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# On $\nabla^{**}$-distance and fixed point theorems in generalized partially ordered $D^*$-metric spaces

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**Abstract**

In this paper, we introduce a new concept on a complete generalized $D^*$-metric space by using the concept of generalized $D^*$-metric space ($D^*$-cone metric space) called $\nabla^{**}$-distance and, by using the concept of the $\nabla^{**}$-distance we prove some new fixed point theorems in complete partially ordered generalized $D^*$-metric space which is the main result of our paper. 

*Keywords:* Fixed point theorem, generalized $D^*$-metric spaces, $\nabla^{**}$-distance.
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## 1. Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a fundamental role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories. The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions, (see [1, 4-6, 8-12, 15]). The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone. This cone naturally induces a partial order in the Banach spaces. In recent times, fixed point theory has developed rapidly in partially ordered metric spaces such as Nieto and Lopez [12, 14], Ran and Reurings [17] and Petruşel and Rus [16] presented some new results for contractions in partially ordered metric spaces. The main idea in [12, 14, 17] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Sedghi et. al., [18] have been

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introduced the concept of $D^*$-metric space which as a probable modification of the definition of $D$-metric introduced by Dhage, [7]. Afterwards, many authors [13, 20-21] proved some fixed point theorems in these spaces. In [3], AL. Jumaili and Yang, used the concept of a $D^*$-metric space and introduced a new concept of the $\nabla^*$-distance on a complete $D^*$-metric space and proved some fixed point theorems in partially ordered $D^*$-metric space. Recently, Aage and Salunke, [2] generalized the concept of $D^*$-metric space by replacing $R$ by a real Banach space in $D^*$-metric spaces. The purpose of this paper is to introduce a new concept called $\nabla^{**}$-distance on a complete generalized $D^*$-metric space which is a generalization of the concept of $\nabla^*$-distance which is proposed by AL. Jumaili and Yang [3], by replacing the set of real numbers by an ordered Banach space. By using the concepts of generalized $D^*$-metric space and $\nabla^{**}$-distance, we prove some new fixed point theorems in complete partially ordered generalized $D^*$-metric space, which is the main result of our paper.

## 2. Preliminaries

Let $E$ be a real Banach space and $P$ a subset of $E$. $P$ is called a cone if and only if:

(a) $P$ is closed, non-empty and $P \neq \{0\}$,
(b) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$,
(c) $P \bigcap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on $E$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$, where $intP$ denotes the interior of $P$. A cone $P$ is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$.
In the following we always suppose that $E$ is a Banach space, $P$ is a cone in $E$ with $intP \neq \{0\}$ and $\leq$ is a partial ordering with respect to $P$.

Now, we state the definitions of generalized $D^*$-metric, $\nabla^{**}$-distance and prove a lemma. For more information on $D^*$-metrics and generalized $D^*$-metric, we refer the reader to [2] and [18] respectively.

**Definition 2.1.** ([2]) Let $X$ be a non empty set. A generalized $D^*$-metric on $X$ is a function, $D^*$: $X \times X \times X \to E$ that satisfies the following conditions for all $x, y, z, a \in X$:

(a) $D^*(x, y, z) \geq 0$,

(b) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(c) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where $p$ is a permutation function,

(d) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$ .

Then the function $D^*$ is called a generalized $D^*$-metric ($D^*$-cone metric) and the pair $(X, D^*)$ is called a generalized $D^*$-metric space ($D^*$-cone metric space).

**Definition 2.2.** ([2]) Let $(X, D^*)$ be a generalized $D^*$-metric space then:

(a) A sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence. If for every $c$ in $E$ with $0 \ll c$, there is $N$ such that for all $m, n, l > N$, $D^*(x_m, x_n, x_l) \ll c$.

(b) If every Cauchy sequence in $X$ is convergent in $X$, then $X$ is called a complete generalized $D^*$-metric.

(c) A sequence $\{x_n\}$ in $X$ is said to be converges to point $x \in X$. If for every $c \in E$ with $0 \ll c$ there is $N$ such that for all $m, n > N$, $D^*(x_m, x_n, x) \ll c$, and a point $x$ is the limit of $\{x_n\}$ and denoted this by $x_n \to x$ $(n \to \infty)$.

