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On the Ulam stability of a quadratic set-valued functional equation

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Abstract

In this paper, we prove the Ulam stability of the following set-valued functional equation by employing the direct method and the fixed point method, respectively,

$$f\left(x-\frac{y+z}{2}\right)\oplus f\left(x+\frac{y-z}{2}\right)\oplus f(x+z)=3f(x)\oplus\frac{1}{2}f(y)\oplus\frac{3}{2}f(z).$$

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1. Introduction and Preliminaries

The investigation of the Ulam stability problems of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms, i.e.,

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there is a homomorphism $H: G_1\to G_2$ such that $d(h(x),H(x))<\epsilon$ for all $x\in G_1$?

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The following year, Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hereafter, the theorem of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [14] for linear mappings by allowing an unbounded Cauchy difference. It should be pointed out that Rassias's work has a great influence on the development of the Ulam stability theory of functional equations. Afterwards, Găvruta [6] generalized the Rassas's theorem by using a general control function. Since then, the Ulam stability of various types of functional equations has been widely and extensively studied. For more details, the reader is referred to [5, 9, 15, 17].

As a generalization of the stability of single-valued functional equations, Lu and Park [11] initiated the study of the Ulam stability of set-valued functional equations, in which the functional inequality is replaced by an appropriate inclusion relation. In the following, various authors considered the Ulam stability problems of several types of set-valued functional equations by using a similar method [12, 13]. Unlike the previous approach, Kenary et al. [10] applied the Hausdorff metric defined on all closed convex subsets of a Banach space to characterize the functional inequality and investigated the Ulam stability of several types of set-valued functional equations by using a fixed point technique, which is used to deal with the stability of single-valued functional equations. Recently, Jang et al. [8] and Chu et al. [3] further studied the Ulam stability problems of some generalized set-valued functional equations in a similar way.

In [18], Shen and Lan constructed the following functional equation:

$$f\left(x-\frac{y+z}{2}\right)+f\left(x+\frac{y-z}{2}\right)+f(x+z)=3f(x)+\frac{1}{2}f(y)+\frac{3}{2}f(z),$$

they proved that the general solution of the preceding functional equation on an Abelian group is equivalent to the solution of the classic quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y),$$

it is natural to say that the above functional equation constructed by Shen and Lan is a quadratic functional equation.

Throughout this paper, unless otherwise stated, let X be a real vector space and Y be a Banach space with the norm $\|\cdot\|_Y$. We denote by $\mathcal{C}_b(Y)$, $\mathcal{C}_c(Y)$ and $\mathcal{C}_{cb}(Y)$ the set of all closed bounded subsets of Y, the set of all closed convex subsets of Y and the set of all closed convex bounded subsets of Y, respectively.

Let A and B be two nonempty subsets of Y, $\lambda \in \mathbb{R}$. The addition and the scalar multiplication can be defined as follows

$$A+B=\{a+b|a\in A, b\in B\}, \qquad \lambda A=\{\lambda a|a\in A\}.$$

Furthermore, for the subsets $A, B \in \mathcal{C}_c(Y)$, we write $A\oplus B=\overline{A+B}$, where $\overline{A+B}$ denotes the closure of $A+B$.

Generally, for arbitrary $\lambda, \mu \in \mathbb{R}^+$, we can obtain that

$$\lambda A+\lambda B=\lambda(A+B), \qquad (\lambda+\mu)A\subseteq \lambda A+\mu A.$$

In particular, if A is convex, then we have $(\lambda+\mu)A=\lambda A+\mu A$.

For $A, B\in \mathcal{C}_b(Y)$, the Hausdorff distance between A and B is defined by

$$h(A,B):=\inf\{\epsilon>0|A\subseteq B+\epsilon\overline{S_1}, B\subseteq A+\epsilon\overline{S_1}\},$$

where $\overline{S_1}$ denotes the closed unit ball in Y, i.e., $\overline{S_1}=\{y\in Y|\|y\|_Y\le 1\}$. Since Y is a Banach space, it is proved that $(\mathcal{C}_{cb}(Y),\oplus,h)$ is a complete metric semigroup [2]. Rådström [16] proved that $(\mathcal{C}_{cb}(Y),\oplus,h)$ can be isometrically embedded in a Banach space.

The main purpose of this paper is to establish the Ulam stability of the following quadratic set-valued functional equation by employing the direct method and the fixed point method, respectively.

$$f\left(x-\frac{y+z}{2}\right)\oplus f\left(x+\frac{y-z}{2}\right)\oplus f(x+z)=3f(x)\oplus\frac{1}{2}f(y)\oplus\frac{3}{2}f(z) \tag{1.1}$$

The following are some properties of the Hausdorff distance.

