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Research Article

Qualitative behavior of vector-borne disease model

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Abstract

We investigate some qualitative behavior of a vector-borne disease model. Specially, we study local as well as global asymptotic stability of both disease-free and endemic equilibria of the model under certain parametric conditions. Furthermore, global behavior of disease-free equilibrium is investigated by constructing Lyapunov function, while global behavior of endemic equilibrium is discussed through geometric approach. Numerical simulations are provided to illustrate the theoretical discussion.

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1. Introduction

Vector-borne diseases are found among both human beings and animals from their origin. In history, great plagues were caused by these infectious diseases. For example, during 14th century "Black Death" occurred in Europe, and yellow fever destroyed the harmony of the most part of our world. In the beginning of 20th century, vector-borne diseases were terrible, both for humans and animals, because there were no precautionary measures and proper treatment for these epidemic diseases. Great efforts were made after the first half of 20th Century to control these epidemics through proper awareness among the people and by applications of insecticides [7].

Usually, vectors are the main source of transmission of vector-borne disease among their hosts, but there are some cases which show that direct transmission of a disease is also possible [9].

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Many infectious diseases can be modeled through systems of nonlinear differential equations. Many authors modeled these infectious diseases by introducing different incidence rates. Arguing as in [2, 3], it is suitable to use standard incidence rates as compared to simple mass action incidence. It is more suitable to construct a mathematical model with time-varying total population as compared to constant population, because for most of the endemic diseases, such as malaria, or those diseases that have high mortality rates (HIV/AIDS in poor countries), it is not appropriate to neglect the change in population size. Otherwise, we can not obtain the desired results with high accuracy.

In [6], authors proved that the period of immunity for malaria depends on repeated exposure. Furthermore, Niger and Gumel [13] investigated the qualitative behavior of malaria model by considering the role of partial immunity. A new mathematical model for malaria was proposed by Wan and Cui [17] by taking into account the partially immune population. Nonlinear incidence with partial immunity was used by Ozair et al. [14] in order to discuss dynamics of a vector-borne model. Recently, Ozair et al. [15] discussed a vector-host disease model with standard incidence and variable human population. Motivated by the above study, we want to modify the model presented in [11] and discuss its global behavior by constructing Laypunov function and using compound matrices. The rest of the paper is organized as follows.

The second section is dedicated to mathematical description of the model. In the third section, we discuss the existence and uniqueness of "endemic" equilibria. In the fourth section, we use Lyapunov function theory to show global stability of "disease-free" equilibrium (DFE) and a geometric approach to prove global stability of "endemic" equilibrium. Finally, discussions and simulations are presented in the last section.

2. Model description and dimensionless formulation

The total host population $N_h(t)$, described by SEIS model, is partitioned into three distinct compartments, susceptibles $S_h(t)$, exposed or infected $E_h(t)$ and infectious $I_h(t)$. The vector population $N_v(t)$ is described by SEI model and it is also divided into three subclasses, namely susceptible $S_v(t)$, exposed $E_v(t)$ and infectious $I_v(t)$ classes. The proposed dynamical system is given by

$$\begin{aligned}
\frac{dS_h(t)}{dt} &= b_1 N_h - \beta_1 \frac{S_h I_h}{N_h} - \beta_2 \frac{S_h I_v}{N_v} - \mu_h S_h + \alpha_h I_h,\\
\frac{dE_h(t)}{dt} &= \beta_1 \frac{S_h I_h}{N_h} + \beta_2 \frac{S_h I_v}{N_v} - \gamma_h E_h - \mu_h E_h,\\
\frac{dI_h(t)}{dt} &= \gamma_h E_h - \alpha_h I_h - \mu_h I_h - \xi_h I_h,\\
\frac{dS_v(t)}{dt} &= dN_v - \beta_3 \frac{S_v I_h}{N_h} - dS_v,\\
\frac{dE_v(t)}{dt} &= \beta_3 \frac{S_v I_h}{N_h} - \gamma_v E_v - dE_v,\\
\frac{dI_v(t)}{dt} &= \gamma_v E_v - dI_v.
\end{aligned} \tag{2.1}$$

In the above, b_1 is the per-capita birth rate of humans that are assumed to be susceptible, μ_h is natural mortality rate of humans and ξ_h is the disease induced death rate. Susceptible humans can be infected through contact with an infected individual and the effective infection rate is represented by β_1. The infectious individuals do not acquire permanent immunity and become susceptible again at the rate α_h. If the vector is infectious, then the average number of contacts per day that results in infection is β_2. Similarly the effective contact rate between susceptible vectors and infectious humans is β_3. Newly infected individuals develop clinical symptoms of the disease and move to the infectious class at the rate γ_h and exposed vectors progress to the infectious class at the rate γ_v. We assume that the birth and death rates of the vector population is equal to d so that it has constant size.

Taking

$$s_h = \frac{S_h}{N_h}, e_h = \frac{E_h}{N_h}, i_h = \frac{I_h}{N_h}, s_v = \frac{S_v}{N_v}, e_v = \frac{E_v}{N_v}, i_v = \frac{I_v}{N_v}, \tag{2.2}$$

we arrive at the following normalized model

$$\begin{aligned}
\frac{ds_h(t)}{dt} &= b_1(1-s_h) - \beta_1 s_h i_h - \beta_2 s_h i_v + \alpha_h i_h + \xi_h s_h i_h,\\
\frac{de_h(t)}{dt} &= \beta_1 s_h i_h + \beta_2 s_h i_v - \gamma_h e_h - b_1 e_h + \xi_h e_h i_h,\\
\frac{di_h(t)}{dt} &= \gamma_h e_h - \alpha_h i_h - \xi_h i_h - b_1 i_h + \xi_h {i_h}^2,\\
\frac{ds_v(t)}{dt} &= d(1-s_v) - \beta_3 s_v i_h,\\
\frac{de_v(t)}{dt} &= \beta_3 s_v i_h - \gamma_v e_v - d e_v,\\
\frac{di_v(t)}{dt} &= \gamma_v e_v - d i_v.
\end{aligned} \tag{2.3}$$

Since

$$s_h + e_h + i_h = 1, s_v + e_v + i_v = 1, \tag{2.4}$$

we can study the following subsystem

$$\begin{aligned}
\frac{de_h(t)}{dt} &= \beta_1(1-e_h-i_h)i_h + \beta_2(1-e_h-i_h)i_v - \gamma_h e_h - b_1 e_h + \xi_h e_h i_h,\\
\frac{di_h(t)}{dt} &= \gamma_h e_h - \alpha_h i_h - \xi_h i_h - b_1 i_h + \xi_h {i_h}^2,\\
\frac{de_v(t)}{dt} &= \beta_3(1-e_v-i_v)i_h - \gamma_v e_v - d e_v,\\
\frac{di_v(t)}{dt} &= \gamma_v e_v - d i_v.
\end{aligned} \tag{2.5}$$