**Proposition 2.3.** ([2]) *If $(X, D^*)$ is a generalized $D^*$-metric space, then for all $x, y \in X$, we have $D^*(x, x, y) = D^*(x, y, y)$.*

*Remark* 2.4. ([2]) If $u_n \geq 0$ then $u \geq 0$. Thus if $u_n \leq v_n$ in $P$ then, $\lim u_n \leq \lim v_n$, provided limit exist.

**Lemma 2.5.** ([2]) *Let $(X, D^*)$ be a generalized $D^*$-metric space, $P$ be a normal cone with normal constant $K$. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $D^*(x_m, x_n, x) \to 0$, $(m, n \to \infty)$.*

**Lemma 2.6.** ([2]) *Let $(X, D^*)$ be a generalized $D^*$-metric space, $P$ be a normal cone with normal constant $K$. Let $\{x_n\}$ be a sequence in $X$. If $\{x_n\}$ converges to $x$ and $\{x_n\}$ converges to $y$, then $x = y$. That is the limit of $\{x_n\}$, if exists, is unique.*

**Lemma 2.7.** ([2]) *Let $(X, D^*)$ be a generalized $D^*$-metric space, $\{x_n\}$ be a sequence in $X$. If $\{x_n\}$ converges to $x$, then $\{x_n\}$ is a Cauchy sequence.*

**Lemma 2.8.** ([2]) *Let $(X, D^*)$ be a generalized $D^*$ metric space and $P$ be a normal cone with normal constant $K$. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $D^*(x_m, x_n, x_l) \to 0$, $(m, n, l \to \infty)$.*

**Lemma 2.9.** ([2]) *Let $(X, D^*)$ be a generalized $D^*$-metric space and $P$ be a normal cone with normal constant $K$. Let $\{x_m\}$, $\{y_n\}$, and $\{z_l\}$ be three sequences in $X$ such that, $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $D^*(x_m, y_n, z_l) \to D^*(x, y, z)$, $(m, n, l \to \infty)$.*

## 3. $\nabla^{**}$-distance on a generalized $D^*$-metric space $(X, D^*)$

Now, we introduce the concept of $\nabla^{**}$-distance on a generalized $D^*$-metric space $(X, D^*)$, which is a generalization of the concept of $\nabla^*$-distance which is proposed by AL. Jumaili and Yang [3]. We start with a definition of $\nabla^{**}$-distance.

**Definition 3.1.** Let $(X, D^*)$ be a generalized $D^*$-metric space. Then a function, $\nabla^{**}$: $X^3 \to E$ is called $\nabla^{**}$-distance on $X$ if the following conditions are satisfied:

(($\Delta_1$)) $\nabla^{**}(x, x, y) \geq 0$ for all $x, y \in X$;

(($\Delta_2$)) $\nabla^{**}(x, y, z) \leq \nabla^{**}(x, y, a) + \nabla^{**}(a, z, z)$ for all $x, y, z, a \in X$;

(($\Delta_3$)) For all $x, y \in X$ and $n \geq 1$, if $\nabla^{**}(x, y, z_n) \leq \delta$ for some $\delta = \delta_x \in P$ and $\nabla^{**}(x, z_n, y) \leq \beta$ for some $\beta = \beta_x \in P$ then, $\nabla^{**}(x, y, z) \leq \delta$ and $\nabla^{**}(x, z, y) \leq \beta$ respectively, whenever $\{z_n\}$ is a sequence in $X$ converges to point $z \in X$;

(($\Delta_4$)) For each $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that, $\nabla^{**}(x, y, a) \ll e$ and $\nabla^{**}(a, z, z) \ll e$ imply $D^*(x, y, z) \ll c$.

*Remark* 3.2. Let $(X, D^*)$ be a $D^*$-metric space, $E = R^+, P = [0, \infty)$ and $(\Delta_3)$ is replaced with the following condition:
(For any $x, y \in X, \nabla^{**}(x, y, .), \nabla^{**}(x, ., y) : X \to R^+$ are lower semi-continuous), then the $\nabla^{**}$-distance is a $\nabla^*$-distance on $X$ which is proposed by AL. Jumaili and Yang [3]. Moreover, it is easy to see that, if $\nabla^{**}(x, y, .), \nabla^{**}(x, ., y)$ are lower semi-continuous, then $(\Delta_3)$ holds. Thus, it is obvious that every $\nabla^*$-distance is a $\nabla^{**}$-distance if $(X, D^*)$ is a $D^*$-metric space, $E = R^+, P = [0, \infty)$, but the converse do not hold. Therefore, the $\nabla^{**}$-distance is a generalization of the $\nabla^*$-distance.

