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Lefschetz type theorems for a class of noncompact mappings

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Abstract

In this paper we present new fixed point results for general compact absorbing type contractions in new extension spaces.

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1. Introduction

In Section 2 we present new Lefschetz fixed point theorems for multivalued maps in extension type spaces. In particular for extension spaces of type $GNES$, $GANES$, $GMNES$ and $GMANES$ and for maps which are general compact absorbing contractions or general approximative compact absorbing contractions. These results improve those in the literature; see [1-3, 5-6, 8-11, 14-19] and the references therein. Our results were motivated in part from ideas in [2, 3, 9, 11-12, 16-19].

For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ (nonempty subsets of X) and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \leq t \leq 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0, 1]$.

The following result can be found in [4, Lemma 1.2 and 4.7].

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Theorem 1.1. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X\left(\overline{F(X)}\right)$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.2. From Theorem 1.1 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values (see [16, 17]) it suffices [5 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [13 pp. 199] so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular) [7 pp. 431] (see also [7 pp. 434]). Note in Theorem 1.1 if F is compact valued then the assumption that X is regular can be removed.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p, q) \sim (p', q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p, q)]$ and is called a morphism from X to Y. We let $M(X, Y)$ be the set of all such morphisms. For any $\phi \in M(X, Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p, q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [10 pp. 364]) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [8 pp. 227].

Theorem 1.3. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Two morphisms $\phi, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X\times[0,1],Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where i_0, $i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [9, pp. 231]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.4. A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [9, 10] (and we write $\phi \in Ads(X,Y)$) provided there exists a selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.

Definition 1.5. A map $\phi \in Ads(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

When we talk about $\phi \in Ads$ it is assumed that we are also considering a specified selected pair (p,q) of ϕ with $\phi(x) = q\,(p^{-1}(x))$.

Remark 1.6. In fact since we specify the pair (p,q) of ϕ it is enough to say ϕ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of ϕ, X known in the literature [9] the more restrictive condition in Definition 1.2 works. We note [9, pp 227] that $\phi_\star$ does not depend on the choice of diagram from $[(p,q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [9, 10]) $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

If we do not wish to specify the selected pair (p,q) of ϕ then we would consider the Lefschetz set $\mathbf{\Lambda}\,(\phi) = \{\Lambda(q_\star\, p_\star^{-1}) : \ \phi = q\,(p^{-1})\}$.

Definition 1.7. A Hausdorff topological space X is said to be a Lefschetz space (for the class Ads) provided every compact $\phi \in Ads(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies ϕ has a fixed point.

Definition 1.8. A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair (p,q) of ϕ.

Definition 1.9. A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \left\{\Lambda(q_\star\, p_\star^{-1}) : \ (p,q) \subset \phi\right\}.$$

Definition 1.10. A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Remark 1.11. Many examples of Lefschetz spaces (for the class Ad or Ads) can be found in [1, 2, 8-12, 14-19]. For example in [8, 14, 18] the extension space ES(compact) or the neighborhood extension space NES(compact) are Lefschetz spaces (for the class Ad or Ads).

2. Asymptotic Fixed Point Theory

By a space we mean a Hausdorff topological space. Let X be a space and $F \in Ad(X,X)$. We say $X \in GNES$ (w.r.t. Ad) if there exists a Lefschetz space (for the class Ad) U, a single valued continuous map $r : U \to X$ and a compact valued map $\Phi \in Ad(X,U)$ with $r\,\Phi = id_X$.

Remark 2.1. This corrects a slight inaccuracy in the definition in [16] for Ad maps (this was corrected in [17]). In fact the definition in [16] is correct provided we restate (see below) the main result in [16]. In [16] we say $X \in GNES$ (w.r.t. Ad and F) (here X is a space and $F \in Ad(X,X)$) if there exists a Lefschetz space (for the class Ad) U, a single valued continuous map $r : U \to K$ and a compact valued map $\Phi \in Ad(K,U)$ with $r\,\Phi = id_K$ (here $K = \overline{F(X)}$). Note for any selected pair (p,q) of F then $(\overline{p},\overline{q}) \subset F|_K$ so $F|_K \in Ad(K,K)$; here $\overline{p}$, $\overline{q} : p^{-1}(K) \to K$ are given by $\overline{p}(z) = p(z)$, $\overline{q}(z) = q(z)$ for $z \in p^{-1}(K)$. The proof in [16] (the reasoning is word for word the same as in [16] except F is replaced by $F|_K$ and $E'' = K'$) immediately guarantees that if $X \in GNES$ (w.r.t. Ad and F) and $F|_K$ is compact then if (α,β) is a selected pair for $F|_K$ then $\beta_\star\,\alpha_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}\,(F|_K)$ is well defined. In addition if $\mathbf{\Lambda}\,(F|_K) \neq \{0\}$ then $F|_K$ has a fixed point. Also we note that in the definition of $GNES$ (w.r.t. Ad and F), $F \in Ad(X,X)$ could be replaced by $F : X \to 2^X$ with $F|_K \in Ad(K,K)$.