This system is defined in the subset $\Gamma \times [0,\infty)$ of ${R^5}_+$, where $\Gamma = \{e_h, i_h, e_v, i_v : 0 \le e_h, i_h, e_v, i_v \le 1, 0 \le e_h + i_h \le 1, 0 \le e_v + i_v \le 1\}$ and the original quantities can be determined from the proportions through (2.2) and (2.4). The Jacobian matrix at DFE E_1 given by (e_h, i_h, e_v, i_v)=$(0,0,0,0)$ is

$$J = \begin{pmatrix}
-(b_1+\gamma_h) & \beta_1 & 0 & \beta_2\\
\gamma_h & -(b_1+\alpha_h+\xi_h) & 0 & 0\\
0 & \beta_3 & -(\gamma_v+d) & 0\\
0 & 0 & \gamma_v & -d
\end{pmatrix}.$$

The characteristic equation for the above Jacobian matrix is given by

$$f(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned}
a_1 &= (2d+\gamma_v) + (2b_1+\gamma_h+\alpha_h+\xi_h),\\
a_2 &= (b_1+\gamma_h)(b_1+\alpha_h+\xi_h) - \beta_1\gamma_h + (2d+\gamma_v)(2b_1+\gamma_h+\alpha_h+\xi_h) + d(\gamma_v+d),\\
a_3 &= (2d+\gamma_v)((b_1+\gamma_h)(b_1+\alpha_h+\xi_h) - \beta_1\gamma_h) + d(\gamma_v+d)(2b_1+\gamma_h+\alpha_h+\xi_h),\\
a_4 &= d(\gamma_v+d)(b_1+\gamma_h)(b_1+\alpha_h+\xi_h)(1-R_0),
\end{aligned} \tag{2.6}$$

and

$$R_0 = \frac{\beta_1\gamma_h}{Q_1Q_3} + \frac{\beta_2\beta_3\gamma_h\gamma_v}{dQ_1Q_2Q_3},$$

where $Q_1 = b_1 + \gamma_h$, $Q_2 = \gamma_v + d$, $Q_3 = b_1 + \alpha_h + \xi_h$. The four eigenvalues of the above Jacobian matrix have negative real parts if they satisfy the Routh–Hurwitz criteria [1], i.e., $a_i > 0$ for $i = 1, 2, 3, 4$, with $a_1a_2a_3 > a_3^2 + a_1^2a_4$. For $R_0 < 1$, we have $(b_1+\gamma_h)(b_1+\alpha_h+\xi_h)-\beta_1\gamma_h > 0$ and so $a_i > 0$ for $i = 1, 2, 3, 4$. It can also be easily verified that $a_1a_2a_3 > a_3^2 + a_1^2a_4$. Thus, all the eigenvalues of the above characteristic equation have negative real parts if and only if $R_0 < 1$, which shows that the DFE E_1 is locally asymptotically stable.

Remark 2.1. If $R_0 > 1$, we have $f(0) < 0$ and $f(\lambda) \to +\infty$ as $\lambda \to +\infty$. Thus there exists at least one $\lambda^* > 0$ such that $f(\lambda^*) = 0$ which proves instability of DFE.

3. Endemic equilibrium

Let $E_2 = (e_h^*, i_h^*, e_v^*, i_v^*)$ represents any arbitrary endemic equilibrium of the model (2.3). Solving system (2.3) at steady state gives

$$\begin{aligned} e_h^* &= \frac{(Q_3-\xi_h i_h^*)i_h^*}{\gamma_h}, \\ e_v^* &= \frac{\beta_3 d i_v^*}{Q_2(\beta_3 i_h^*+d)}, \\ i_v^* &= \frac{\beta_3\gamma_v i_h^*}{Q_2(\beta_3 i_h^*+d)}, \end{aligned} \tag{3.1}$$

where i_h^* is a root of the following cubic equation

$$g(i_h^*) = m_3 {i_h^*}^3 + m_2 {i_h^*}^2 + m_1 i_h^* + m_0 = 0, \tag{3.2}$$

with

$$\begin{aligned} m_3 &= Q_2\beta_3\xi_h(\beta_1 - \xi_h), \\ m_2 &= \beta_1Q_2d\xi_h + \beta_2\beta_3\gamma_v\xi_h + b_1Q_2\xi_h\beta_3 - (\beta_1 - \xi_h)(\gamma_hQ_2\beta_3 + Q_2Q_3\beta_3), \\ m_1 &= (\beta_3Q_2\gamma_h - \gamma_hQ_2d - Q_2Q_3d)(\beta_1 - \xi_h) - b_1Q_2Q_3\beta_3 - \beta_2\beta_3\gamma_v\gamma_h - \beta_2\beta_3\gamma_vQ_3 + b_1Q_2\xi_hd, \\ m_0 &= dQ_1Q_2Q_3(R_0 - 1). \end{aligned} \tag{3.3}$$

Assuming $R_0 > 1$, we have:
(1) If $\beta_1 > \xi_h$, then $m_3 > 0$, so we have $g(-\infty) < 0$, $g(\infty) > 0$ and $g(0) = m_0 > 0$. Further, $g(1) < 0$ if $\frac{b_1}{2} > \beta_1 > \xi_h + \gamma_h$, so there exists a **unique** $i_h^* \in (0, 1)$ such that $g(i_h^*) = 0$ (see Fig. 1).
(2) If $\beta_1 = \xi_h$, then $m_3 = 0$ and $g(i_h^*) = m_2{i_h^*}^2 + m_1i_h^* + m_0$, where $m_2 = \beta_1Q_2d\xi_h + \beta_2\beta_3\gamma_v\xi_h + b_1Q_2\xi_h\beta_3 > 0$. Also, $g(-\infty) > 0$, $g(\infty) > 0$ and $g(0) = m_0 > 0$.

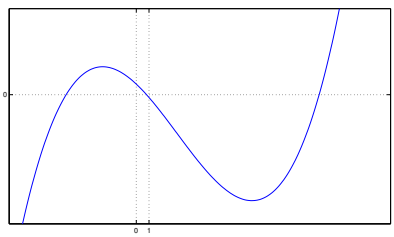


Figure 1: $(\frac{b_1}{2} > \beta_1 > \xi_h + \gamma_h)$

Moreover, $g(1) < 0$ if $\frac{b_1}{2} > \beta_1 = \xi_h$. Therefore, there exists a **unique** $i_h^* \in (0, 1)$ such that $g(i_h^*) = 0$ (see Fig. 2).