**Example 3.3.** Let $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$, we consider the generalized $D^*$-metric and defined a mapping, $D^*$: $X^3 \to E$ by:
$D^*(x, y, z) = (| x-y | + | y-z | + | x-z |, \alpha(| x-y | + | y-z | + | x-z |))$ where $\alpha \geq 0$ is a constant (see [2]). Then the mapping, $\nabla^{**}$: $X^3 \to E$ which defined by: $\nabla^{**}(x, y, z) =| z-x | + | x-y |$ for all $x, y, z \in R$ is a $\nabla^{**}$-distance on $R$.

*Proof.* The proofs of parts $(\Delta_1)$ , $(\Delta_2)$ and $(\Delta_3)$ are immediate. Now, we prove $(\Delta_4)$, let $c \in E$ with $0 \ll c$ be given and put $e = \frac{c}{3}$.
Suppose that $\nabla^{**}(x, y, a) \ll e$ and $\nabla^{**}(a, z, z) \ll e$, then we have, $| x-a | \ll e, | y-a | \ll e$ and $| a-z | \ll e$, which imply that,
$D^*(x, y, z) \leq \nabla^{**}(x, y, a) + \nabla^{**}(a, z, z) =| x-a | + | y-a | + | a-z | \ll e+e+e = 3e = c$. This shows that $\nabla^{**}$ satisfies $(\Delta_4)$ and hence $\nabla^{**}$ is a $\nabla^{**}$-distance. □

**Example 3.4.** Let $(X, D^*)$ be a generalized $D^*$-metric space and $P$ be a normal cone. Let $D^*$: $X^3 \to E$ defined by:

$$D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for all $x, y, z \in X$. Then $\nabla^{**} = D^*$ is a $\nabla^{**}$-distance on $X$.

*Proof.* The proofs of a parts $(\Delta_1)$ and $(\Delta_2)$ are direct. By Lemma 2.9 we have that $(\Delta_3)$ holds, let $c \in E$ with $0 \ll c$ be given and put $e = \frac{c}{2}$. Suppose that $\nabla^{**}(x, y, a) \ll e$ and $\nabla^{**}(a, z, z) \ll e$, then we have, $D^*(x, y, z) \leq \nabla^{**}(x, y, a) + \nabla^{**}(a, z, z)$, which implies that $D^*(x, y, z) \leq e + e = 2e = c$. This shows that $\nabla^{**}$ satisfies $(\Delta_4)$ and hence $\nabla^{**}$ is a $\nabla^{**}$-distance. □

**Example 3.5.** Let $(X, D^*)$ be a generalized $D^*$-metric space. Then the mapping $\nabla^{**} : X \times X \times X \to E$ defined by: $\nabla^{**}(x, y, z) = t$, for all $x, y, z \in X$ is a $\nabla^{**}$-distance on $X$, where $t$ is a positive real number.

*Proof.* The proofs of $(\Delta_1)$ and $(\Delta_2)$ are immediate, Lemma 2.9 show that a part $(\Delta_3)$ holds, now to show $(\Delta_4)$, for each $c \in E$ with $0 \ll c$, put $e = \frac{c}{2}$. If $\nabla^{**}(x, y, a) \ll e$ and $\nabla^{**}(a, z, z) \ll e$, imply that,

$$D^*(x, y, z) \leq \nabla^{**}(x, y, a) + \nabla^{**}(a, z, z) \ll e + e = 2e = c.$$

□

**Example 3.6.** Let $(X, D^*)$ be a generalized $D^*$-metric space and $P$ be a normal cone. Let $D^*$: $X^3 \to E$ defined by:

$$D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$$

for all $x, y, z \in X$. Then $\nabla^{**} = D^*$ is a $\nabla^{**}$-distance on $X$.