Lemma 1.1 (Castaing and Valadier [2]). *For any $A_1, A_2, B_1, B_2, C \in \mathcal{C}_{cb}(Y)$ and $\lambda \in \mathbb{R}^+$, the following expressions hold*
(i) $h(A_1 \oplus A_2, B_1 \oplus B_2) \leq h(A_1, B_1) + h(A_2, B_2)$;
(ii) $h(\lambda A_1, \lambda B_1) = \lambda h(A_1, B_1)$;
(iii) $h(A_1 \oplus C, B_1 \oplus C) = h(A_1, B_1)$.

In the following, we recall an fundamental result in the fixed point theory to be used.

Lemma 1.2 (Diaz and Margolis [4]). *Let (X, d) be a complete generalized metric space, i.e., one for which d may assume infinite values. Suppose that $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for every element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all $n \geq 0$ or there exists an $n_0 \in \mathbb{N}$ such that
(i) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(ii) The sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(iii) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

2. Ulam stability of the quadratic set-valued functional equation (1.1): The direct method

In this section, we shall consider the Ulam stability of the set-valued equation (1.1) by employing the direct method.

Theorem 2.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi(x, y, z) = \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k y, 2^k z) < \infty \tag{2.1}$$

for all $x, y, z \in X$. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is the mapping with $f(0) = \{0\}$ and satisfies

$$h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2} f(y) \oplus \frac{3}{2} f(z)\Big) \leq \varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

exists for every $x \in X$ and defines a unique quadratic mapping $Q : X \to \mathcal{C}_{cb}(Y)$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4} \Phi(x, x, x) \tag{2.3}$$

for all $x \in X$.

Proof. Putting $y = z = x$ in (2.2). Since $f(0) = \{0\}$, by Lemma 1.1, we can get that

$$h\Big(\frac{1}{4} f(2x), f(x)\Big) \leq \frac{1}{4} \varphi(x, x, x) \tag{2.4}$$

for all $x \in X$. Replacing x by $2^{n-1} x$ and dividing by 4^{n-1} in (2.4), we have

$$h\Big(\frac{1}{4^n} f(2^n x), \frac{1}{4^{n-1}} f(x)\Big) \leq \frac{1}{4^n} \varphi(2^{n-1} x, 2^{n-1} x, 2^{n-1} x) \tag{2.5}$$

for all $x \in X$ and $n \in \mathbb{N}$. From (2.4) and (2.5), it follows that

$$h\Big(f(x), \frac{1}{4^n} f(2^n x)\Big) \leq \sum_{k=1}^{n} \frac{1}{4^k} \varphi(2^{k-1}x, 2^{k-1}x, 2^{k-1}x) \tag{2.6}$$

for all $x \in X$ and $n \in \mathbb{N}$. Now we claim that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in $(\mathcal{C}_{cb}(Y), h)$. Indeed, for all $m, n \in \mathbb{N}$, by (2.6), we can obtain that

$$\begin{aligned}
& h\Big(\frac{1}{4^{n+m}} f(2^{n+m}x), \frac{1}{4^m} f(2^m x)\Big) \\
&= \frac{1}{4^m} h\Big(\frac{1}{4^n} f(2^{n+m}x), f(2^m x)\Big) \\
&\leq \frac{1}{4^m} \sum_{k=1}^{n} \frac{1}{4^k} \varphi(2^{m+k-1}x, 2^{m+k-1}x, 2^{m+k-1}x) \\
&= \frac{1}{4^m} \sum_{k=0}^{n-1} \frac{1}{4^{k+1}} \varphi(2^{m+k}x, 2^{m+k}x, 2^{m+k}x)
\end{aligned} \tag{2.7}$$

for all $x \in X$. From the condition (2.1), it follows that the last expression tends to zero as $m \to \infty$. Then, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is Cauchy. Therefore, the completeness of $\mathcal{C}_{cb}(Y)$ implies that the following expression is well-defined, that is, we can define

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

Next, we show that Q satisfies the set-valued equality (1.1). Replacing x, y, z by $2^n x, 2^n y, 2^n z$ in (2.2), respectively, and dividing both sides by 4^n, we get

$$\begin{aligned}
&\frac{1}{4^n} h\Big(f\Big(2^n\Big(x - \frac{y+z}{2}\Big)\Big) \oplus f\Big(2^n\Big(x + \frac{y-z}{2}\Big)\Big) \oplus f(2^n(x+z)), \\
&\quad 3f(2^n x) \oplus \frac{1}{2} f(2^n y) \oplus \frac{3}{2} f(2^n z)\Big) \leq \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z).
\end{aligned}$$

By letting $n \to \infty$, since the right-hand side in the preceding expression tends to zero, we obtain that Q is a quadratic set-valued mapping. Moreover, letting $n \to \infty$ in (2.6), we get the desired inequality (2.3).