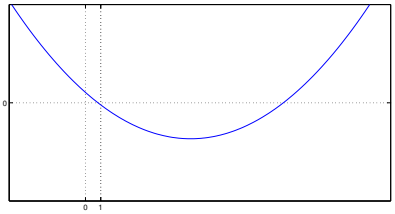


Figure 2: $(\frac{b_1}{2} > \beta_1 = \xi_h)$

(3) If $\beta_1 < \xi_h$, then $m_2 > 0$, $m_3 < 0$, so we have $g(-\infty) > 0$, $g(\infty) < 0$ and $g(0) = m_0 > 0$. Thus there exists at least one positive root or three positive roots, according to whether m_1 is positive or negative. We know that $g(i_h^*) = 0$ has three real roots if and only if $\frac{a^2}{4} + \frac{b^3}{27} \leq 0$, where

$$a = \frac{m_1}{m_3} - \frac{(m_2)^2}{3(m_3)^2}, \qquad b = \frac{m_0}{m_3} - \frac{m_1 m_2}{3(m_3)^2} + \frac{2(m_2)^3}{27(m_3)^3},$$

or

$$\hat{R_0} = \frac{18m_0m_1m_2m_3 - 4m_0(m_2)^3 - 4(m_1)^3m_3 + (m_1)^2(m_2)^2}{27(m_0)^2(m_3)^2} \geq 1.$$

If $\hat{R_0} < 1$, there is a unique i_h^* such that $g(i_h^*) = 0$ in the feasible interval.

If $\hat{R_0} > 1$, there are three different real roots for $g(i_h^*) = 0$ say $i_{h1}^*, i_{h2}^*, i_{h3}^* (i_{h1}^* < i_{h2}^* < i_{h3}^*)$.

Also, $g'(i_h^*) = 3m_3 {i_h^*}^2 + 2m_2 i_h^* + m_1$.

The three different real roots for $g(i_h^*) = 0$ are in the feasible interval if and only if the following inequalities are satisfied

$$\begin{aligned} 0 < \frac{-m_2}{3m_3} < 1, \\ g'(0) = m_1 < 0, \\ g'(1) = 3m_3 + 2m_2 + m_1 < 0. \end{aligned} \tag{3.4}$$

If $\hat{R_0} = 1$, then there are three real roots for $g(i_h^*) = 0$, among which at least two are identical. Similarly, if inequalities (3.4) are satisfied, then there are three real roots for $g(i_h^*) = 0$ in the feasible interval, say $i_{h1}^*, i_{h2}^*, i_{h3}^* (i_{h1}^* = i_{h2}^*)$.

For $R_0 = 1$, we have the following two cases.

(1) If $\beta_1 = \xi_h$, then $m_3 = 0$ and (3.2) reduces to $i_h^* \bar{g}(i_h^*) = 0$, where $\bar{g}(i_h^*) = (m_2 i_h^* + m_1)$. This implies that $i_h^* = 0$ or $i_h^* = \frac{-m_1}{m_2}$, which is positive but lies outside the interval $(0, 1)$ if $(\frac{b_1}{2} > \beta_1 = \xi_h)$ because $\bar{g}(1) = (m_2 + m_1)$.

(2) If $\beta_1 > \xi_h$, then $m_3 > 0$, so we have $i_h^*(m_3 {i_h^*}^2 + m_2 i_h^* + m_1) = 0$ which implies that $i_h^* = 0$ or i_h^* is the solution of the equation

$$\tilde{g}(i_h^*) = m_3 {i_h^*}^2 + m_2 i_h^* + m_1 = 0.$$

$\tilde{g}(-\infty) > 0$, $\tilde{g}(\infty) > 0$, $\tilde{g}(0) = m_1 < 0$ and $\tilde{g}(1) < 0$ if $\frac{b_1}{2} > \beta_1 > \xi_h + \gamma_h$. Therefore, there exists no i_h^* such that $\tilde{g}(i_h^*) = 0$ in the interval $(0, 1)$ if $\frac{b_1}{2} > \beta_1 > \xi_h + \gamma_h$. We summarize the discussion below.

Theorem 3.1. *Suppose that*

$$\frac{b_1}{2} > \beta_1 > \xi_h + \gamma_h$$

or

$$\frac{b_1}{2} > \beta_1 = \xi_h.$$

Then there is always a DFE for system (2.5)*; if* $R_0 > 1$*, then there is a unique "endemic" equilibrium* $E_2(s_h^*, i_h^*, i_v^*)$ *with coordinates satisfying* (3.1) *and* (3.2) *besides the DFE.*

Remark 3.2. : The global behavior of the equilibria is carried out under the assumptions given in Theorem 3.1.

4. Global dynamics

In this section, we discus the global stability of DFE and global stability of endemic equilibrium.

4.1. Global stability of the disease-free equilibrium

In this subsection, we analyze the global behavior of the equilibria system (2.3). The following theorem provides a global property of the disease-free equilibrium E_1 of the system.

Theorem 4.1. *If $R_c \leq 1$, then the infection-free equilibrium E_1 is globally asymptotically stable in the interior of Γ, where $R_c = \frac{\beta_1}{Q_3} + \frac{\beta_2\beta_3}{dQ_3}$.*

Proof. To establish the global stability of the disease-free equilibrium, we construct the following Lyapunov function:

$$L(t) = e_h(t) + i_h(t) + \frac{\beta_2}{d}e_v(t) + \frac{\beta_2}{d}i_v(t).$$

Calculating the time derivative of L along (2.5), we obtain

$$\begin{aligned}
L'(t) &= e_h'(t) + i_h'(t) + \frac{\beta_2}{d}e_v'(t) + \frac{\beta_2}{d}i_v'(t) \\
&= \beta_1(1-e_h-i_h)i_h + \beta_2(1-e_h-i_h)i_v - \gamma_h e_h - b_1 e_h + \xi_h e_h i_h + \gamma_h e_h - \alpha_h i_h - \xi_h i_h - b_1 i_h + \xi_h {i_h}^2 \\
&\quad + \frac{\beta_2}{d}[\beta_3(1-e_v-i_v)i_h - \gamma_v e_v - d e_v] + \frac{\beta_2}{d}[\gamma_v e_v - d i_v] \\
&= \beta_1 i_h - \beta_1 e_h i_h - \beta_1 {i_h}^2 + \beta_2 i_v - \beta_2 e_h i_v - \beta_2 i_h i_v - \gamma_h e_h - b_1 e_h + \xi_h e_h i_h + \gamma_h e_h - \alpha_h i_h - \xi_h i_h \\
&\quad - b_1 i_h + \xi_h {i_h}^2 + \frac{\beta_2}{d}[\beta_3 i_h - \beta_3 e_v i_h - \beta_3 i_v i_h - \gamma_v e_v - d e_v] + \frac{\beta_2}{d}[\gamma_v e_v - d i_v] \\
&= \beta_1 i_h - (\beta_1 - \xi_h) e_h i_h - (\beta_1 - \xi_h){i_h}^2 + \beta_2 i_v - \beta_2 e_h i_v - \beta_2 i_h i_v - b_1 e_h - Q_3 i_h \\
&\quad + \frac{\beta_2\beta_3}{d}i_h - \frac{\beta_2\beta_3}{d}e_v i_h - \frac{\beta_2\beta_3}{d}i_v i_h - \frac{\beta_2}{d}\gamma_v e_v - \beta_2 e_v + \frac{\beta_2}{d}\gamma_v e_v - \beta_2 i_v \\
&= (\beta_1 + \frac{\beta_2\beta_3}{d} - Q_3)i_h - (\beta_1 - \xi_h)e_h i_h - (\beta_1 - \xi_h){i_h}^2 - \beta_2 e_h i_v - \beta_2 i_h i_v - b_1 e_h - \frac{\beta_2\beta_3}{d}e_v i_h \\
&\quad - \frac{\beta_2\beta_3}{d}i_v i_h - \beta_2 e_v \\
&= Q_3(R_c - 1)i_h - (\beta_1 - \xi_h)e_h i_h - (\beta_1 - \xi_h){i_h}^2 - \beta_2 e_h i_v - \beta_2 i_h i_v - b_1 e_h - \frac{\beta_2\beta_3}{d}e_v i_h \\
&\quad - \frac{\beta_2\beta_3}{d}i_v i_h - \beta_2 e_v.
\end{aligned}$$

We can see that for $R_c < 1$, L' is negative. Again $L' = 0$ if and only if $e_h = 0$, $i_h = 0$ and $e_v = 0$. Therefore the largest compact invariant set in $\{(e_h, i_h, e_v, i_v) \in \Gamma, L' = 0\}$, when $R_c \leq 1$, is the singelton $\{E_1\}$. Hence, LaSalle's invariance principle [10] implies that "E_1" is globally asymptotically stable in Γ. This completes the proof. □

Remark 4.2. This above result is of outmost importance because it shows that if at any time, through appropriate interventions, we are able to lower R_0 and R_c to less than unity, then the disease will disappear. Obviously, $R_0 < R_c$.

4.2. Global stability of endemic equilibrium

Here we apply the result given on page 59 of [5] to establish the global asymptotic stability of the unique "endemic" equilibrium $E^*(s_h^*, i_h^*, i_v^*)$. The Lozinskiĭ measure for an $n \times n$ matrix A is defined as

$$\tilde{\mu}(A) = \inf\{\rho : D_+\|Z\| \leq \rho\|Z\| \text{ for all solutions of } Z' = AZ\},$$

where D_+ is the right-hand derivative [12]. The unique endemic equilibrium is globally asymptotically stable if there exists a norm on R^6 which is associated with the Lozinskiĭ measure and satisfies $\tilde{\mu}(A) < 0$ for all $x \in int(\Gamma)$ if $R_0 > 1$. The Jacobian matrix at endemic equilibrium point is given by

$$J = \begin{pmatrix} g_{11} & \beta_1(1-e_h-i_h) - \beta_1 i_h - \beta_2 i_v + \xi_h e_h & 0 & \beta_2(1-e_h-i_h) \\ \gamma_h & -(b_1+\alpha_h+\xi_h) + 2\xi_h i_h & 0 & 0 \\ 0 & \beta_3(1-e_v-i_v) & -\beta_3 i_h - (\gamma_v + d) & -\beta_3 i_h \\ 0 & 0 & \gamma_v & -d \end{pmatrix},$$

where $g_{11} = -\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h)$.

The second compound matrix [8] is

$$J^{[2]} = \begin{pmatrix} j_{11} & 0 & 0 & 0 & -\beta_2(1-e_h-i_h) & 0 \\ \beta_3(1-e_v-i_v) & j_{22} & -\beta_3 i_h & j_{24} & 0 & -\beta_2(1-e_h-i_h) \\ 0 & \gamma_v & j_{33} & 0 & j_{35} & 0 \\ 0 & \gamma_h & 0 & j_{44} & -\beta_3 i_h & 0 \\ 0 & 0 & \gamma_h & \gamma_v & j_{55} & 0 \\ 0 & 0 & 0 & 0 & \beta_3(1-e_v-i_v) & j_{66} \end{pmatrix},$$

where

$$\begin{aligned} j_{11} &= -\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - (b_1 + \alpha_h + \xi_h) + 2\xi_h i_h \\ j_{22} &= -\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - \beta_3 i_h - (\gamma_v + d) \\ j_{33} &= -\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - d \\ j_{44} &= -(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - \beta_3 i_h - (\gamma_v + d) \\ j_{55} &= -(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - d \\ j_{66} &= -\beta_3 i_h - (\gamma_v + d) - d \\ j_{24} &= \beta_1(1-e_h-i_h) - \beta_1 i_h - \beta_2 i_v + \xi_h e_h \\ j_{35} &= \beta_1(1-e_h-i_h) - \beta_1 i_h - \beta_2 i_v + \xi_h e_h. \end{aligned}$$

Let $P = diag(\frac{1}{i_h}, \frac{1}{i_v}, \frac{1}{i_v}, \frac{1}{i_v}, \frac{1}{i_v}, \frac{1}{i_v})$. Then we have

$$K = P_f P^{-1} + P J^{[2]} P^{-1}$$

where

$$K = \begin{pmatrix} j_{11} - \frac{i_h'}{i_h} & 0 & 0 & 0 & -\beta_2(1-e_h-i_h)\frac{i_v}{i_h} & 0 \\ \beta_3(1-e_v-i_v)\frac{i_h}{i_v} & j_{22} - \frac{i_v'}{i_v} & -\beta_3 i_h & j_{24} & 0 & -\beta_2(1-e_h-i_h) \\ 0 & \gamma_v & j_{33} - \frac{i_v'}{i_v} & 0 & j_{35} & 0 \\ 0 & \gamma_h & 0 & j_{44} - \frac{i_v'}{i_v} & -\beta_3 i_h & 0 \\ 0 & 0 & \gamma_h & \gamma_v & j_{55} - \frac{i_v'}{i_v} & 0 \\ 0 & 0 & 0 & 0 & \beta_3(1-e_v-i_v) & j_{66} - \frac{i_v'}{i_v} \end{pmatrix}.$$

