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New fixed point results for contractive maps involving dominating auxiliary functions

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Abstract

In this paper, we establish certain new fixed point theorems for contractive inequalities using an auxiliary function which dominates the ordinary metric function. As application, we derive some recent known results as corollaries. Certain interesting consequences of our results are also presented. An example is given to illustrate the usability of the obtained results. ©2016 All rights reserved.

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1. Introduction and Preliminaries

In the last two decades, the theory of fixed point and related topics emerged as a rapidly growing area of research because of its applications in nonlinear analysis, optimization, economics, game theory, etc. The Banach contraction principle is an important result in fixed point theory due to its vast applications. Consequently, a number of extensions of this result appeared in the literature (see [1, 2, 4, 5, 7–9, 12, 14, 15] and references therein).

Recently, Wardowski [16] introduced a new concept of a contraction map. Given $k > 0$, denote by Δ_k the set of all functions $F : R_+^0 \to R$ satisfying the following conditions:

(W1) F is strictly increasing;
(W2) for any sequence (α_n) in R_+^0, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

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(W3) $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We denote $\Delta = \cup\{\Delta_k; k \in (0,1)\}$. Any F in the class Δ will be called a *Wardowski function*. Now, taking the metric space (X,d) and $F \in \Delta$ and $\tau > 0$, let us say that the self-mapping $T : X \to X$ is a *F-contraction*, provided

$$x, y \in X, d(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x,y)). \tag{Wa}$$

On the other hand, Samet et al. [13] introduced the concepts of α-ψ-contractive and α-admissible mappings and established fixed point theorems for such mappings defined on complete metric spaces. Afterwards, Salimi et al.[11] and Hussain et al. [9] modified the notions of α-ψ-contractive and α-admissible mappings and established certain fixed point theorems.

Let Ψ be the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

We present now the necessary definitions and results which will be useful in the sequel.

Definition 1.1 ([13]). Let T be a self-mapping on X and let $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Definition 1.2 ([10]). Let T be an α-admissible mapping. We say that T is a triangular α-admissible mapping if, $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ implies that $\alpha(x,z) \geq 1$.

Lemma 1.3 ([10]). *Let T be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Definition 1.4 ([11]). Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x,y) = 1$ then this definition reduces to Definition 1.1.

In this paper, we prove certain new results for contractive inequalities involving dominating auxiliary function instead of ordinary metric function. Further, we derive some recent results as corollaries.

2. Main Results

Now we state and prove our first main result.

Theorem 2.1. *Let $\alpha : X \times X \to [0, \infty)$ be a mapping and (X, d) be a complete metric space. Let T be a self-mapping on X and the following assertions hold:*

(i) *T is α-admissible mapping with respect to d;*

(ii) *either T is continuous or,*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\lim_{n \to \infty} \alpha(x_n, x) = 0$ and $\lim_{n \to \infty} \alpha(x_n, Tx) \geq d(x, Tx)$,*

(iv) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$,*

(v) there exists $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(Tx, Ty) \leq \psi\big(\alpha(x, y)\big). \tag{2.1}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping with respect to d and $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2 x_0) \geq d(Tx_0, T^2 x_0) = d(x_1, x_2)$. By continuing this process, we get $\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T and the result is proved. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, then

$$\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

By taking $x = x_{n-1}$, $y = x_n$ in (v) we get,

$$\alpha(Tx_{n-1}, Tx_n) \leq \psi(\alpha(x_{n-1}, x_n)),$$

and hence by induction, we have

$$\alpha(x_n, x_{n+1}) \leq \psi^n(\alpha(x_0, x_1)).$$

Now using (2.2) we have,

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1}) \leq \psi^n(\alpha(x_0, x_1)).$$

Fix $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(\alpha(x_0, x_1)) < \epsilon \qquad \text{for all } n \in \mathbb{N}.$$

Let $m, n \in \mathbb{N}$ with $m > n \geq N$. Then by triangular inequality we get

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{n \geq N} \psi^n(\alpha(x_0, x_1)) < \epsilon.$$