*Proof.* In fact, the proofs of $(\Delta_1)$ and $(\Delta_2)$ are obvious immediate. Lemma 2.9 shows that a part $(\Delta_3)$ holds, let $c \in E$ with $0 \ll c$ be given and put $e = \frac{c}{2}$. If $\nabla^{**}(x, y, a) \ll e$ and $\nabla^{**}(a, z, z) \ll e$, we have, $d(x, a) \ll e, d(y, a) \ll e, d(a, z) \ll e$ and $d(y, z) \ll e$, respectively, which imply that,

$$D^*(x, y, z) \leq \nabla^{**}(x, y, a) + \nabla^{**}(a, z, z) \ll e + e = 2e = c.$$

This shows that $D^*$ satisfies $(\Delta_4)$ and hence $D^*$ is a $\nabla^{**}$-distance. □

**Example 3.7.** Let $(X, D^*)$ be a generalized $D^*$-metric space and $P$ be a normal cone. Let $D^*$: $X^3 \to E$ defined by: $D^*(x, y, z) = \{d(u, y), d(u, z), d(u, x)\}$ for all $x, y, z \in X$ where $u \in X$ is a fixed. Then $\nabla^{**} = D^*$ is a $\nabla^{**}$-distance on $X$.

*Proof.* The proofs of $(\Delta_1)$ and $(\Delta_3)$ immediate. Since $D^*(u,y,z) \leq D^*(u,y,a) + D^*(a,z,z)$ this means,

$$\nabla^{**}(x,y,z) \leq \nabla^{**}(x,y,a) + \nabla^{**}(a,z,z),$$

this shows that $(\Delta_2)$ holds.

Now, let $c \in E$ with $0 \ll c$ be given and put $e = \frac{c}{2}$. Suppose that $\nabla^{**}(x,y,a) \ll e$ and

$$\nabla^{**}(a,z,z) \ll e$$

then we have, $D^*(x,y,z) \leq D^*(u,y,a) + D^*(a,z,z) = \nabla^{**}(x,y,a) + \nabla^{**}(a,z,z) \ll e + e = 2e = c$. This shows that $(\Delta_4)$ holds. Thus $\nabla^{**}$ is a $\nabla^{**}$-distance. □

**Lemma 3.8.** *Let $(X, D^*)$ be a generalized $D^*$-metric space and $\nabla^{**}$ be a $\nabla^{**}$-distance on $X$. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in $X$, $\{\delta_n\}$ and $\{\beta_n\}$ be a sequences in $P$ converging to zero and let $x, y, z, a \in X$. Then the following hold:*

(a) *If $\nabla^{**}(y,y,x_n) \leq \delta_n$ and $\nabla^{**}(x_n,z,z) \leq \beta_n$ for $n \in N$, then $D^*(y,y,z) \ll c$ and $y = z$,*

(b) *If $\nabla^{**}(y_n,y_m,x_n) \leq \delta_n$ and $\nabla^{**}(x_n,z,z) \leq \beta_n$ for any $m > n \in N$, then $D^*(y_n,y_m,z)$ Converges to 0 and hence $\{y_n\}$ converges to $z$,*

(c) *If $\nabla^{**}(x_n,x_m,x_l) \leq \delta_n$ for any $n,m,l \in N$ with $n \leq m \leq l$, then $\{x_n\}$ is a $D^*$-Cauchy Sequence in $X$,*

(d) *If $\nabla^{**}(x_n,x_m,a) \leq \delta_n$ for any $n \in N$, then $\{x_n\}$ is a $D^*$-Cauchy sequence in $X$.*

*Proof.* First, we prove a part (b). Let $c \in E$ with $0 \ll c$ be given. Then there exists $\lambda > 0$ such that $c - x \in intP$ for all $x \in P$ with $\| x \| < \lambda$. From the definition of $\nabla^{**}$-distance, there exists $e \in E$ with $0 \ll e$ such that $\nabla^{**}(u,v,a) \ll e$ and $\nabla^{**}(a,z,z) \ll e$ imply $D^*(u,v,z) \ll c$. Since a sequences are $\{\delta_n\}$ and $\{\beta_n\}$ converges to zero, there exists a positive integer $n_0$ such that $\| \delta_n \| < \lambda$ and $\| \beta_n \| < \lambda$ for all $n \geq n_0$ and so $c - \delta_n \in intP$ and $c - \beta_n \in intP$ (i.e.) $\delta_n \ll c$ and $\beta_n \ll c$ for all $n \geq n_0$. Therefore by $(\Delta_4)$ with e = c, for all $m > n \geq n_0$, we have, $\nabla^{**}(y_n,y_m,x_n) \ll \delta_n \ll c, \nabla^{**}(x_n,z,z) \ll \beta_n \ll c$, and hence $D^*(y_n,y_m,z) \ll c$ thus we obtain $\{y_n\}$ converges to $z$. It follows from (b) that (a) holds.