Let $X \in GNES$ (w.r.t. Ad) and $F \in Ad(X,X)$ a compact map. Let (p,q) be a selected pair for F. In [16] we showed (the proof is word for word the same as in [16] with K replaced by X in one place (there $K = \overline{F(X)}$)) that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}\,(F)$ is well defined. In addition we showed if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.

Remark 2.2. From the proof in [16] we see that we can replace the condition that U is a Lefschetz space with the assumption that the compact map $\Phi\,F\,r \in Ad(U,U)$ is a Lefschetz map and $\mathbf{\Lambda}(\Phi\,F\,r) \neq \{0\}$ implies $\Phi\,F\,r$ has a fixed point.

Let X be a Hausdorff topological space. A map $F \in Ad(X,X)$ is said to be a general compact absorbing contraction (written $F \in GCAC(X,X)$ or $F \in GCAC(X)$) if there exists $Y \subseteq X$ such that

(i). $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ad(Y,Y)$ (automatically satisfied) is a compact map with $Y \in GNES$ (w.r.t. Ad);

(iii). for any selected pair (p,q) of F, $q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

Remark 2.3. Of course condition (ii) above could be replaced by the more general abstract assumption that $F|_Y \in Ad(Y,Y)$ is a Lefschetz map and if $\mathbf{\Lambda}(F|_Y) \neq \{0\}$ then $F|_Y$ has a fixed point.

Remark 2.4. For a discussion on compact absorbing contractions see the papers [2, 3, 11, 16, 17] and the books [9, Section 42] and [12, Section 15.5]. For example a single valued generalized compact absorbing contraction with respect to h as defined in [3, 11] and the obvious extension to admissible maps are particular examples of generalized compact absorbing contractions in this paper; for admissible maps the obvious extension of a generalized compact absorbing contraction with respect to G (here $G \in Ad(X,X)$) is if (iii) above is replaced by: for every compact $K \subset X$ there exists an integer $n = n_K$ such that $F^n(G(K)) \subset Y$ (or $G(F^n(K)) \subset Y$ and $F(G^{-1}(Y)) \subset G^{-1}(Y)$) and there exists a selected pair (α,β) of G such that $\beta_\star\,\alpha_\star^{-1} : H(X,Y) \to H(X,Y)$ is an epimorphism (or $\beta_\star\,\alpha_\star^{-1} : H(X,Y) \to H(X,Y)$ is an monomorphism).

Theorem 2.5. *Let X be a Hausdorff topological space and $F \in GCAC(X,X)$. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.*

Proof. Let Y be as described above. Let (p,q) be a selected pair for F so in particular $q\,p^{-1}(Y) \subseteq F(Y)$. Consider $F|_Y$ and let q', $p' : p^{-1}(Y) \to Y$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$. Notice (p',q') is a selected pair for $F|_Y$. Now since $Y \in GNES$ (w.r.t. Ad) then as mentioned above $q'_\star\,(p')_\star^{-1}$ is a Leray endomorphism. Now (iii) and [9, Property 11.8, pp 53] guarantees that $q''_\star\,(p'')_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q''_\star\,(p'')_\star^{-1}) = 0$. Also [9, Property 11.5, pp 52] guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism (with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$) so $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$. Then there exists a selected pair (p,q) of F with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let (p',q') be as described above with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$. Then $\Lambda\,(q'_\star\,(p')_\star^{-1}) \neq 0$ so since $Y \in GNES$ (w.r.t. Ad) there exists $x \in Y$ with $x \in F|_Y\,(x)$ i.e. $x \in F\,x$. □

Let X be a space and $F \in Ad(X,X)$. We say $X \in GANES$ (w.r.t. Ad) if for each $\alpha \in Cov_X\,(X)$ there exists a Lefschetz space (for the class Ad) U_α, a single valued continuous map $r_\alpha : U_\alpha \to X$ and a compact valued map $\Phi_\alpha \in Ad(X, U_\alpha)$ such that $r_\alpha\,\Phi_\alpha : X \to X$ and $i : X \to X$ are α-close (by this we mean for each $x \in X$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \in V_x$ and $x = i(x) \in V_x$) and α-homotopic.