Consequently $\lim_{m,n,\to+\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$. At first we assume that T is continuous then we have

$$Tz = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = z.$$

So z is a fixed point of T. Now assume, (iii) holds. That is, $\lim_{n \to \infty} \alpha(x_n, z) = 0$ and $\lim_{n \to \infty} \alpha(x_{n+1}, Tz) = d(z, Tz)$. From (v) with $x = x_n$ and $y = z$ we have,

$$\alpha(x_{n+1}, Tz) = \alpha(Tx_n, Tz) \leq \psi(\alpha(x_n, z)).$$

Therefore by taking limit as $n \to \infty$ in the above inequality we have,

$$d(z, Tz) \leq \lim_{n \to \infty} \alpha(x_{n+1}, Tz) = \lim_{n \to \infty} \alpha(Tx_n, Tz) \leq \psi(\lim_{n \to \infty} \alpha(x_n, z)) = \psi(0) = 0.$$

That is, $z = Tz$. □

Remark 2.2. Notice that if in (v) of Theorem 2.1 we put $\psi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$, then we obtain generalized version of Banach contraction principle.

Example 2.3. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ be a metric on X. Define $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \frac{1}{4}x, & \text{if } x \in [0, 1] \\ 2^{x^2} + 1, & \text{if } x \in (1, \infty) \end{cases}, \alpha(x, y) = \begin{cases} d(x, y), & \text{if } x, y \in [0, 1] \\ 0, & \text{otherwise} \end{cases} \text{ and } \psi(t) = \frac{1}{2}t,$$

clearly, $\alpha(0, T0) \geq d(0, T0)$. Let, $\alpha(x, y) \geq d(x, y)$. Then, $x, y \in [0, 1]$ and so, $\alpha(Tx, Ty) \geq d(Tx, Ty)$. That is, T is α-admissible mapping with respect to d. Let $\{x_n\}$ be a sequence such that, $\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1})$ and $x_n \to x$ as $n \to \infty$. Then $\{x_n\} \subset [0, 1]$ and so, $x \in [0, 1]$, which implies, $Tx \in [0, 1]$. Hence, $\lim_{n\to\infty} \alpha(x_n, x) = 0$ and $\lim_{n\to\infty} \alpha(x_{n+1}, Tx) \geq d(x, Tx)$. Also,

$$\alpha(Tx, Ty) = \begin{cases} d(Tx, Ty) & \text{if } Tx, Ty \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} d(Tx, Ty) & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$\leq \begin{cases} \frac{1}{2}d(x, y) & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases} = \psi(\alpha(x, y))$$

for all $x, y \in X$. Therefore all conditions of Theorem 2.1 are satisfied and thus T has a fixed point.

Theorem 2.4. *Assume that all the hypothesis of Theorem 2.1 except the assertion (v) hold. If there exists $F \in \Delta$ and $\tau > 0$ such that for all $x, y \in X$,*

$$\alpha(Tx, Ty) > 0 \Rightarrow \tau + F(\alpha(Tx, Ty)) \leq F(\alpha(x, y))$$

holds. Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. As in proof of Theorem 2.1 we have,

$$\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

By taking $x = x_{n-1}$, $y = x_n$ in the inequality of the hypothesis we obtain,

$$\tau + F(\alpha(x_n, x_{n+1})) = \tau + F(\alpha(Tx_{n-1}, Tx_n)) \leq F(\alpha(x_{n-1}, x_n)) \tag{2.4}$$

and so we deduce that,

$$F\big(\alpha(x_n, x_{n+1})\big) \leq F\big(\alpha(x_{n-1}, x_n)\big) - \tau.$$

Therefore,

$$F\big(\alpha(x_n, x_{n+1})\big) \leq F\big(\alpha(x_{n-1}, x_n)\big) - \tau \leq F\big(\alpha(x_{n-2}, x_{n-1})\big) - 2\tau \leq \ldots \leq F(\alpha(x_0, x_1)) - n\tau. \tag{2.5}$$

By taking limit as $n \to \infty$ in (2.5) we have, $\lim_{n\to\infty} F\big(\alpha(x_n, x_{n+1})\big) = -\infty$, and since, $F \in \Delta$ we obtain,