Now we will prove part (c). Let $c \in E$ with $0 \ll c$ be given. As in the proof of part (b), choose $e \in E$ with $0 \ll e$. Then there exists a positive integer $n_0$. Such that $\nabla^{**}(x_n,x_m,x_{n+1}) \ll \delta_n \ll e, \nabla^{**}(x_{n+1},x_l,x_l) \ll \delta_{n+1} \ll e$ for any $l \geq m > n \geq n_0$, therefore $D^*(x_n,x_m,x_l) \ll c$. This implies that $\{x_n\}$ is a $D^*$-Cauchy sequence in $X$. Since a part (d) is a special case of a part (c). So as in the proof of (c), we can prove (d).This completes the proof. □

**Definition 3.9.** Let $(X, D^*) \to (X^*, D^{**})$ be generalized $D^*$-metric spaces, then a function $f\colon X \to X^*$ is said to be $D^*$-continuous at a point $x \in X$ (see [2]), if and only if it is $D^*$-sequentially continuous at $x$, that is, whenever $\{x_n\}$ is $D^*$-convergent to $x$ we have $\{f(x_n)\}$ is $D^*$-convergent to $f(x)$.

*Remark* 3.10. $X$ is said to be $\nabla^{**}$-bounded if there is a constant $M > 0$ such that, $\nabla^{**}(x,y,z) \leq M$ for all $x, y, z \in X$.

## 4. Fixed point theorems and $\nabla^{**}$-distance in a complete partially ordered generalized $D^*$-metric spaces

In this section, we prove some new fixed point theorems by using the concept of $\nabla^{**}$-distance in a complete partially ordered generalized $D^*$-metric space.

**Definition 4.1.** ([19]) Suppose $(X, \leq)$ is a partially ordered set and $T : X \to X$ is a mapping of $X$ into itself. We say that $T$ is non-decreasing if for $x, y \in X$, $x \leq y$ implies $T(x) \leq T(y)$.

**Theorem 4.2.** *Let $(X, \leq)$ be a partially ordered set and suppose that $(X, D^*)$ is a complete generalized $D^*$-metric space and $P$ is a normal cone with normal constant $K$. Let $\nabla^{**}$ is a $\nabla^{**}$-distance on $X$ and $T : X \to X$ be a non-decreasing mapping with respect to $\leq$. Let $X$ be $\nabla^{**}$-bounded.*
*Suppose that for all $x \leq T_x$ and $w \in X$ there exists $h \in [0, 1)$ such that, $\nabla^{**}(T_x, T_x^2, T_w) \leq h\nabla^{**}(x, T_x, w)$. Also, $\inf\{\| \nabla^{**}(x, y, x) \| + \| \nabla^{**}(x, y, T_x) \| + \| \nabla^{**}(x, T_x^2, y) \|: x \leq T_x\} > 0$ for every $x, y \in X$ with $y \neq T_y$. If there is an $x_0 \in X$ with $x_0 \leq T_{x_0}$, then $T$ has a fixed point. Moreover, if $v = T_v$, then, $\nabla^{**}(v, v, v) = 0$.*

*Proof.* We will discuss two cases (a) $T_{x_0} = x_0$, $(b)T_{x_0} \neq x_0$.
(a) If $T_{x_0} = x_0$, then $x_0$ is a fixed point of $T$ and the proof in this case finished.
(b) Suppose that $T_{x_0} \neq x_0$. Since $x_0 \leq T_{x_0}$ and $T$ is non-decreasing mapping, we obtain:

$$x_0 \leq T_{x_0} \leq T_{x_0}^2 \leq ..... \leq T_{x_0}^n \leq T_{x_0}^{n+1} \leq .....$$

For all $n \in N$ and $t \geq 0$,

$$\nabla^{**}(T_{x_0}^n, T_{x_0}^{n+1}, T_{x_0}^{n+t}) \leq h\nabla^{**}(T_{x_0}^{n-1}, T_{x_0}^n, T_{x_0}^{n+t-1}) \leq ..... \leq h^n\nabla^{**}(x_0, T_{x_0}, T_{x_0}^t).$$