Remark 2.6. This corrects a slight inaccuracy in the definition in [16] for Ad maps (this was corrected in [17]). In fact the definition in [16] is correct provided we restate (see below) the main result in [16]. In [16] we say $X \in GANES$ (w.r.t. Ad and F) (here X is a space and $F \in Ad(X,X)$) if for each $\alpha \in Cov_X\,(K)$ (here $K = \overline{F(X)}$) there exists a Lefschetz space (for the class Ad) U_α, a single valued continuous map $r_\alpha : U_\alpha \to K$ and a compact valued map $\Phi_\alpha \in Ad(K, U_\alpha)$ such that $r_\alpha\,\Phi_\alpha : K \to K$ and $i : K \to K$ are α-close (by this we mean for each $x \in K$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \in V_x$ and $x = i(x) \in V_x$) and α-homotopic. The proof in [16] (the reasoning is word for word the same as in [16] except F is replaced by $F|_K$ and $E'' = K'$) immediately guarantees that if $X \in GANES$ (w.r.t. Ad and F) is a uniform space and $F|_K$ is compact then if (α,β) is a selected pair for $F|_K$ then $\beta_\star\,\alpha_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}\,(F|_K)$ is well defined. In addition if $\mathbf{\Lambda}\,(F|_K) \neq \{0\}$ then $F|_K$ has a fixed point. Also we note that in the definition of $GANES$ (w.r.t. Ad and F), $F \in Ad(X,X)$ could be replaced by $F : X \to 2^X$ with $F|_K \in Ad(K,K)$.

Now assume $X \in GANES$ (w.r.t. Ad) is a uniform space and $F \in Ad(X,X)$ is a compact map. Let (p,q) be a selected pair for F. In [16] we showed (the proof is word for word the same as in [16] with $\overline{F(X)}$ replaced by X) that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}\,(F)$ is well defined. In addition we showed if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then F has a fixed point.

Remark 2.7. From the proof in [16] we see that we can replace the condition that U_α is a Lefschetz space for each $\alpha \in Cov_X\,(X)$ with the assumption that for each $\alpha \in Cov_X\,(X)$ the compact map $\Phi_\alpha\,F\,r_\alpha \in Ad(U_\alpha, U_\alpha)$ is a Lefschetz map and $\mathbf{\Lambda}(\Phi_\alpha\,F\,r_\alpha) \neq \{0\}$ implies $\Phi_\alpha\,F\,r_\alpha$ has a fixed point.

Remark 2.8. In the definition of $GANES$ (w.r.t. Ad) it is easy to see (see [16]) that one could replace the assumption that $r_\alpha\,\Phi_\alpha : X \to X$ and $i : X \to X$ are α-close and α-homotopic with the assumption that $r_\alpha\,\Phi_\alpha : X \to 2^X$ and $i : X \to X$ are strongly α-close (by this we mean for each $x \in X$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \subseteq V_x$ and $x = i(x) \in V_x$) and $(r_\alpha)_\star\,(q^1_\alpha)_\star\,(p^1_\alpha)_\star^{-1} = i_\star$ for any selected pair (p^1_α, q^1_α) of Φ_α. Also as in Remark 2.5 in the definition of $GANES$ (w.r.t. Ad and F) it is easy to see that one could replace the assumption that $r_\alpha\,\Phi_\alpha : K \to K$ and $i : K \to K$ are α-close and α-homotopic with the assumption that $r_\alpha\,\Phi_\alpha : K \to 2^K$ and $i : K \to K$ are strongly α-close (by this we mean for each $x \in K$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \subseteq V_x$ and $x = i(x) \in V_x$) and $(r_\alpha)_\star\,(q^1_\alpha)_\star\,(p^1_\alpha)_\star^{-1} = i_\star$ for any selected pair (p^1_α, q^1_α) of Φ_α.

Let X be a Hausdorff topological space. A map $F \in Ad(X,X)$ is said to be a general approximative compact absorbing contraction (written $F \in GACAC(X,X)$ or $F \in GACAC(X)$) if there exists $Y \subseteq X$ such that

(i). Y is a uniform space and $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ad(Y,Y)$ (automatically satisfied) is a compact map with $Y \in GANES$ (w.r.t. Ad);

(iii). for any selected pair (p,q) of F, $q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

The same reasoning as in Theorem 2.5 establishes the following result (the only difference in the proof is that $GNES$ is replaced by $GANES$ in two places).