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1}) = 0. \tag{2.6}$$

Now from (W3), there exists $0 < k < 1$ such that,

$$\lim_{n\to\infty} [\alpha(x_n, x_{n+1})]^k F\big(\alpha(x_n, x_{n+1})\big) = 0. \tag{2.7}$$

By (2.5) we have,

$$\lim_{n\to\infty}[\alpha(x_n,x_{n+1})]^k[F\big(\alpha(x_n,x_{n+1})\big) - F(\alpha(x_0,x_1))] \leq -n\tau[\alpha(x_n,x_{n+1})]^k \leq 0. \tag{2.8}$$

By taking limit as $n \to \infty$ in (2.8) and applying (2.6) and (2.7) we have,

$$\lim_{n\to\infty} n[\alpha(x_n,x_{n+1})]^k = 0.$$

Now from (2.3) we obtain,

$$0 \leq \lim_{n\to\infty} n[d(x_n,x_{n+1})]^k \leq \lim_{n\to\infty} n[\alpha(x_n,x_{n+1})]^k = 0.$$

That is,

$$\lim_{n\to\infty} n[d(x_n,x_{n+1})]^k = 0. \tag{2.9}$$

It follows from (2.9) that there exists, $n_1 \in \mathbb{N}$ such that,

$$n[d(x_n,x_{n+1})]^k \leq 1$$

and thus

$$d(x_n,x_{n+1}) \leq \frac{1}{n^k}$$

for all $n > n_1$. Now for $m > n > n_1$ we have,

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^k}.$$

Since, $0 < k < 1$, then $\sum_{i=n}^{\infty} \frac{1}{i^k}$ converges. Therefore, $d(x_n,x_m) \to 0$ as $m,n \to \infty$. That is, $\{x_n\}$ is a Cauchy sequence. By the completeness of X there exists $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now assume that T is continuous. Then as in proof of Theorem 2.1 we can deduce that T has a fixed point. Now assume (iii) holds. Again as in proof of Theorem 2.1 we have, $\lim_{n\to\infty} \alpha(x_n,z) = 0$ and $\lim_{n\to\infty} \alpha(x_{n+1},Tz) = d(z,Tz)$. Now from (v) we have,

$$F(\alpha(x_{n+1},Tz)) = F(\alpha(Tx_n,Tz)) \leq \tau + F(\alpha(Tx_n,Tz)) \leq F(\alpha(x_n,z)),$$

which implies,

$$\alpha(x_{n+1},Tz) \leq \alpha(x_n,z).$$

Taking limit as $n \to \infty$ in the above inequality we get,

$$d(z,Tz) \leq \lim_{n\to\infty} \alpha(x_{n+1},Tz) \leq \lim_{n\to\infty} \alpha(x_n,z) = 0$$

and so, $z = Tz$ as required. $\square$

Consistent with Jleli and Samet [6], we denote by Δ_θ the set of all functions $\theta : (0,\infty) \to [1,\infty)$ satisfying following conditions:

(θ_1) θ is increasing;

(θ_2) for all sequence $\{\alpha_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} \theta(\alpha_n) = 1$;

(θ_3) there exist $0 < r < 1$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Theorem 2.5. *Assume that all the hypothesis of Theorem 2.1 except the assertion (v) hold. If there exists $\theta \in \Delta_\theta$ and $0 \leq k < 1$ such that for all $x, y \in X$,*

$$\alpha(Tx,Ty) > 0 \Rightarrow \theta(\alpha(Tx,Ty)) \leq [\theta(\alpha(x,y))]^k$$

holds. Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = T x_{n-1}$ for all $n \in \mathbb{N}$. As in proof of Theorem 2.1 we have,

$$\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.10}$$

By taking $x = x_{n-1}$, $y = x_n$ in (v) we obtain,

$$\theta\big(\alpha(x_n, x_{n+1})\big) \leq \theta\big(\alpha(x_{n-1}, x_n)\big)^k.$$

Therefore,

$$1 \leq \theta\big(\alpha(x_n, x_{n+1})\big) \leq \theta\big(\alpha(x_{n-1}, x_n)\big)^k \leq \theta\big(\alpha(x_{n-2}, x_{n-1})\big)^{k^2} \leq \ldots \leq \theta(\alpha(x_0, x_1))^{k^n}. \tag{2.11}$$