To prove the uniqueness of Q. Assume that Q' is another quadratic set-valued mapping satisfying the inequality (2.3). Thus we can infer that

$$\begin{aligned}
h(Q(x), Q'(x)) &= \frac{1}{4^n} h(Q(2^n x), Q'(2^n x)) \\
&\leq \frac{1}{4^n} (h(Q(2^n x), f(2^n x)) + h(f(2^n x)) + Q'(2^n x)) \\
&\leq \frac{2}{4^{n+1}} \Phi(2^n x, 2^n x, 2^n x).
\end{aligned}$$

It is easy to see from the condition (2.1) that the last expression tends to zero as $n \to \infty$. Then, we obtain that $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. □

Corollary 2.2. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies*

$$\begin{aligned}
&h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2} f(y) \oplus \frac{3}{2} f(z)\Big) \\
&\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)
\end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and

$$h(f(x), Q(x)) \leq \frac{3\theta\|x\|^p}{4-2^p}$$

for all $x \in X$.

Proof. Letting $\varphi(x,y,z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and the result follows directly from Theorem 2.1. □

Corollary 2.3. *Let $0 < p < \frac{2}{3}$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies*

$$h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z)\Big) \\ \leq \theta\|x\|^p\|y\|^p\|z\|^p$$

for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and

$$h(f(x), Q(x)) \leq \frac{\theta\|x\|^{3p}}{4-2^{3p}}$$

for all $x \in X$.

Proof. Letting $\varphi(x,y,z) = \theta\|x\|^p\|y\|^p\|z\|^p$ and the result follows directly from Theorem 2.1. □

Theorem 2.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that*

$$\Psi(x,y,z) = \sum_{k=0}^{\infty} 4^k\psi(2^{-k}x, 2^{-k}y, 2^{-k}z) < \infty \tag{2.8}$$

for all $x, y, z \in X$. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is the mapping satisfying

$$h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z)\Big) \leq \psi(x,y,z) \tag{2.9}$$

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n\to\infty} 4^n f(2^{-n}x)$$

exists for every $x \in X$ and defines a unique quadratic mapping $Q : X \to \mathcal{C}_{cb}(Y)$ such that

$$h(f(x), Q(x)) \leq \Psi\Big(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\Big) \tag{2.10}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.9), we get $f(0) = \{0\}$, since the condition $\Psi(0,0,0) = \sum_{k=0}^{\infty} 4^k\psi(0,0,0)$ implies that $\psi(0,0,0) = 0$.

Setting $y = z = x$ in (2.9), we have

$$h(4f(x), f(2x)) \leq \psi(x,x,x) \tag{2.11}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.11), we get

$$h\Big(4f\Big(\frac{x}{2}\Big), f(x)\Big) \leq \psi\Big(\frac{x}{2}, \frac{x}{2}, \frac{x}{2}\Big) \tag{2.12}$$

for all $x \in X$. Replacing x by $\frac{x}{2^{n-1}}$ and multiplying both sides by 4^{n-1} in (2.12), we can obtain that

$$h\Big(4^n f\Big(\frac{x}{2^n}\Big), 4^{n-1}f(\frac{x}{2^{n-1}})\Big) \leq 4^{n-1}\psi\Big(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\Big) \tag{2.13}$$

for all $x \in X$ and $n \in \mathbb{N}$. Combing the inequalities (2.12) and (2.13) gives

$$h\Big(4^n f\Big(\frac{x}{2^n}\Big), f(x)\Big) \leq \sum_{k=0}^{n-1} 4^k \psi\Big(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\Big) \tag{2.14}$$

for all $x \in X$ and $n \in \mathbb{N}$. The rest of the proof is analogous to the proof of Theorem 2.1. □

Corollary 2.5. *Let $p > 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping satisfying*

$$\begin{aligned} &h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z)\Big) \\ &\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and

$$h(f(x), Q(x)) \leq \frac{3\theta\|x\|^p}{2^p - 4}$$

for all $x \in X$.