Theorem 2.9. *Let X be a Hausdorff topological space and $F \in GACAC(X,X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.*

Now we discuss a more general situation considered in [17]. Let X be a space and $F \in Ad(X,X)$. We say $X \in GMNES$ (w.r.t. Ad and F) if there exists a Lefschetz space (for the class Ad) U, a compact map $\Phi \in Ad(U,X)$, a compact valued map $\Psi \in Ad(X,U)$ with $\Phi\,\Psi(x) \subseteq F(x)$ for $x \in X$, and such that if (p,q) is a selected pair of F then there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with $(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}$.

Now assume $X \in GMNES$ (w.r.t. Ad and F) and $F \in Ad(X,X)$. Let (p,q) be a selected pair for F. In [17] we showed that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}(F)$ is well defined. In addition we showed if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.

Remark 2.10. From the proof in [17] we see that we can replace the condition that U is a Lefschetz space with the assumption that the compact map $\Psi\,\Phi \in Ad(U,U)$ is a Lefschetz map and $\mathbf{\Lambda}(\Psi\,\Phi) \neq \{0\}$ implies $\Psi\,\Phi$ has a fixed point.

Remark 2.11. Suppose we change the above definition as follows. We say $X \in GMNES$ (w.r.t. Ad and F) (here X is a space and $F \in Ad(X,X)$) if there exists a Lefschetz space (for the class Ad) U, a compact map $\Phi \in Ad(U,K)$, a compact valued map $\Psi \in Ad(K,U)$ with $\Phi\,\Psi(x) \subseteq F(x)$ for $x \in K$ (here $K = \overline{F(X)}$), and such that if (p,q) is a selected pair of $F|_K$ then there exists a selected pair (p_1,q_1) of Φ and a selected pair (p',q') of Ψ with $(q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}$. The proof in [17] (the reasoning is word for word the same as in [17] except F is replaced by $F|_K$ and $E'' = K'$) immediately guarantees that if $X \in GMNES$ (w.r.t. Ad and F) then if (α,β) is a selected pair for $F|_K$ then $\beta_\star\,\alpha_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}(F|_K)$ is well defined. In addition if $\mathbf{\Lambda}(F|_K) \neq \{0\}$ then $F|_K$ has a fixed point. Also we note that in the definition of $GMNES$ (w.r.t. Ad and F), $F \in Ad(X,X)$ could be replaced by $F: X \to 2^X$ with $F|_K \in Ad(K,K)$.

Let X be a Hausdorff topological space. A map $F \in Ad(X,X)$ is said to be a general absorbing contraction (written $F \in GAC(X,X)$ or $F \in GAC(X)$) if there exists $Y \subseteq X$ such that

(i). $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ad(Y,Y)$ (automatically satisfied) with $Y \in GMNES$ (w.r.t. Ad and $F|_Y$);

(iii). for any selected pair (p,q) of F, $q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

Theorem 2.12. *Let X be a Hausdorff topological space and $F \in GAC(X,X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.*

Proof. Let Y be as described above. Let (p,q) be a selected pair for F. Consider $F|_Y$ and let $q', p' : p^{-1}(Y) \to Y$ be given by $p'(u) = p(u)$ and $q'(u) = q(u)$. Now since $Y \in GMNES$ (w.r.t. Ad and $F|_Y$) then $q'_\star\,(p')_\star^{-1}$ is a Leray endomorphism. Now (iii) guarantees that $q''_\star\,(p'')_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q''_\star\,(p'')_\star^{-1}) = 0$. Thus $q_\star\,p_\star^{-1}$ is a Leray endomorphism (with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$) so $\mathbf{\Lambda}(F)$ is well defined. Next suppose $\mathbf{\Lambda}(F) \neq \{0\}$. Then there exists a selected pair (p,q) of F with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let (p',q') be as described above with $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,(q'_\star\,(p')_\star^{-1})$. Then $\Lambda\,(q'_\star\,(p')_\star^{-1}) \neq 0$ so since $Y \in GMNES$ (w.r.t. Ad and $F|_Y$) there exists $x \in Y$ with $x \in F|_Y(x)$ i.e. $x \in F\,x$. □

Let X be a space and $F \in Ad(X,X)$. We say $X \in GMANES$ (w.r.t. Ad and F) if for each $\alpha \in Cov_X(X)$ there exists a Lefschetz space (for the class Ad) U_α, a compact map $\Phi_\alpha \in Ad(U_\alpha,X)$, a compact valued map $\Psi_\alpha \in Ad(X,U_\alpha)$ such that for each $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$ there exists $U_{x,y} \in \alpha$ with $y \in U_{x,y}$ and $F(y) \cap U_{x,y} \neq \emptyset$ and such that if (p,q) is a selected pair of F then there exists a selected pair $(p_{1,\alpha},q_{1,\alpha})$ of Φ_α and a selected pair (p'_α,q'_α) of Ψ_α with $(q_{1,\alpha})_\star\,(p_{1,\alpha})_\star^{-1}\,(q'_\alpha)_\star\,(p'_\alpha)_\star^{-1} = q_\star\,p_\star^{-1}$.