By taking limit as $n \to \infty$ in (2.11) we have, $\lim_{n\to\infty} \theta\big(\alpha(x_n, x_{n+1})\big) = 1$, and since, $\theta \in \Delta_\theta$ we obtain,

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1}) = 0. \tag{2.12}$$

Now from (θ_3), there exists $0 < r < 1$ and $0 < \ell \leq \infty$ such that,

$$\lim_{n\to\infty} \frac{\theta\big(\alpha(x_n, x_{n+1})\big) - 1}{[\alpha(x_n, x_{n+1})]^r} = \ell. \tag{2.13}$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2}$. From the definition of the limit there exists $n_0 \in \mathbb{N}$ such that,

$$|\frac{\theta\big(\alpha(x_n, x_{n+1})\big) - 1}{[\alpha(x_n, x_{n+1})]^r} - \ell| \leq B \text{ for all } n \geq n_0,$$

which implies,

$$\frac{\theta\big(\alpha(x_n, x_{n+1})\big) - 1}{[\alpha(x_n, x_{n+1})]^r} \geq \ell - B = B \text{ for all } n \geq n_0$$

and so,

$$n[\alpha(x_n, x_{n+1})]^r \leq nA[\theta\big(\alpha(x_n, x_{n+1})\big) - 1] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Now assume that $\ell = \infty$. Let $C > 0$ be a given number. From the definition of limit there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{\theta\big(\alpha(x_n, x_{n+1})\big) - 1}{[\alpha(x_n, x_{n+1})]^r} \geq C \text{ for all } n \geq n_0,$$

which implies

$$n[\alpha(x_n, x_{n+1})]^r \leq nA[\theta\big(\alpha(x_n, x_{n+1})\big) - 1] \text{ for all } n \geq n_0,$$

where $A = \frac{1}{C}$. Hence in all cases there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that,

$$n[\alpha(x_n, x_{n+1})]^r \leq nA[\theta\big(\alpha(x_n, x_{n+1})\big) - 1] \text{ for all } n \geq n_0.$$

From (2.11) we have,

$$n[\alpha(x_n, x_{n+1})]^r \leq nA[\theta(\alpha(x_0, x_1))^{k^n} - 1] \text{ for all } n \geq n_0.$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} n[\alpha(x_n, x_{n+1})]^r = 0,$$

and then using (2.10) we obtain,

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^r = 0. \tag{2.14}$$

Now following the proof of Theorem 2.1 we can get a Cauchy sequence $\{x_n\}$ in the complete space X. Then there exists $x^* \in X$ such that, $x_n \to x^*$ as $n \to \infty$. Now assume (iii) holds. Again as in proof of Theorem 2.1 we have, $\lim_{n\to\infty} \alpha(x_n, z) = 0$ and $\lim_{n\to\infty} \alpha(x_{n+1}, Tz) \geq d(z, Tz)$. Thus, we have

$$\theta(\alpha(x_{n+1}, Tz)) = \theta(\alpha(Tx_n, Tz)) \leq [\theta(\alpha(x_n, z))]^k$$

and so,

$$\ln\left(\theta(\alpha(x_{n+1}, Tz))\right) \leq k \ln[\theta(\alpha(x_n, z))] \leq \ln[\theta(\alpha(x_n, z))].$$

and from property of θ we have,

$$\alpha(x_{n+1}, Tz) \leq \alpha(x_n, z)$$

Taking limit as $n \to \infty$ in the above inequality we get,

$$d(z, Tz) \leq \lim_{n\to\infty} \alpha(x_{n+1}, Tz) \leq \lim_{n\to\infty} \alpha(x_n, z) = 0$$

and so, $z = Tz$ as required. □

Definition 2.6. Let T be an α-admissible mapping with respect to η. We say T is triangular α−admissible mapping with respect to η, if $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$, implies, $\alpha(x, z) \geq \eta(x, z)$.