Let $Z = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6)^T$ be the solution of the linear homogeneous system $\frac{dZ}{dt} = KZ$, where

$$\begin{aligned}
Z_1{}' &= (j_{11} - \frac{i_h'}{i_h})Z_1 + (-\beta_2(1-e_h-i_h)\frac{i_v}{i_h})Z_5, \\
Z_2{}' &= (\beta_3(1-e_v-i_v)\frac{i_h}{i_v})Z_1 + (j_{22} - \frac{i_v'}{i_v})Z_2 - \beta_3 i_h Z_3 + j_{24}Z_4 - \beta_2(1-e_h-i_h)Z_6, \\
Z_3{}' &= \gamma_v Z_2 + (j_{33} - \frac{i_v'}{i_v})Z_3 + j_{35}Z_5, \\
Z_4{}' &= \gamma_h Z_2 + (j_{44} - \frac{i_v'}{i_v})Z_4 - \beta_3 i_h Z_5, \\
Z_5{}' &= \gamma_h Z_3 + \gamma_v Z_4 + (j_{55} - \frac{i_v'}{i_v})Z_5, \\
Z_6{}' &= \beta_3(1-e_v-i_v)Z_5 + (j_{66} - \frac{i_v'}{i_v})Z_6.
\end{aligned}$$

It can be easily seen from (2.5) that

$$\begin{aligned}
\frac{e_h{}'}{e_h} &= \beta_1(1-e_h-i_h)\frac{i_h}{e_h} + \beta_2(1-e_h-i_h)\frac{i_v}{e_h} - (\gamma_h + b_1 - \xi_h i_h), \\
\frac{i_h{}'}{i_h} &= \gamma_h\frac{e_h}{i_h} - \alpha_h - \xi_h - b_1 + \xi_h i_h, \\
\frac{e_v{}'}{e_v} &= \beta_3(1-e_v-i_v)\frac{i_h}{e_v} - \gamma_v - d, \\
\frac{i_v{}'}{i_v} &= \gamma_v\frac{e_v}{i_v} - d.
\end{aligned} \tag{4.1}$$

Theorem 4.3. *Suppose that $R_0 > 1$. The unique endemic equilibrium E_2 is globally asymptotically stable in Γ^o if the following inequalities are satisfied:*

$$\begin{aligned}
&b_1 > \xi_h + \gamma_h, \\
&r\beta_3 < \gamma_v + d, \\
&b_1 + d > \beta_1 + \gamma_v.
\end{aligned} \tag{4.2}$$

Proof. We consider the following norms on Z [16]

$$\|Z\| = \begin{cases} \max\{|Z_1|, i_v(|Z_2|+|Z_3|), i_v(|Z_4|+|Z_5|), i_v|Z_6|\}, & \text{if } sgn(Z_1)=sgn(Z_2)=sgn(Z_3),\\ & sgn(Z_4)=sgn(Z_5)=sgn(Z_6)\\ \max\{i_h|Z_1|, |Z_2|+|Z_3|, |Z_4|+|Z_5|, |Z_6|\}, & \text{if } -sgn(Z_1)=sgn(Z_2)=sgn(Z_3),\\ & sgn(Z_4)=sgn(Z_5)=sgn(Z_6)\\ \max\{|Z_1|, i_v|Z_2|, |Z_3|, |Z_4|+|Z_5|, |Z_6|\}, & \text{if } sgn(Z_1)=-sgn(Z_2)=sgn(Z_3),\\ & sgn(Z_4)=sgn(Z_5)=sgn(Z_6)\\ \max\{|Z_1|, i_v|Z_2|, |Z_3|, |Z_4|+|Z_5|, |Z_6|\}, & \text{if } sgn(Z_1)=sgn(Z_2)=-sgn(Z_3),\\ & sgn(Z_4)=sgn(Z_5)=sgn(Z_6)\\ \max\{|Z_1|, i_v(|Z_2|+|Z_3|), |Z_4|, |Z_5|, |Z_6|\}, & \text{if } sgn(Z_1)=sgn(Z_2)=sgn(Z_3),\\ & -sgn(Z_4)=sgn(Z_5)=sgn(Z_6)\\ \max\{i_h|Z_1|, i_v|Z_2|, i_v|Z_3|, i_v|Z_4|, |Z_5|, |Z_6|\}, & \text{if } sgn(Z_1)=sgn(Z_2)=sgn(Z_3),\\ & sgn(Z_4)=-sgn(Z_5)=sgn(Z_6)\\ \max\{|Z_1|, i_v(|Z_2|+|Z_3|), i_v(|Z_4|+|Z_5|), |Z_6|\}, & \text{if } sgn(Z_1)=sgn(Z_2)=sgn(Z_3),\\ & sgn(Z_4)=sgn(Z_5)=-sgn(Z_6). \end{cases} \tag{4.3}$$

If we take $sgn(Z_1)=sgn(Z_2)=sgn(Z_3), sgn(Z_4)=sgn(Z_5)=sgn(Z_6)$, then $\|Z\|=max\{|Z_1|, i_v(|Z_2|+|Z_3|), i_v(|Z_4|+|Z_5|), i_v|Z_6|\}$. We can discuss here the following four cases.

Case 1: $|Z_1| > \{i_v(|Z_2|+|Z_3|), i_v(|Z_4|+|Z_5|), i_v|Z_6|\}$. We have $\|Z\|=|Z_1|=Z_1$ and

$$\begin{aligned} D_+\|Z\| &= Z_1'\\ &= (j_{11}-\frac{i_h'}{i_h})Z_1-\beta_2(1-e_h-i_h)\frac{i_v}{i_h}Z_5\\ &= (-\beta_1 i_h-\beta_2 i_v-(b_1+\gamma_h-\xi_h i_h)-(b_1+\alpha_h+\xi_h)+2\xi_h i_h-\gamma_h\frac{e_h}{i_h}+(b_1+\alpha_h+\xi_h)\\ &\quad -\xi_h i_h)Z_1-\beta_2(1-e_h-i_h)\frac{i_v}{i_h}Z_5\\ &\le (-(\beta_1-\xi_h)i_h-\beta_2 i_v-(b_1-\xi_h)-(\gamma_h i_h+\gamma_h\frac{e_h}{i_h}))|Z_1|-\beta_2(1-e_h-i_h)\frac{i_v}{i_h}|Z_5|\\ &< (-(\beta_1-\xi_h)i_h-\beta_2 i_v-(b_1-\xi_h)-(\gamma_h i_h+\gamma_h\frac{e_h}{i_h}))|Z_1|\\ &= -\rho_1\|Z\|, \end{aligned}$$

where

$$\rho_1=(\beta_1-\xi_h)i_h+\beta_2 i_v+(b_1-\xi_h)+(\gamma_h i_h+\gamma_h\frac{e_h}{i_h}).$$