Thus, for any $l > m > n$ in which $m = n + k$ and $l = m + t(t, k \in N)$, we have:

$$\begin{aligned}
&\nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^l)\\
&\leq \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^{n+1}) + \nabla^{**}(T_{x_0}^{n+1}, T_{x_0}^l, T_{x_0}^l)\\
&\leq \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^{n+1}) + \nabla^{**}(T_{x_0}^{n+1}, T_{x_0}^{m+1}, T_{x_0}^{n+2}) + \nabla^{**}(T_{x_0}^{n+2}, T_{x_0}^l, T_{x_0}^l)\\
&\leq \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^{n+1}) + \nabla^{**}(T_{x_0}^{n+1}, T_{x_0}^{m+1}, T_{x_0}^{n+2})\\
&+\nabla^{**}(T_{x_0}^{m-2}, T_{x_0}^m, T_{x_0}^{m-1}) + \nabla^{**}(T_{x_0}^{m-1}, T_{x_0}^l, T_{x_0}^l)\\
&\leq \textstyle\sum_{j=n}^{m-1} Mh^j \leq (\frac{h^n}{1-h})M.
\end{aligned}$$

By part (c) of Lemma 3.8, $\{T_{x_0}^n\}$ is a $D^*$-Cauchy sequence in $X$. Since $(X, D^*)$ is a complete generalized $D^*$-metric space, then, there exists a point $z \in X$ such that $\{T_{x_0}^n\}$ converges to $z$, i.e., $(T_{x_0}^n \to z)$ as $n \to \infty$. Let $n \in N$ be a fixed point. Then, by condition $(\Delta_3)$, for $m > n$ we have,

$$\nabla^{**}(T_{x_0}^n, T_{x_0}^m, z) \leq \lim_{p\to\infty} \inf \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^p) \leq (\frac{h^n}{1-h})M$$

and for $l \geq n$, we get:

$$\nabla^{**}(T_{x_0}^n, z, T_{x_0}^l) \leq \lim_{p\to\infty} \inf \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^p) \leq (\frac{h^n}{1-h})M.$$

Since $P$ is a normal cone with normal constant $K$, we have:

$$\| \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^l) \| \leq (\frac{Kh^n}{1-h}) \| M \|$$

and for $m > n$ we obtain:

$\| \nabla^{**}(T_{x_0}^n, T_{x_0}^m, z) \| \leq \lim_{p\to\infty} \inf\{\| \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^p) \|\} \leq (\frac{Kh^n}{1-h}) \| M \|$ and for $l \geq n$, we get:

$\| \nabla(T_{x_0}^n, z, T_{x_0}^l) \| \leq \lim_{p\to\infty} \inf\{\| \nabla^{**}(T_{x_0}^n, T_{x_0}^m, T_{x_0}^p) \|\} \leq (\frac{Kh^n}{1-h}) \| M \|.$

Suppose that $T_z \neq z$. Since $T_{x_0}^n \leq T_{x_0}^{n+1}$, then by hypothesis, we have:

$0 \leq \inf\{\| \nabla^{**}(T^n_{x_0}, z, T^n_{x_0}) \| + \| \nabla^{**}(T^n_{x_0}, z, T^{n+1}_{x_0}) \| + \| \nabla^{**}(T^n_{x_0}, T^{n+2}_{x_0}, z)\| : n \in N\}$

$\leq \inf\{(\frac{Kh^n}{1-h}) \| M \| +(\frac{Kh^n}{1-h}) \| M \| +(\frac{Kh^n}{1-h}) \| M \|: n \in N\}$

$= \inf\{3(\frac{Kh^n}{1-h}) \| M \|: n \in N\} = 0.$

This is a contradiction. Thus, we obtain $Tz = z$. Now suppose that, $T_v = v$ holds. Then we have,

$$\nabla^{**}(v, v, v) = \nabla^{**}(T_v, T^2_v, T^3_v) \leq h\nabla^{**}(v, T_v, T^2_v) = h\nabla^{**}(v, v, v),$$

therefore we get $\nabla^{**}(v, v, v) = 0$. This completes the proof. □

**Theorem 4.3.** *Let $(X, \leq)$ be a partially ordered set and suppose that $(X, D^*)$ is a complete generalized $D^*$-metric space and $P$ is a normal cone with normal constant $K$. Let $\nabla^{**}$ is a $\nabla^{**}$-distance on $X$ and $T : X \to X$ be a non-decreasing mapping with respect to $\leq$. Let $X$ be $\nabla^{**}$-bounded. Assume that there exists an $h \in [0, 1)$ such that $\nabla^{**}(T_x, T^2_x, T_w) \leq h\nabla^{**}(x, T_x, w)$ for all $x \leq T_x$ and $w \in X$. Suppose that either of the following conditions hold:*