Lemma 2.7. *Let T be a triangular α-admissible mapping with respect to η. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n) \geq \eta(x_m, x_n) \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Proof. Proof is similar to that of Lemma 1.3, so is omitted. □

Theorem 2.8. *Let $\alpha : X \times X \to [0, \infty)$ be a mapping and (X, d) be a complete metric space. Let T be a self mapping on X and the following assertions hold:*

(i) T is triangular α-admissible mapping with respect to $d(x, y)$,

(ii) either T is continuous or,

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\lim_{n\to\infty} \alpha(x_n, Tx) \geq d(x, Tx)$,

(iv) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$,

(v) assume that there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and for all $x, y \in X$

$$\alpha(Tx, Ty) \leq \beta(\alpha(x, y))d(x, y). \tag{2.15}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq d(x_0, Tx_0)$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Then from Lemma 2.7 we have,

$$\alpha(x_m, x_n) \geq d(x_m, x_n) \text{ for all } m, n \in \mathbb{N} \text{ with } m < n. \tag{2.16}$$

By the inequality (v) we have

$$\alpha(x_n, x_{n+1}) = \alpha(Tx_{n-1}, Tx_n) \leq \beta(\alpha(x_{n-1}, x_n))d(x_{n-1}, x_n).$$

From (2.16) we have,

$$d(x_n, x_{n+1}) \leq \beta(\alpha(x_{n-1}, x_n))d(x_{n-1}, x_n), \tag{2.17}$$

which implies $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$. It follows that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing. Thus, there exists $s \in \mathbb{R}_+$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = s$. We shall prove that $s = 0$. From (2.17) we have

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(\alpha(x_{n-1}, x_n)) \leq 1,$$

which implies $\lim_{n\to\infty} \beta(\alpha(x_{n-1}, x_n)) = 1$. Regarding the property of the function β, we conclude that

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1}) = 0$$

and so from (2.16) we have,

$$0 \leq \lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} \alpha(x_n, x_{n+1}) = 0.$$

That is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.18}$$

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, we have

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

By the triangle inequality, we derive that

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$
$$< \varepsilon + d(x_{m(k)-1}, x_{m(k)})$$

$k \in \mathbb{N}$. Taking the limit as $k \to +\infty$ in the above inequality and regarding the limit (2.18), we get

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.19}$$

Again, by the triangle inequality, we find that

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}).$$

Taking the limit of the inequality above as $k \to +\infty$, together with (2.18) and (2.19), we deduce that

$$\lim_{k\to+\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.20}$$

From (v) we get,

$$\alpha(x_{n(k)+1}, x_{m(k)+1}) \leq \beta(\alpha(x_{n(k)}, x_{m(k)}))d(x_{n(k)}, x_{m(k)})$$

and so by (2.16) we deduce,

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq \beta(\alpha(x_{n(k)}, x_{m(k)}))d(x_{n(k)}, x_{m(k)}).$$

Hence,

$$\frac{d(x_{n(k)+1}, x_{m(k)+1})}{d(x_{n(k)}, x_{m(k)})} \leq \beta(\alpha(x_{n(k)}, x_{m(k)})) \leq 1.$$

Letting $k \to \infty$ in the inequality above, we get

$$\lim_{n \to \infty} \beta(\alpha(x_{n(k)}, x_{m(k)})) = 1.$$

That is, $\lim_{k \to \infty} \alpha(x_{n(k)}, x_{m(k)}) = 0$. Again by (2.16) we obtain,

$$\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) \leq \lim_{k \to \infty} \alpha(x_{n(k)}, x_{m(k)}) = 0.$$

That is, $\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, so there is $z \in X$ such that $x_n \to z$. If T is continuous then we have

$$Tz = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n+1} = z.$$

So z is a fixed point of T. Next, we suppose that T is not continuous and (iii) holds. As in proof of Theorem 2.1 we have, $\lim_{n \to \infty} \alpha(x_{n+1}, Tz) \geq d(z, Tz)$. Now from (v) we have,

$$\alpha(x_{n+1}, Tz) \leq \beta(\alpha(x_n, z))d(x_n, z).$$

By taking limit as $n \to \infty$ in the above inequality we get,

$$d(z, Tz) \leq \lim_{n \to \infty} \alpha(x_{n+1}, Tz) \leq \lim_{n \to \infty} (\beta(\alpha(x_n, z))d(x_n, z)) = 0.$$

Then $d(z, Tz) = 0$. That is, $z = Tz$. □

Remark 2.9. If in Theorems 2.1, 2.4, 2.8, we take $\alpha(x, y) = d(x, y)$ for all $x, \ y \in X$, then we obtain the well known results of Boyd and Wong [3], Wardowski [16], Samet and jleli Corollary 2.1 [6] and the classical result of Geraghty [5], respectively.

