given $t \in \mathbb{R}_+$, $j \in \mathbb{Z}_+$,*

(A_4) $\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) < -\beta_k(t), \quad k \geq j - 1, \quad 0 \leq s \leq 1;$

(A_5) $\frac{\partial \tau_{k+1}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) \geq -\alpha_{k+1}(t), \quad k \geq j - 1, \quad 0 \leq s \leq 1.$

Then each solution $x(t) = x(t, t_0, x_0)$ of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$ meets every surface S_{j+2n} exactly once and does not meet S_{j+2n+1}, $n = 0, 1, 2, \cdots$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$. Let $\Gamma(t) = t - \tau_j(t, x(t))$. Note that $\Gamma(t_0) = t_0 - \tau_j(t_0, x_0) < 0$, then $\Gamma(t) - \Gamma(t_0) = \Gamma^{'}(\xi)(t - t_0) \geq (1 - \mu)(t - t_0) \to +\infty, t \to +\infty$. Then it is clear that there is a unique $t_1 > t_0$ such that

$$t_1 = \tau_j(t_1, x(t_1)) \ \text{ and } \ \mathrm{t} < \tau_\mathrm{j}(\mathrm{t}, \mathrm{x}(\mathrm{t}))\ \forall \mathrm{t} \in (\mathrm{t}_0, \mathrm{t}_1).$$

Let $T(t) = t - \tau_{j-1}(t, x(t, t_0, x_0^+))$, in view of (A_2), it is obvious that $T(t)$ is nondecreasing in (t_0, t_1) and considering $T(t_0) = t_0 - \tau_{j-1}(t_0, x_0) > 0$, we get

$$t > \tau_{j-1}(t, x(t, t_0, x_0^+)), \qquad t \in [t_0, t_1].$$

Therefore, $x(t)$ meets the surface S_j first at $t = t_1$ before hitting any other surface. Setting $x_1 = x(t_1), x_1^+ = x_1 + I_j(x_1)$, it follows from (A_4) that

$$t_1 = \tau_j(t_1, x_1) > \tau_j(t_1, x_1^+) + \beta_j(t_1) \geq \tau_{j+1}(t_1, x_1^+).$$

On the other hand, (A_5) implies that

$$t_1 = \tau_j(t_1, x_1) < \tau_{j+1}(t_1, x_1) \leq \tau_{j+1}(t_1, x_1^+) + \alpha_{j+1}(t_1) \leq \tau_{j+2}(t_1, x_1^+),$$

which leads to

$$\tau_{j+1}(t_1, x_1^+) < t_1 < \tau_{j+2}(t_1, x_1^+).$$

Proceeding as before, there exists a unique $t_2 > t_1$ such that

$$t_2 = \tau_{j+2}(t_2, x(t_2, t_1, x_1^+)) \ \text{ and } \ \mathrm{t} < \tau_{\mathrm{j}+2}(\mathrm{t}, \mathrm{x}(\mathrm{t}, \mathrm{t}_1, \mathrm{x}_1^+))\ \ \forall \mathrm{t} \in (\mathrm{t}_1, \mathrm{t}_2).$$

Let $T^1(t) = t - \tau_{j+1}(t, x(t, t_1, x_1^+))$, then it follows from condition (A_2) that function

$$T^1(t) = t - \tau_{j+1}(t, x(t, t_1, x_1^+))$$

is nondecreasing in (t_1, t_2). Thus, in view of $T^1(t_1) > 0$, we get

$$t > \tau_{j+1}(t, x(t, t_1, x_1^+)), \quad t \in [t_1, t_2].$$

Thus, $x(t)$ meets S_{j+2} first at $t = t_2$ after t_1 before hitting any other surface. Setting again $x_2 = x(t_2)$, $x_2^+ = x_2 + I_{j+2}(x_2)$ and considering conditions (A_4) and (A_5), we obtain

$$\tau_{j+3}(t_2, x_2^+) < t_2 < \tau_{j+4}(t_2, x_2^+).$$

Then arguing as before, we obtain that there exists a $t_3 = \tau_{j+4}(t_3, x(t_3))$ such that $x(t)$ meets S_{j+4} first at $t = t_3$ after t_2. Repeating this process, one can prove the stated claim and therefore the proof is complete. □

Remark 3.2. In [1, 3, 6, 9], the pulse phenomena of system (2.1) with impulsive surface $S_k : t = \tau_k(x)$ has been studied and some sufficient conditions that guarantee the absence or presence of pulse phenomena were derived. But one may note that those results only apply when function τ_k in the surface S_k is independent of time t. In other words, those results cannot be applied to some general cases such as $\tau_k(t, x) = k + \sin t + x$, or $\tau_k(t, x) = k + \ln t + x^2$. In this paper, we cover the shortage and propose some sufficient conditions for the case of impulsive surface $S_k : t = \tau_k(t, x)$. Hence, our development results improve and generalize the existing results in [1, 3, 6, 9].