Case 2: $i_v(|Z_2|+|Z_3|) > \{|Z_1|, i_v(|Z_4|+|Z_5|), i_v|Z_6|\}$. We have $\|Z\|=i_v(|Z_2|+|Z_3|)=i_v(Z_2+Z_3)$ and

$$\begin{aligned} D_+\|Z\| &= i_v(\frac{i_v'}{i_v}Z_2+\frac{i_v'}{i_v}Z_3+Z_2'+Z_3')\\ &= i_v[(\beta_3(1-e_v-i_v)\frac{i_h}{i_v})Z_1+j_{22}Z_2-\beta_3 i_h Z_3+j_{24}Z_4-\beta_2(1-e_h-i_h)Z_6+\gamma_v Z_2+j_{33}Z_3\\ &\quad +j_{35}Z_5]\\ &= \beta_3 i_h(1-e_v-i_v)|Z_1|+j_{22}i_v|Z_2|-\beta_3 i_h i_v|Z_3|+j_{24}i_v|Z_4|-\beta_2 i_v(1-e_h-i_h)|Z_6|+\gamma_v i_v|Z_2|\\ &\quad +j_{33}i_v|Z_3|+j_{35}i_v|Z_5|\\ &< \beta_3 i_h(1-e_v-i_v)|Z_1|+j_{22}i_v|Z_2|-\beta_3 i_h i_v|Z_3|+j_{24}i_v|Z_4|+\gamma_v i_v|Z_2|+j_{33}i_v|Z_3 i_v \end{aligned}$$

$$
\begin{aligned}
&+ j_{35} i_v |Z_5| \\
&= \beta_3 i_h |Z_1| - \beta_3 i_h (e_v + i_v)|Z_1| + (-\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - \beta_3 i_h - (\gamma_v + d)) i_v |Z_2| \\
&\quad - \beta_3 i_h i_v |Z_3| + (\beta_1 (1 - e_h - i_h) - \beta_1 i_h - \beta_2 i_v + \xi_h e_h) i_v |Z_4| + \gamma_v i_v |Z_2| \\
&\quad + (-\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - d) i_v |Z_3| + (\beta_1 (1 - e_h - i_h) - \beta_1 i_h - \beta_2 i_v + \xi_h e_h) i_v |Z_5| \\
&< (\beta_3 i_h - \beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - \beta_3 i_h - (\gamma_v + d) + \gamma_v) i_v |Z_2| \\
&\quad + (\beta_1 + (\beta_1 - \xi_h) e_h - \beta_1 i_h - \beta_1 i_h - \beta_2 i_v) i_v |Z_4| + (\beta_3 i_h - \beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) \\
&\quad - d - \beta_3 i_h) i_v |Z_3| + (\beta_1 + (\beta_1 - \xi_h) e_h - \beta_1 i_h - \beta_1 i_h - \beta_2 i_v) i_v |Z_5| \\
&= (-\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - d) i_v |Z_2| + (-\beta_1 i_h - \beta_2 i_v - (b_1 + \gamma_h - \xi_h i_h) - d) i_v |Z_3| + \\
&\beta_1 (i_v |Z_4| + i_v |Z_5|) - (\xi_h e_h + \beta_1 i_h + \beta_1 i_h + \beta_2 i_v)(i_v |Z_4| + i_v |Z_5|) \\
&< (-(\beta_1 - \xi_h) i_h - \beta_2 i_v - (b_1 - \beta_1) - \gamma_h - d) i_v |Z_2| + (-(\beta_1 - \xi_h) i_h - \beta_2 i_v - (b_1 - \beta_1) - \gamma_h - d) \\
&i_v |Z_3| \\
&= -\rho_2 (i_v |Z_2| + i_v |Z_3|) \\
&= -\rho_2 \|Z\|,
\end{aligned}
$$

where

$$\rho_2 = (\beta_1 - \xi_h) i_h + \beta_2 i_v + (b_1 - \beta_1) + \gamma_h + d.$$

Case 3: $i_v(|Z_4| + |Z_5|) > \{|Z_1|, i_v(|Z_2| + |Z_3|), i_v|Z_6|\}$. We have $\|Z\| = i_v(|Z_4| + |Z_5|) = i_v(Z_4 + Z_5)$ and

$$
\begin{aligned}
D_+ \|Z\| &= i_v \left(\frac{i_v'}{i_v} Z_4 + \frac{i_v'}{i_v} Z_5 + Z_4' + Z_5'\right) \\
&= i_v (\gamma_h Z_2 + j_{44} Z_4 - \beta_3 i_h Z_5 + \gamma_h Z_3 + \gamma_v Z_4 + j_{55} Z_5) \\
&= i_v (\gamma_h Z_2 + (-(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - \beta_3 i_h - (\gamma_v + d)) Z_4 - \beta_3 i_h Z_5 + \gamma_h Z_3 + \gamma_v Z_4 \\
&\quad + (-(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - d) Z_5) \\
&= \gamma_h i_v (|Z_2| + |Z_3|) + (-(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - \beta_3 i_h - (\gamma_v + d) + \gamma_v)|Z_4| \\
&\quad + (-(b_1 + \alpha_h + \xi_h) + 2\xi_h i_h - d - \beta_3 i_h)|Z_5| \\
&\leq -[(b_1 - \xi_h - \gamma_h) + \alpha_h + \beta_3 i_h + d] i_v (|Z_4| + |Z_5|) \\
&= -\rho_3 \|Z\|,
\end{aligned}
$$

where

$$\rho_3 = (b_1 - \xi_h - \gamma_h) + \alpha_h + \beta_3 i_h + d.$$

Case 4: $i_v|Z_6| > \{|Z_1|, i_v(|Z_2| + |Z_3|), i_v(|Z_4| + |Z_5|)\}$. We have $\|Z\| = i_v|Z_4| = i_v Z_6$ and

$$
\begin{aligned}
D_+ \|Z\| &= i_v \left(\frac{i_v'}{i_v} Z_6 + Z_6'\right) \\
&= i_v (\beta_3 (1 - e_v - i_v) Z_5 + j_{66} Z_6) \\
&\leq \beta_3 i_v |Z_5| - \beta_3 (e_v + i_v)|Z_5| + (-\beta_3 i_h - (\gamma_v + d) - d) i_v |Z_6| \\
&< (\beta_3 - \beta_3 i_h - (\gamma_v + d) - d) i_v |Z_6| \\
&= -\rho_4 \|Z\|,
\end{aligned}
$$