3. Fixed Point Results On Indirected Metric Spaces

As an application of our results we deduce further results in different settings.

Definition 3.1. Let X be a non-empty set and $\mathcal{A} : X \times X \to [0, \infty)$ be a function. We say the function $\mathcal{A}$ is an indirected metric function if

- there exists a metric function d on X such that $d(x, y) \leq \mathcal{A}(x, y)$ for all $x, y \in X$.

Then we say the pair $(X, \mathcal{A})$ is an indirected metric space with respect to d.

Lemma 3.2. *Let $(X, \mathcal{A})$ be indirected metric space with respect to d, and $T : X \to X$ be a given function. Then,*

- *T is triangular $\mathcal{A}$-admissible mapping with respect to d,*

- *$\lim_{n \to \infty} \mathcal{A}(x_n, Tx) \geq d(x, Tx)$ for any sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$,*

- *there exists $x_0 \in X$ such that, $\mathcal{A}(x_0, Tx_0) \geq d(x_0, Tx_0)$.*

Proof. Evidently, $d(x, y) \leq \mathcal{A}(x, y)$ for all $x, y \in X$ because of Definition 3.1. So, also $d(Tx, Ty) \leq \mathcal{A}(Tx, Ty)$ holds for all $x, y \in X$. Then, T is $\mathcal{A}$-admissible mapping with respect to $\eta(x, y) = d(x, y)$. As $d(x, y) \leq \mathcal{A}(x, y)$ for all $x, y \in X$, so T is triangular $\mathcal{A}$-admissible mapping with respect to $\eta(x, y) = d(x, y)$. Assume $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Now, since

$$\mathcal{A}(x_n, Tx) \geq d(x_n, Tx) \text{ for all } n \in \mathbb{N}$$

and d is continuous, then

$$\lim_{n \to \infty} \mathcal{A}(x_n, Tx) \geq d(x, Tx).$$

Clearly, there exists $x_0 \in X$ such that, $\mathcal{A}(x_0, Tx_0) \geq d(x_0, Tx_0)$ by Definition 3.1. □

Definition 3.3. An indirected metric space $(X, \mathcal{A})$ with respect to (X, d) is called complete indirected metric space if (X, d) is a complete metric space.

Example 3.4. Let (X, d) be a complete metric space and L is a positive real number. Define, $\mathcal{A} : X \times X \rightarrow [0, \infty)$ with $\mathcal{A}(x, y) = d(x, y) + L$. Then, $(X, \mathcal{A})$ is a complete indirected metric space.

Example 3.5. Let (X, D) be a metric space. Assume there exists a complete metric space (X, d) such that $d(x, y) \leq D(x, y)$ for all $x, y \in X$. Define, $\mathcal{A} : X \times X \rightarrow [0, \infty)$ with $\mathcal{A}(x, y) = D(x, y)$. Then, $(X, \mathcal{A})$ is a complete indirected metric space.

Example 3.6. Let (X, p) be a complete partial metric space. Define, $\mathcal{A} : X \times X \rightarrow [0, \infty)$ with $\mathcal{A}(x, y) = p(x, y)$. Then, $(X, \mathcal{A})$ is a complete indirected metric space. Indeed, if we chose a metric function d on X with $d(x, y) = \begin{cases} p(x, y), & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Then $d(x, y) \leq p(x, y)$ for all $x, y \in X$.

By using Lemma 3.2 and Theorem 2.8 we obtain following result.