Remark 3.3. One can find that the assumption of boundedness of τ_k is necessary to derive the results for absence/presence impulsive phenomena in [1, 3, 6, 9]. In this paper, we drop the assumption completely. In other words, our results can be applied to cases such as $\tau_k(t, x) = \cos t + x + k$. This is the advantage of the results in this paper compared with those in [1, 3, 6, 9].

Theorem 3.4. *Assume that assumptions $(A_1) - (A_3)$ hold. Suppose further that $x + I_k(x) \in \Omega$ for $x \in \Omega$ and for any given $t \in \mathbb{R}_+$, $j \in \mathbb{Z}_+$, $N \in \mathbb{Z}_+$,*

(A_6) $\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) < -\sum_{i=0}^{N-2} \beta_{k+i}(t), \quad k \geq j-1, \quad 0 \leq s \leq 1;$

(A_7) $\frac{\partial \tau_{k+N-1}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) \geq -\alpha_{k+N-1}(t), \quad k \geq j-1, \quad 0 \leq s \leq 1.$

Then each solution $x(t) = x(t, t_0, x_0)$ of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$ meets every surface S_{j+iN} in turn, $i = 0, 1, 2, \cdots$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$. Arguing as Theorem 3.1, considering function Γ and the continuity of $\tau_j(t, x)$ on $\mathbb{D}$, we obtain that $x(t)$ meets S_j first at $t = t_1 (t_1 > t_0)$ before hitting any other surface. Setting $x_1 = x(t_1), x_1^+ = x_1 + I_j(x_1)$, it then follows from (A_6) that

$$\tau_j(t_1, x_1^+) - \tau_j(t_1, x_1) < -\sum_{i=0}^{N-2} \beta_{j+i}(t_1),$$

i.e.,

$$\tau_j(t_1, x_1) > \tau_j(t_1, x_1^+) + \sum_{i=0}^{N-2} \beta_{j+i}(t_1).$$

Moreover,

$$\begin{aligned}
\tau_{j+N-1}(t_1, x_1^+) - \tau_j(t_1, x_1^+) &= \tau_{j+N-1}(t_1, x_1^+) - \tau_{j+N-2}(t_1, x_1^+) + \tau_{j+N-2}(t_1, x_1^+) - \tau_{j+N-3}(t_1, x_1^+) \\
&\quad + \cdots + \tau_{j+1}(t_1, x_1^+) - \tau_j(t_1, x_1^+) \\
&\leq \beta_{j+N-2}(t_1) + \beta_{j+N-3}(t_1) + \cdots + \beta_j(t_1) \\
&= \sum_{i=0}^{N-2} \beta_{j+i}(t_1),
\end{aligned}$$

i.e.,

$$\tau_j(t_1, x_1^+) + \sum_{i=0}^{N-2} \beta_{j+i}(t_1) \geq \tau_{j+N-1}(t_1, x_1^+),$$

which implies that

$$t_1 = \tau_j(t_1, x_1) > \tau_{j+N-1}(t_1, x_1^+).$$

On the other hand, condition (A_7) implies that

$$\tau_{j+N-1}(t_1, x_1^+) - \tau_{j+N-1}(t_1, x_1) \geq -\alpha_{j+N-1}(t_1),$$

i.e.,

$$\tau_{j+N}(t_1, x_1^+) \geq \tau_{j+N-1}(t_1, x_1^+) + \alpha_{j+N-1}(t_1) \geq \tau_{j+N-1}(t_1, x_1) > \tau_j(t_1, x_1) = t_1,$$

which leads to

$$\tau_{j+N-1}(t_1, x_1^+) < t_1 < \tau_{j+N}(t_1, x_1^+).$$

Proceeding as before, there exists a unique $t_2 > t_1$ such that

$$t_2 = \tau_{j+N}(t_2, x(t_2, t_1, x_1^+)) \;\text{ and }\; \mathrm{t} < \mathrm{\tau_{j+N}(t, x(t, t_1, x_1^+))} \;\; \forall \mathrm{t} \in (\mathrm{t_1}, \mathrm{t_2}).$$