(a) *If $T_y \neq y$, then,*
*$\inf\{\| \nabla^{**}(x, y, x) \| + \| \nabla^{**}(x, y, T_x) \| + \| \nabla^{**}(x, T^2_x, y) \|: x \leq Tx\} > 0$, for every $x \in X$,*

(b) *If $\{x_n\}$ and $\{T_{x_n}\}$ convergent sequences to $y$ and $\nabla^{**}(v, w, .) = \nabla^{**}(w, v, .)$ for every $v, w \in X$, then $T_y = y$,*

(c) *$T$ is continuous mapping and $\nabla^{**}(v, w, .) = \nabla^{**}(w, v, .)$ for every $v, w \in X$. If there exists $x_0 \in X$ with $x_0 \leq T_{x_0}$, then $T$ has a fixed point. Furthermore, if $T_v = v$, then, $\nabla^{**}(v, v, v) = 0$.*

*Proof.* (a): It was proved in Theorem (4.2).
$(a) \Rightarrow (b)$. Suppose that there exists $y \in X$ with $T_y \neq y$ and,

$\inf\{\| \nabla^{**}(x, y, x) \| + \| \nabla^{**}(x, y, T_x) \| + \| \nabla^{**}(x, T^2_x, y) \|\} = 0$. Then there exists a sequence $\{x_n\}$ in $X$ such that $x_n \leq T_{x_n}$ and,

$\lim_{n\to\infty}\{\| \nabla^{**}(x_n, y, x_n) \| + \| \nabla^{**}(x_n, y, T_{x_n}) \| + \| \nabla^{**}(x_n, T^2_{x_n}, y) \|\} = 0$. So, we get:

$\lim_{n\to\infty} \nabla^{**}(y, y, x_n) = \lim_{n\to\infty} \nabla^{**}(x_n, x_n, y) = \lim_{n\to\infty} \nabla^{**}(x_n, y, x_n) = 0, \lim_{n\to\infty} \nabla^{**}(x_n, y, T_{x_n}) = 0$, and $\lim_{n\to\infty} \nabla^{**}(x_n, T^2_{x_n}, y) = 0$, Thus, by part (a) of Lemma 3.8 we obtain,

$\lim_{n\to\infty} D^*(y, y, T_{x_n}) = 0$ and $\lim_{n\to\infty} D^*(y, y, T^2_{x_n}) = 0$.

By using the continuity of generalized $D^*$-metric, we get: $\lim_{n\to\infty} T_{x_n} = \lim_{n\to\infty} T^2_{x_n} = y$. Also we have:

$\lim_{n\to\infty} \nabla^{**}(T_y, T_y, T_{x_n}) \leq h \lim_{n\to\infty} \nabla^{**}(y, y, x_n) = 0,$
$\lim_{n\to\infty} \nabla^{**}(T_{x_n}, y, T_y) \leq \lim_{n\to\infty} \inf\{\| \nabla^{**}(T_{x_n}, T^2_{x_n}, Ty) \|\}$
$\leq h \lim_{n\to\infty} \inf\{\| \nabla^{**}(x_n, T_{x_n}, y) \|\}$
$\leq h \lim_{n\to\infty} \inf\{\| \nabla^{**}(x_n, T^2_{x_n}, y) \|\} = 0.$

Therefore, by part (a) of Lemma 3.8, we obtain, $D^*(T_y, y, T_y) = 0$ and hence $T_y = y$, and $(a) \Rightarrow (b)$.
Now, we show that $(c) \Rightarrow (b)$. Let $T$ be continuous mapping. Further suppose that $\{x_n\}$ and $\{T_{x_n}\}$ converge to $y$. Then we have $T_y = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T_{x_n} = y$. □

Next, we recall the following example to shows the validity of Theorem 4.2.