Theorem 3.7. *Let $(X, \mathcal{A})$ be a complete indirected metric space with respect to d. Let T be a self mapping such that there exists a function $\beta : [0, \infty) \rightarrow [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and*

$$\mathcal{A}(Tx, Ty) \leq \beta(\mathcal{A}(x, y))d(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

By using Example 3.4 and Theorem 3.7 we get the following corollary.

Corollary 3.8. *Let (X, d) be a complete metric space. Let T be a self mapping of X, there exists a function $\beta : [0, \infty) \rightarrow [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and*

$$d(Tx, Ty) + L \leq \beta(d(x, y) + L)d(x, y)$$

for all $x, y \in X$ where $L \geq 0$. Then T has a fixed point.

By using Example 3.5 and Theorem 3.7 we obtain the following corollary.

Corollary 3.9. *Let (X, D) be a metric space. Assume there exists a complete metric space (X, d) such that $d(x, y) \leq D(x, y)$ for all $x, y \in X$. Let T be a self mapping of X such that there exists a function $\beta : [0, \infty) \rightarrow [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and*

$$D(Tx, Ty) \leq \beta(D(x, y))d(x, y)$$

for all $x, y \in X$. Then T has a fixed point.

By using Example 3.6 and Theorem 3.7 we get following corollary.

Corollary 3.10. *Let (X, p) be a complete partial metric space. Let T be a self mapping of X such that there exists a function $\beta : [0, \infty) \rightarrow [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and*

$$p(Tx, Ty) \leq \beta(p(x, y))d(x, y)$$

for all $x, y \in X$ where $d(x, y) = \begin{cases} p(x, y), & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$. Then T has a fixed point.

4. Further Consequences

We denote by Λ the family of functions $\lambda : [0, \infty) \rightarrow [0, \infty)$ such that λ is continuous, $\lambda(t) > t$ for all $t > 0$ and $\lambda(t) = 0$ iff $t = 0$.

Lemma 4.1. *Let (X, d) be a metric space and T be a given self mapping on X. Define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \lambda(d(x, y))$ for some $\lambda \in \Lambda$. Then T is an α-admissible mapping with respect to $\eta(x, y) = d(x, y)$. Further, if $\{x_n\}$ is a sequence with $x_n \to x$ as $n \to \infty$, then $\lim_{n \to \infty} \alpha(x_n, x) = 0$ and $\lim_{n \to \infty} \alpha(x_n, Tx) \geq d(x, Tx)$.*

Proof. Since $\alpha(x, y) = \lambda(d(x, y)) \geq d(x, y)$ for all $x, y \in X$. Then T is an α-admissible mapping with respect to $\eta(x, y) = d(x, y)$. Further, let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to \infty$. Then,

$$\lim_{n \to \infty} \alpha(x_n, x) = \lim_{n \to \infty} \lambda(d(x_n, x)) = \lambda(d(x, x)) = 0$$

and

$$\lim_{n \to \infty} \alpha(x_n, Tx) = \lim_{n \to \infty} \lambda(d(x_n, Tx)) = \lambda(d(x, Tx)) \geq d(x, Tx).$$

$\square$

By using Lemma 4.1 and our main results, we deduce the following new fixed point theorems.

Theorem 4.2. *Let (X, d) be a complete metric space. Let T be a self mapping on X such that,*

$$\lambda(d(Tx, Ty)) \leq \psi\big(\lambda(d(x, y))\big)$$

holds for all $x, y \in X$ where $\psi \in \Psi$ and $\lambda \in \Lambda$. Then T has a fixed point.

Theorem 4.3. *Let (X, d) be a complete metric space. Let T be a self mapping on X such that there exists $F \in \Delta$ and $\tau > 0$ such that,*

$$\lambda(d(Tx, Ty)) > 0 \Rightarrow \tau + F(\lambda(d(Tx, Ty))) \leq F(\lambda(d(x, y)))$$

holds for all $x, y \in X$ where $\lambda \in \Lambda$. Then T has a fixed point.

Theorem 4.4. *Let (X, d) be a complete metric space. Let T be a self mapping on X such that there exists $\theta \in \Delta_\theta$ and $0 \leq k < 1$ such that,*

$$\lambda(d(Tx, Ty)) > 0 \Rightarrow \theta(\lambda(d(Tx, Ty))) \leq [\theta(\lambda(d(x, y)))]^k$$

holds for all $x, y \in X$ where $\lambda \in \Lambda$. Then T has a fixed point.