Define $T^2(t) = t - \tau_{j+N-1}(t, x(t, t_1, x_1^+))$. Then condition (A_2) implies that function T^2 is nondecreasing in (t_1, t_2), and it then follows from $T^2(t_1) > 0$ that

$$t > \tau_{j+N-1}(t, x(t, t_1, x_1^+)), \qquad t \in [t_1, t_2].$$

Hence, $x(t)$ meets S_{j+N} first at $t = t_2$ after t_1 before hitting any other surface. Setting again $x_2 = x(t_2)$, $x_2^+ = x_2 + I_{j+N}(x_2)$ and considering conditions (A_6) and (A_7), we obtain

$$\tau_{j+2N-1}(t_2, x_2^+) < t_2 < \tau_{j+2N}(t_2, x_2^+).$$

Then arguing as before, there exists a $t_3 = \tau_{j+2N}(t_3, x(t_3))$ such that $x(t)$ meets S_{j+2N} first at $t = t_3$ after t_2. Repeating this process, we obtain Theorem 3.4 and the proof is complete. □

Next we present some sufficient conditions that guarantee the presence of pulse phenomena of system (2.1).

Theorem 3.5. *Assume that assumptions $(A_1) - (A_3)$ hold. Suppose further that $x + I_k(x) \in \Omega$ for $x \in \Omega$ and for any given $t \in \mathbb{R}_+$, $j \in \mathbb{Z}_+$,*

(A_8) $\frac{\partial \tau_j}{\partial x}(t, x + sI_j(x)) \cdot I_j(x) < \alpha_{j-1}(t), \quad 0 \leq s \leq 1;$

(A_9) $\frac{\partial \tau_{j+1}}{\partial x}(t, x + sI_j(x)) \cdot I_j(x) \geq \beta_j(t), \quad 0 \leq s \leq 1.$

Then each solution $x(t) = x(t, t_0, x_0)$ of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$ meets the surface S_j several times.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$. Arguing as before, it is easy to derive that $x(t)$ meets S_j first at $t = t_1 (t_1 > t_0)$ before hitting any other surface. Setting $x_1 = x(t_1), x_1^+ = x_1 + I_j(x_1)$, in view of (A_8), we get

$$t_1 = \tau_j(t_1, x_1) > \tau_j(t_1, x_1^+) - \alpha_{j-1}(t_1) \geq \tau_{j-1}(t_1, x_1^+).$$

On the other hand, (A_9) implies that

$$t_1 = \tau_j(t_1, x_1) < \tau_{j+1}(t_1, x_1) \le \tau_{j+1}(t_1, x_1^+) - \beta_j(t_1) \le \tau_j(t_1, x_1^+),$$

which yields that

$$\tau_{j-1}(t_1, x_1^+) < t_1 < \tau_j(t_1, x_1^+).$$

Then there exists a unique $t_2 > t_1$ such that $t_2 = \tau_j(t_2, x(t_2, t_1, x_1^+))$, which shows that $x(t)$ meets S_j secondly at $t = t_2$ after t_1 before hitting any other surface. Repeating this process, Theorem 3.5 can be derived and the proof is complete. □

Theorem 3.6. *Assume that assumptions $(A_1) - (A_3)$ hold. Suppose further that $x + I_k(x) \in \Omega$ for $x \in \Omega$ and for any given $t \in R_+$ and $j, m, n \in \mathbb{Z}_+$,*

(i) for $k = 2n - 1$,

- (a_1) $\frac{\partial \tau_{k-1}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) < -\beta_{k-1}(t), \quad k \ge j-1, \quad 0 \le s \le 1;$
- (a_2) $\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) \ge -\alpha_k(t), \quad k \ge j-1, \quad 0 \le s \le 1,$

(ii) for $k = 2n$,

- (b_1) $\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) < -\sum_{i=0}^{1} \beta_{k+i}(t), \quad k \ge j-1, \quad 0 \le s \le 1;$
- (b_2) $\frac{\partial \tau_{k+2}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) \ge -\alpha_{k+2}(t), \quad k \ge j-1, \quad 0 \le s \le 1.$