**Example 4.4.** Consider Example 3.3 define a mapping $T : R \to R$ as follows:
$T_x = \frac{x}{2}$ for all $x \in R$. Then $T$ is continuous and non-decreasing with respect to $\leq$. Then we have:

$\nabla^{**}(T_x, T_x^2, T_w) = (\mid T_x^2 - T_x \mid + \mid T_x - T_w \mid)$
$= (\mid \frac{x}{4} - \frac{x}{2} \mid + \mid \frac{x}{2} - \frac{w}{2} \mid)$
$= \frac{1}{2}(\mid \frac{x}{2} - x \mid) + \mid x - w \mid)$
$= \frac{1}{2}\nabla^{**}(x, T_x, w)$
for all $x, w \in R$, since $y \neq T_y$ implies $y \neq 0$, we obtain:

$$\inf\{\| \nabla^{**}(x, y, x) \| + \| \nabla^{**}(x, y, T_x) \| + \| \nabla^{**}(x, T_x^2, y) \| : x \leq T_x\} > 0.$$

Hence, all the conditions of Theorem 4.2 are satisfied and $x = 0$ is a fixed point of $T$. Furthermore, $\nabla^{**}(0, 0, 0) = 0$.

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## References

[1] R. P. Agarwal, M. A. El-Gebeily, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., **87** (2008), 1–8.
[2] C. T. Aage, J. N. Salunke, *Some fixed points theorems in generalized $D^*$-metric spaces*, Appl. Sci., **12** (2010), 1–13.
[3] A. M. AL. Jumaili, X. S. Yang, *Fixed point theorems and $\nabla^*$-distance in partially ordered $D^*$-metric spaces*, Int. J. Math. Anal., **6** (2012), 2949–2955.
[4] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273.
[5] L. B. Ćirić, *Coincidence and fixed points for maps on topological spaces*, Topology Appl., **154** (2007), 3100–3106.
[6] L. B. Ćirić, S. N. Jesić, M. M. Milovanović, J. S. Ume, *On the steepest descent approximation method for the zeros of generalized accretive operators*, Nonlinear Anal.-TMA., **69** (2008), 763–769.
[7] B. C. Dhage, *Generalized metric spaces and mappings with fixed point*, Bull. Calcutta Math, Soc., **84** (1992), 329–336.
[8] J. X. Fang, Y. Gao, *Common fixed point theorems under strict contractive conditions in Menger spaces*, Nonlinear Anal.-TMA. **70** (2009), 184–193.
[9] T. Gnana Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal.-TMA., **65** (2006), 1379–1393.
[10] T. Gnana Bhaskar, V. Lakshmikantham, J. Vasundhara Devi, *Monotone iterative technique for functional differential equations with retardation and anticipation*, Nonlinear Anal.-TMA., **66** (2007), 2237–2242.
[11] N. Hussain, *Common fixed points in best approximation for Banach operator pairs with Ćirić type I-contractions*, J. Math. Anal. Appl., **338** (2008), 1351–1363.
[12] J. J. Nieto, R. R. Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order. **22** (2005), 223–239.
[13] V. L. Nguyen, X. T. Nguyen, *Common fixed point theorem in compact $D^*$-metric spaces*, Int. Math. Forum, **6** (2011), 605–612.
[14] J. J. Nieto, R. R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math. Sin. Eng. Ser., **23** (2007), 2205–2212.
[15] D. O'Regan, R. Saadati, *Nonlinear contraction theorems in probabilistic spaces*, Appl. Math. Comput., **195** (2008), 86–93.
[16] A. Petruşel, I. A. Rus, *Fixed point theorems in ordered L-spaces*, Proc. Amer. Math. Soc., **134** (2006), 411–418.
[17] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132** (2004), 1435–1443.
[18] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem in $D^*$-metric spaces*, Fixed Point Theory and Applications. Article ID 27906, (2007), 13 pages.

[19] R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Mathematical and Computer Modelling, **52** (2010), 797–801.

[20] T. Veerapandi, A. M. Pillai, *Some common fixed point theorems in $D^*$- metric spaces*, African J. Math. Computer Sci. Research, **4** (2011), 357–367.

[21] T. Veerapandi, A. M. Pillai, *A common fixed point theorems in $D^*$- metric spaces*, African J. Math. Computer Sci. Research, **4** (8) (2011), 273–280.