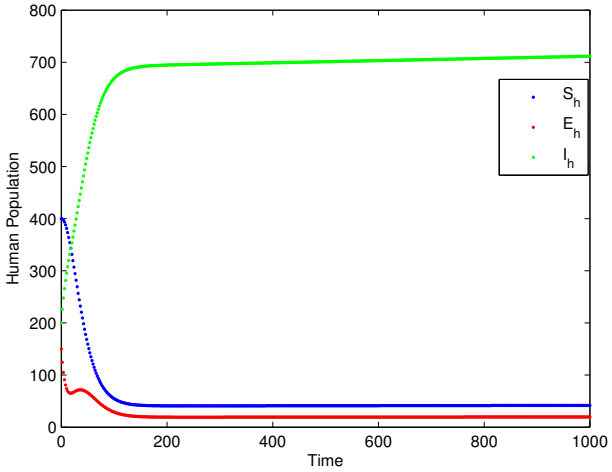
where

$$\rho_4 = \beta_3 i_h + (\gamma_v + d) - \beta_3 + d.$$

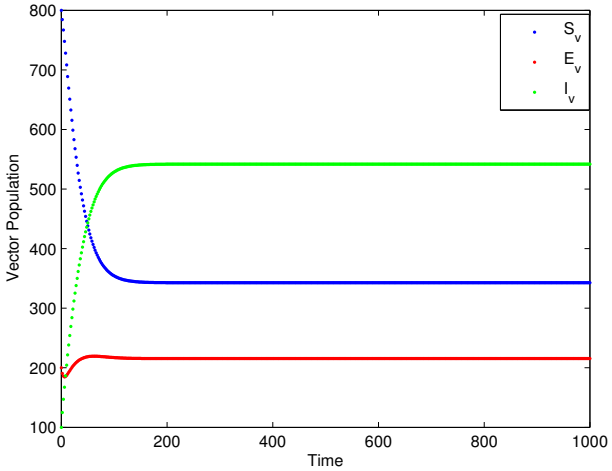
Applying the same technique for other cases, after some calculation, we get $\rho_5, \rho_6, ..., \rho_{31}, \tilde{\rho}_{32}, \tilde{\rho}_{33}$. Take $\rho = min\{\rho_1, \rho_2, \rho_3, ..., \rho_{31}, \tilde{\rho}_{32}, \tilde{\rho}_{33}\}$ and $\rho > 0$ under the conditions in (4.2) and we have the Lozinskiĭ measure $\tilde{\mu}(K) < 0$. By applying the result of [5, p.59], the unique "endemic" equilibrium is globally asymptotically stable, which completes the proof. □

5. Discussions and simulations

This paper deals with a vector-host disease model with standard incidence which allows a direct mode of transmission and varying human population as well as exposed class in humans and vectors. It concerns diseases with long duration and substantial mortality rate (e.g., malaria). Figure (3) shows graphs of a typical solution of the model (2.1) for malaria disease. We used the parametric values from [4] for low malaria transmission. We analyzed the global dynamics of the normalized model. Moreover, we constructed Lyapunov function to show the global stability of DFE. For proving the global stability of endemic equilibrium, compound matrices and the geometric approach is used. By defining some suitable norms, it is proved that the Lozinskiĭ measure of homogeneous system is negative under some conditions. Numerically, it is seen that if $b_1 < \xi_h + \gamma_h$, then exposed and infectious individuals and vectors will also approach to endemic level for different initial conditions (Fig. 4). Furthermore, it is also has shown that the infected classes will also approach the endemic level if $\beta_3 > \gamma_v + d$ (Fig. 5). The same phenomena were observed for the case $b_1 + d < \beta_1 + \gamma_v$ (Fig. 6). From these observations we conclude that the conditions given in (4.2) are not necessary for global asymptotic stability. One can take other forms of $\|Z\|$, which may lead to sufficient conditions different from (4.2).