Then each solution $x(t) = x(t, t_0, x_0)$ of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$ meets every surface $S_{j+4(l-1)}$, S_{j+4l-3} in turn when $j = 2m - 1$, $l = 1, 2, 3, \cdots$, and meets every surface $S_{j+4(l-1)}$, S_{j+4l-1} in turn, when $j = 2m$, $l = 1, 2, 3, \cdots$.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$. Arguing as before, it can be deduced that $x(t)$ meets S_j first at $t = t_1 (t_1 > t_0)$ before hitting any other surface. Setting $x_1 = x(t_1), x_1^+ = x_1 + I_j(x_1)$. Now there are two cases: $j = 2m - 1$ and $j = 2m$. First we consider $j = 2m - 1$. It follows from (a_1) that

$$\tau_{j-1}(t_1, x_1^+) - \tau_{j-1}(t_1, x_1) < -\beta_{j-1}(t_1),$$

i.e.,

$$t_1 = \tau_j(t_1, x_1) > \tau_{j-1}(t_1, x_1) > \tau_{j-1}(t_1, x_1^+) + \beta_{j-1}(t_1) \ge \tau_j(t_1, x_1^+).$$

On the other hand, (a_2) implies that

$$\tau_j(t_1, x_1^+) - \tau_j(t_1, x_1) \ge -\alpha_j(t_1),$$

i.e.,

$$t_1 = \tau_j(t_1, x_1) \le \tau_j(t_1, x_1^+) + \alpha_j(t_1) \le \tau_{j+1}(t_1, x_1^+),$$

which yields that

$$\tau_j(t_1, x_1^+) < t_1 < \tau_{j+1}(t_1, x_1^+).$$

Proceeding as before, there exists a unique $t_2 > t_1$ such that

$$t_2 = \tau_{j+1}(t_2, x(t_2, t_1, x_1^+)) \text{ and } \mathrm{t} < \tau_{\mathrm{j+1}}(\mathrm{t}, \mathrm{x}(\mathrm{t}, \mathrm{t}_1, \mathrm{x}_1^+)) \ \ \forall \mathrm{t} \in (\mathrm{t}_1, \mathrm{t}_2).$$

Then it is easy to derive that

$$t > \tau_j(t, x(t, t_1, x_1^+)), \quad t \in [t_1, t_2].$$

Hence, $x(t)$ meets S_{j+1} first at $t = t_2$ after t_1 before hitting any other surface. Setting again $x_2 = x(t_2)$, $x_2^+ = x_2 + I_{j+1}(x_2)$ and considering $j+1 = 2m$, we obtain by (b_1)

$$\tau_{j+1}(t_2, x_2^+) - \tau_{j+1}(t_2, x_2) < -\sum_{i=0}^{1} \beta_{j+1+i}(t_2),$$

i.e.,

$$t_2 = \tau_{j+1}(t_2, x_2) > \tau_{j+1}(t_2, x_2^+) + \sum_{i=0}^{1} \beta_{j+1+i}(t_2).$$

Moreover,

$$\begin{aligned}\tau_{j+3}(t_2, x_2^+) - \tau_{j+1}(t_2, x_2^+) &= \tau_{j+3}(t_2, x_2^+) - \tau_{j+2}(t_2, x_2^+) + \tau_{j+2}(t_2, x_2^+) - \tau_{j+1}(t_2, x_2^+) \\ &\le \beta_{j+2}(t_2) + \beta_{j+1}(t_2) \\ &= \sum_{i=0}^{1} \beta_{j+1+i}(t_2),\end{aligned}$$

i.e.,

$$\tau_{j+1}(t_2, x_2^+) + \sum_{i=0}^{1} \beta_{j+1+i}(t_2) \ge \tau_{j+3}(t_2, x_2^+),$$

which implies that

$$t_2 = \tau_{j+1}(t_2, x_2) > \tau_{j+3}(t_2, x_2^+).$$

On the other hand, condition (b_2) implies that

$$\tau_{j+3}(t_2, x_2^+) - \tau_{j+3}(t_2, x_2) \ge -\alpha_{j+3}(t_2),$$

i.e.,

$$t_2 = \tau_{j+1}(t_2, x_2) < \tau_{j+3}(t_2, x_2) \le \tau_{j+3}(t_2, x_2^+) + \alpha_{j+3}(t_2) \le \tau_{j+4}(t_2, x_2^+),$$

which implies that

$$\tau_{j+3}(t_2, x_2^+) < t_2 < \tau_{j+4}(t_2, x_2^+).$$

Then arguing as before, there exists a $t_3 = \tau_{j+4}(t_3, x(t_3))$ such that $x(t)$ meets S_{j+4} first at $t = t_3$ after t_2 before hitting any other surface. In this way, it can be deduced that $x(t) = x(t, t_0, x_0)$ meets every surface $S_{j+4(l-1)}$, S_{j+4l-3} in turn for $j = 2m-1$. In addition, when $j = 2m$, we can apply the same argument as $j = 2m-1$. The proof is repetitive and thus omitted here. The proof of Theorem 3.6 is therefore complete. □