Now assume $X \in GMANES$ (w.r.t. Ad and F) is a uniform space and $F \in Ad(X,X)$ is a compact map. Let (p,q) be a selected pair for F. In [17] we showed that $q_\star\, p_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}(F)$ is well defined. In addition we showed if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.

Remark 2.13. From the proof in [17] we see that we can replace the condition that U_α is a Lefschetz space for each $\alpha \in Cov_X(X)$ with the assumption that for each $\alpha \in Cov_X(X)$ the compact map $\Psi_\alpha\,\Phi_\alpha \in Ad(U_\alpha, U_\alpha)$ is a Lefschetz map and $\mathbf{\Lambda}(\Psi_\alpha\,\Phi_\alpha) \neq \{0\}$ implies $\Psi_\alpha\,\Phi_\alpha$ has a fixed point.

Remark 2.14. Suppose we change the above definition as follows. We say $X \in GMANES$ (w.r.t. Ad and F) (here X is a space and $F \in Ad(X,X)$) if for each $\alpha \in Cov_X(K)$ there exists a Lefschetz space (for the class Ad) U_α, a compact map $\Phi_\alpha \in Ad(U_\alpha, K)$, a compact valued map $\Psi_\alpha \in Ad(K, U_\alpha)$ (here $K = \overline{F(X)}$) such that for each $x \in U_\alpha$ and $y \in \Phi_\alpha(x)$ with $x \in \Psi_\alpha(y)$ there exists $U_{x,y} \in \alpha$ with $y \in U_{x,y}$ and $F|_K(y) \cap U_{x,y} \neq \emptyset$ and such that if (p,q) is a selected pair of $F|_K$ then there exists a selected pair $(p_{1,\alpha}, q_{1,\alpha})$ of Φ_α and a selected pair (p'_α, q'_α) of Ψ_α with $(q_{1,\alpha})_\star\,(p_{1,\alpha})_\star^{-1}\,(q'_\alpha)_\star\,(p'_\alpha)_\star^{-1} = q_\star\,p_\star^{-1}$. The proof in [17] (the reasoning is word for word the same as in [17] except F is replaced by $F|_K$ and $E'' = K'$) immediately guarantees that if $X \in GMANES$ (w.r.t. Ad and F) is a uniform space and $F|_K$ is compact then if (α,β) is a selected pair for $F|_K$ then $\beta_\star\,\alpha_\star^{-1}$ is a Leray endomorphism and so $\mathbf{\Lambda}(F|_K)$ is well defined. In addition if $\mathbf{\Lambda}(F|_K) \neq \{0\}$ then $F|_K$ has a fixed point. Also we note that in the definition of $GMANES$ (w.r.t. Ad and F), $F \in Ad(X,X)$ could be replaced by $F : X \to 2^X$ with $F|_K \in Ad(K,K)$.

Let X be a Hausdorff topological space. A map $F \in Ad(X,X)$ is said to be a general approximative absorbing contraction (written $F \in GAAC(X,X)$ or $F \in GAAC(X)$) if there exists $Y \subseteq X$ such that

(i). Y is a uniform space and $F(Y) \subseteq Y$;

(ii). $F|_Y \in Ad(Y,Y)$ (automatically satisfied) is a compact map with $Y \in GMANES$ (w.r.t. Ad and $F|_Y$);

(iii). for any selected pair (p,q) of F, $q''_\star\,(p'')_\star^{-1} : H(X,Y) \to H(X,Y)$ is a weakly nilpotent endomorphism (here p'', $q'' : (\Gamma, p^{-1}(Y)) \to (X,Y)$ are given by $p''(u) = p(u)$ and $q''(u) = q(u)$).

The same reasoning as in Theorem 2.5 establishes the following result.

Theorem 2.15. *Let X be a Hausdorff topological space and $F \in GAAC(X,X)$. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point.*

Remark 2.16. In all the results in this section it is possible to replace the admissible maps Ad with permissible maps $\mathcal{P}$ provided some technical assumptions are added (see [16, 17]).

Remark 2.17. It is very easy to extend the fixed point theory in [15, Section 4] using the definitions and results in this section. We leave the details to the reader.

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