Theorem 4.5. *Let (X, d) be a complete metric space. Let T be a self mapping on X, there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and*

$$\lambda(d(Tx, Ty)) \leq \beta(\lambda(d(x, y)))d(x, y)$$

for all $x, y \in X$ where $\lambda \in \Lambda$. Then T has a fixed point.

Lemma 4.6. *Let T be an $\alpha-$admissible mapping in a metric space (X, d). Define $\alpha_1 : X \times X \to [0, \infty)$ by $\alpha_1(x, y) = \alpha(x, y)d(x, y)$. Then T is an α_1-admissible mapping with respect to $\eta(x, y) = d(x, y)$.*

Proof. Suppose that $\alpha_1(x, y) \geq \eta(x, y)$ with $x \neq y$. That is, $\alpha(x, y)d(x, y) \geq d(x, y)$ which implies, $\alpha(x, y) \geq 1$. Now since T is an $\alpha-$admissible mapping so $\alpha(Tx, Ty) \geq 1$. That is, $\alpha_1(Tx, Ty) = \alpha(Tx, Ty)d(Tx, Ty) \geq d(Tx, Ty) = \eta(Tx, Ty)$. Also, clearly, for all $x = y \in X$ we have, $\alpha_1(Tx, Ty) = \alpha(Tx, Ty)d(Tx, Ty) \geq 0 = d(Tx, Ty) = \eta(Tx, Ty)$. Hence for all $x, y \in X$ with $\alpha_1(x, y) \geq \eta(x, y)$ we have, $\alpha_1(Tx, Ty) \geq \eta(Tx, Ty)$. $\square$

Remark 4.7. Notice that if T is triangular $\alpha-$admissible mapping in a metric space (X, d), then clearly T is triangular α_1-admissible mapping with respect to $\eta(x, y) = d(x, y)$.

By using Lemma 4.6 and our main results, we deduce the following fixed point theorems.

Theorem 4.8. *Let $\alpha : X \times X \to [0, \infty)$ be a mapping and (X, d) be a complete metric space. Let T be a self mapping on X and the following assertions hold:*

(i) T is α-admissible mapping,

(ii) either T is continuous or α is continuous in it's first variable,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(iv) there exists $\psi \in \Psi$ such that for all $x, y \in X$,

$$\alpha(Tx, Ty)d(Tx, Ty) \leq \psi\big(\alpha(x, y)d(x, y)\big).$$

Then T has a fixed point.

Theorem 4.9. *Assume that all the hypothesis of Theorem 4.5 except the assertion (iv) hold. If there exist $F \in \Delta$ and $\tau > 0$ such that for all $x, y \in X$,*

$$\alpha(Tx, Ty)d(Tx, Ty) > 0 \Rightarrow \tau + F(\alpha(Tx, Ty)d(Tx, Ty)) \leq F(\alpha(x, y)d(x, y))$$

holds. Then T has a fixed point.

Theorem 4.10. *Assume that all the hypothesis of Theorem 4.5 except the assertion (iv) hold. If there exist $\theta \in \Delta_\theta$ and $0 \leq k < 1$ such that for all $x, y \in X$,*

$$\alpha(Tx, Ty)d(Tx, Ty) > 0 \Rightarrow \theta(\alpha(Tx, Ty)d(Tx, Ty)) \leq [\theta(\alpha(x, y)d(x, y))]^k$$

holds. Then T has a fixed point.

Theorem 4.11. *Let $\alpha : X \times X \to [0, \infty)$ be a mapping and (X, d) be a complete metric space. Let T be a self mapping on X and the following assertions holds:*

(i) T is triangular α-admissible mapping,

(ii) either T is continuous or α is continuous in it's first variable,

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(iv) assume that there exists a function $\beta : [0, \infty) \to [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$ and for all $x, y \in X$

$$\alpha(Tx, Ty)d(Tx, Ty) \leq \beta(\alpha(x, y)d(x, y))d(x, y).$$

Then T has a fixed point.

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