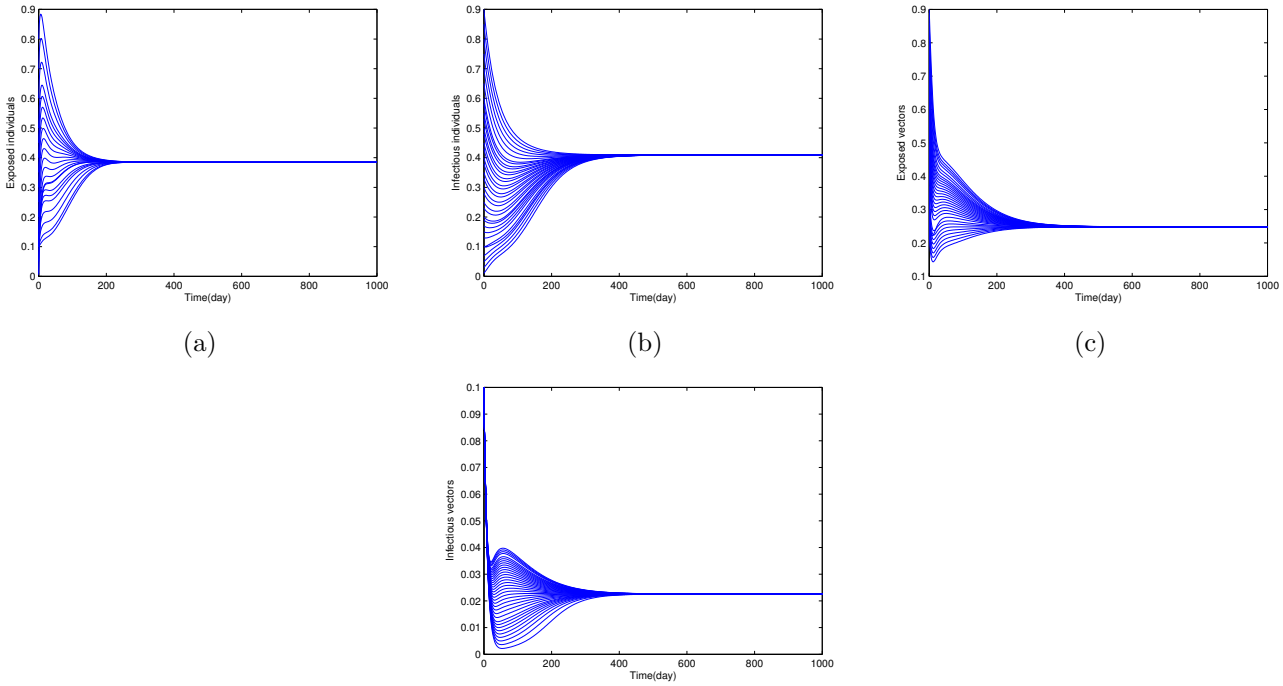


(a) Variation of Human Population for malaria disease



(b) Variation of Vector Population for malaria disease

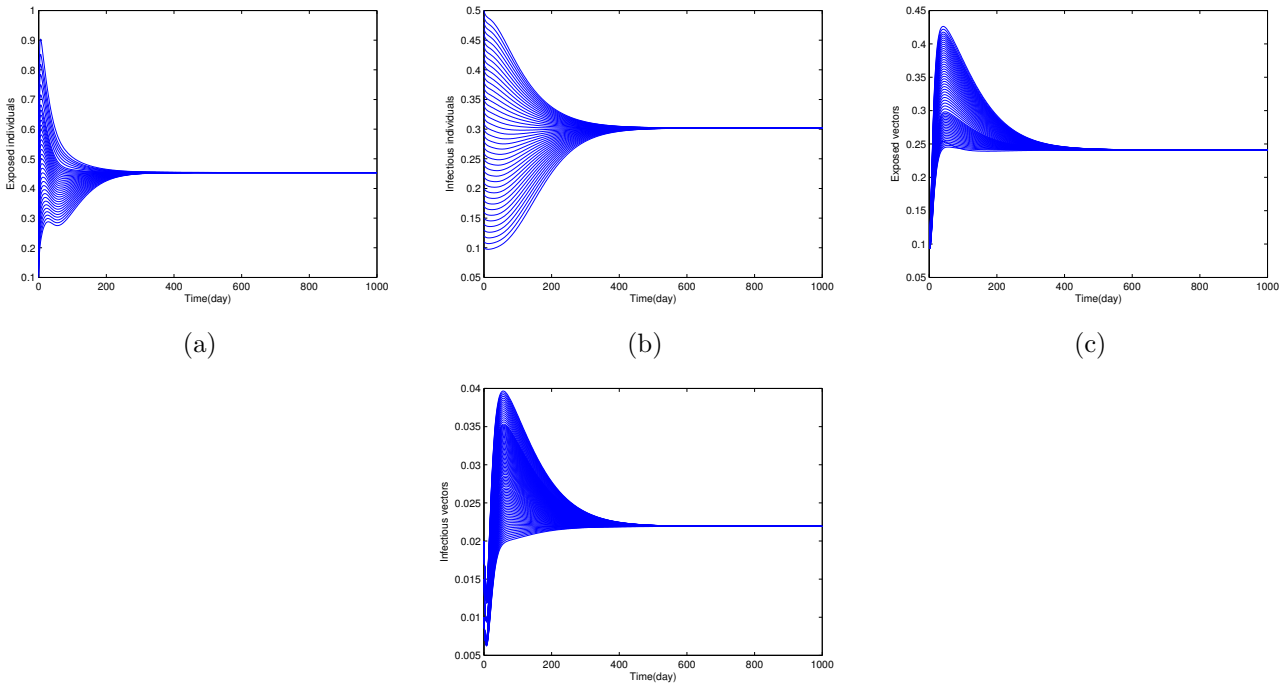
Figure 3



(a) (b) (c)

(d)

Figure 4: Exposed and infectious individuals and vectors approach unique endemic level when $b_1 < \xi_h + \gamma_h$.



(a) (b) (c)

(d)

Figure 5: Exposed and infectious individuals and vectors approach unique endemic level when $\beta_3 > \gamma_v + d$.

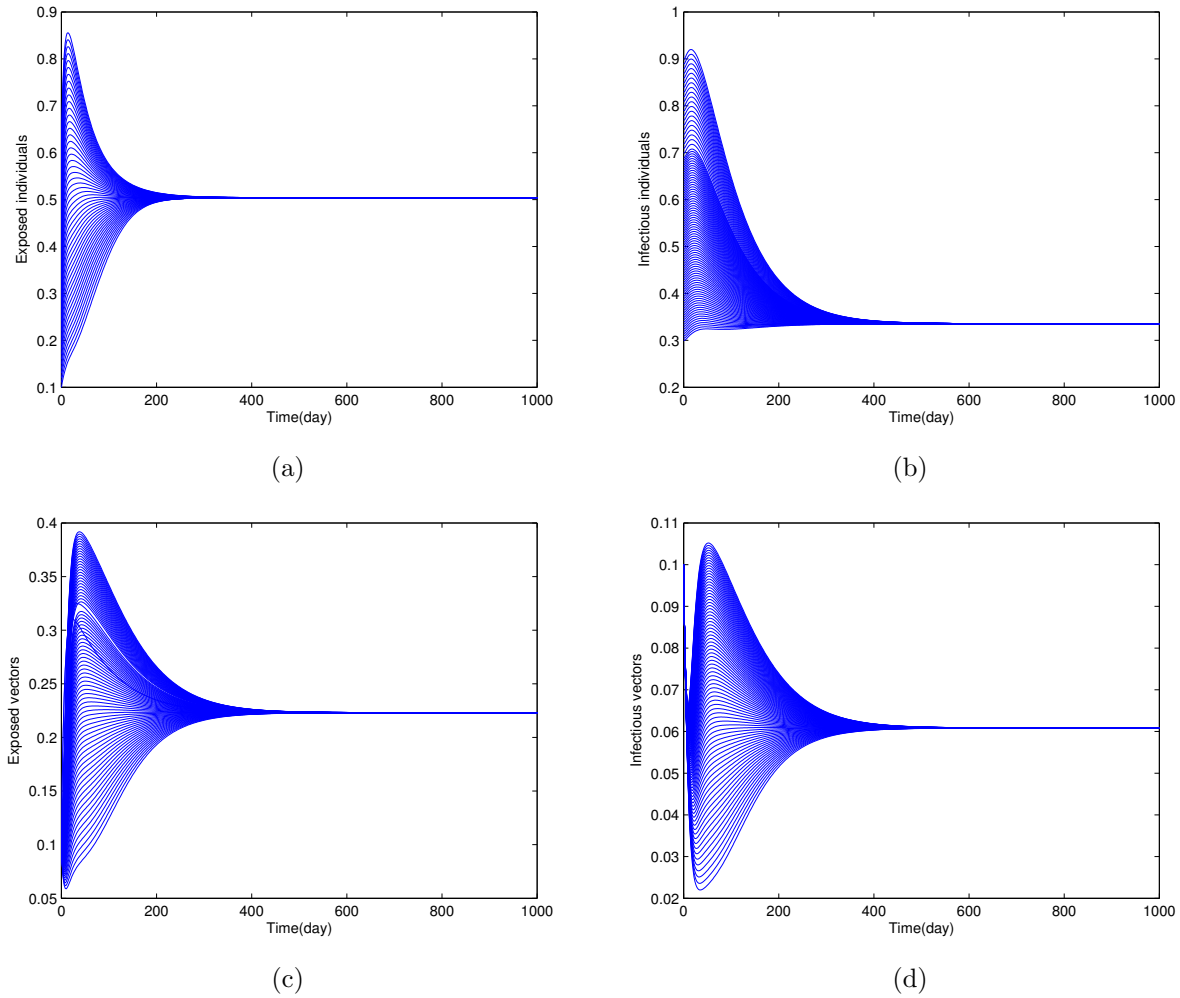


Figure 6: Exposed and infectious individuals and vectors approach unique endemic level when $\beta_3 > \gamma_v + d$.

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