4. Examples

Example 4.1. Consider the impulsive system

$$\begin{cases} \dot{x}(t) = \frac{1}{2} sat(x) + \cos t, & t \ne \tau_k(t, x),\ t \ge 0 \\ \Delta x = -(2k+2), & t = \tau_k(t, x), \end{cases} \tag{4.1}$$

where $sat(x) = \frac{1}{2}[|x+1| - |x-1|]$, $\tau_k(t, x) = x + k^2 - \sin t$. Next we show that each solution $x(t) = x(t, t_0, x_0)$ of (4.1) with $\tau_0(t_0, x_0) < t_0 < \tau_1(t_0, x_0)$ meets every surface S_{2n+1} exactly once and does not meet S_{2n+2}, $n = 0, 1, 2, \cdots$.

In fact, let $\mu = 1/2$, then it is clear that

$$\frac{\partial \tau_k(t,x)}{\partial t} + \frac{\partial \tau_k(t,x)}{\partial x} \cdot f(t,x) = -\cos t + \cos t + \frac{1}{2} sat(x) = \frac{1}{2} sat(x) \leq \mu < 1.$$

Choose $\alpha_k = \beta_k = 2k+1$, then it follows that

$$\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) = -(2k+2) < -(2k+1) = -\beta_k,$$
$$\frac{\partial \tau_{k+1}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) = -(2k+2) \geq -(2k+3) = -\alpha_{k+1}.$$

Thus it is easy to check that conditions $(A_1) - (A_5)$ are satisfied and thus by Theorem 3.1 we know that each solution $x(t) = x(t, t_0, x_0)$ of (4.1) with $\tau_0(t_0, x_0) < t_0 < \tau_1(t_0, x_0)$ meets every surface S_{2n+1} exactly once and does not meet S_{2n+2}, $n = 0, 1, 2, \cdots$.

Example 4.2. Consider the impulsive system

$$\begin{cases} \dot{x}(t) = \frac{2}{3} sat(x) - \sin t, & t \neq \tau_k(t,x), \quad t \geq 0 \\ \Delta x = -2^{k-1} \cdot 7, & t = \tau_k(t,x), \end{cases} \tag{4.2}$$

where $\tau_k(t,x) = x + 2^k - \cos t$. Note that $\tau_k < \tau_{k+1}$ and $\lim_{k\to\infty} \tau_k = \infty$ for given $(t,x) \in \mathbb{D}$. Since $\tau_{k+1} - \tau_k = 2^{k+1} - 2^k = 2^k$, one may choose $\alpha_k = \beta_k = 2^k$ such that condition (A_3) holds. Let $\mu = 2/3$, then it holds that

$$\frac{\partial \tau_k(t,x)}{\partial t} + \frac{\partial \tau_k(t,x)}{\partial x} \cdot f(t,x) = \sin t + \frac{2}{3} sat(x) - \sin t \leq \mu < 1.$$

Moreover,

$$\frac{\partial \tau_k}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) = -2^k \cdot \frac{7}{2} < -2^k \cdot 3 = -(2^k + 2^{k+1}) = -\sum_{i=0}^{1} \beta_{k+i},$$
$$\frac{\partial \tau_{k+2}}{\partial x}(t, x + sI_k(x)) \cdot I_k(x) = -2^k \cdot \frac{7}{2} \geq -2^k \cdot 4 = -2^{k+2} = -\alpha_{k+2}.$$

Hence, all conditions in Theorem 3.4 hold and thus each solution $x(t) = x(t, t_0, x_0)$ of (4.2) with $\tau_{j-1}(t_0, x_0) < t_0 < \tau_j(t_0, x_0)$ meets every surface S_{j+iN} in turn, $i = 0, 1, 2, \cdots$.

5. Conclusion

This paper dealt with the pulse phenomena of nonlinear systems with state-dependent impulses, where the surfaces functions in impulsive perturbations are time-dependent. Based on impulsive control theory, some sufficient conditions which guarantee the absence or presence of pulse phenomena have been presented, which are more general than those mentioned in the literature. The results are helpful to control the absence or present pulse phenomena of nonlinear systems with state-dependent impulses and can be applied to design the stability of networks modeling when the surfaces functions in impulsive perturbations are time-dependent. In addition, we point out that so far there is little work on pulse phenomena of delay impulsive systems. More methods and tools should be explored and developed in the future.

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