Proof. Letting $\psi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and the result follows directly from Theorem 2.4. □

Corollary 2.6. *Let $p > \frac{2}{3}$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping satisfying*

$$\begin{aligned} &h\Big(f\Big(x - \frac{y+z}{2}\Big) \oplus f\Big(x + \frac{y-z}{2}\Big) \oplus f(x+z), 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z)\Big) \\ &\leq \theta\|x\|^p\|y\|^p\|z\|^p \end{aligned}$$

for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and

$$h(f(x), Q(x)) \leq \frac{\theta\|x\|^{3p}}{2^{3p} - 4}$$

for all $x \in X$.

Proof. Letting $\psi(x, y, z) = \theta\|x\|^p\|y\|^p\|z\|^p$ and the result follows directly from Theorem 2.4. □

3. Ulam stability of the quadratic set-valued functional equation (1.1): The fixed point method

In this section, we will investigate the Ulam stability of the set-valued functional equation (1.1) by using the fixed point technique.

Theorem 3.1. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists a positive constant $L < 1$ satisfying*

$$\varphi(2x, 2y, 2z) \leq 4L\varphi(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$. Assume that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies the inequality (2.2) for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping Q defined by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4(1-L)}\varphi(x, x, x) \tag{3.2}$$

for all $x \in X$.

Proof. Consider the set $S = \{g | g : X \to \mathcal{C}_{cb}(Y), g(0) = \{0\}\}$ and introduce the generalized metric d on S, which is defined by

$$d(g_1, g_2) = \inf\{\mu \in (0, \infty) | h(g_1(x), g_2(x)) \leq \mu\varphi(x, x, x), \forall\, x \in X\},$$

where, as usual, $\inf \emptyset = \infty$. It can easily be verified that (S, d) is a complete generalized metric space (see [10]).

Now, we define an operator $T : S \to S$ by

$$Tg(x) = \frac{1}{4}g(2x)$$

for all $x \in X$.

Let $g_1, g_2 \in S$ be given such that $d(g_1, g_2) = \epsilon$. Then

$$h(g_1(x), g_2(x)) \leq \epsilon\varphi(x, x, x)$$

for all $x \in X$. Thus, we can obtain that

$$\begin{aligned} h(Tg_1(x), Tg_2(x)) &= h\Big(\frac{1}{4}g_1(2x), \frac{1}{4}g_2(2x)\Big) \\ &= \frac{1}{4}h(g_1(2x), g_2(2x)) \\ &\leq \frac{1}{4}\epsilon\varphi(2x, 2x, 2x) \\ &\leq L\epsilon\varphi(x, x, x) \end{aligned}$$

for all $x \in X$. Hence, $d(g_1, g_2) = \epsilon$ implies that $d(Tg_1, Tg_2) \leq L\epsilon$. Therefore, we know that $d(Tg_1, Tg_2) \leq Ld(g_1, g_2)$, which means that T is a strictly contractive mapping with the Lipschitz constant $L < 1$. Moreover, we can infer from (2.4) that $d(Tf, f) \leq \frac{1}{4}$. By Lemma 1.2, there exists a set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ satisfying the following:

(i) Q is a fixed point of T, i.e., $4Q(x) = Q(2x)$ for all $x \in X$. Further, Q is the unique fixed point of T in the set $\{g \in S | d(f, g) < \infty\}$, which means that there exists an $\eta \in (0, \infty)$ such that

$$h(f(x), Q(x)) \leq \eta\varphi(x, x, x)$$

for all $x \in X$.

(ii) $d(T^n f, Q) \to 0$ as $n \to \infty$. Then we get

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x)$$

for all $x \in X$.

(iii) $d(f, Q) \leq \frac{1}{1-L}d(f, Tf)$. Then we have $d(f, Q) \leq \frac{1}{4(1-L)}$, which implies the inequality (3.2) holds.

Finally, we replace x, y, z by $2^n x, 2^n y, 2^n z$ in (2.2), respectively, and divide both sides by 4^n, we obtain that

$$\begin{aligned} &\frac{1}{4^n}h\Big(f\Big(2^n\Big(x - \frac{y+z}{2}\Big)\Big) \oplus f\Big(2^n\Big(x + \frac{y-z}{2}\Big)\Big) \oplus f(2^n(x+z)), \\ &\quad 3f(2^n x) \oplus \frac{1}{2}f(2^n y) \oplus \frac{3}{2}f(2^n z)\Big) \\ &\quad \leq \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z) \\ &\quad \leq \frac{1}{4^n} \cdot 4^n L^n \varphi(x, y, z) \\ &\quad = L^n\varphi(x, y, z). \end{aligned}$$

Since $L < 1$, the last expression tends to zero as $n \to \infty$. By (ii), we conclude that Q is a quadratic set-valued mapping satisfying (1.1). □

Remark 3.2. Based on Theorem 3.1, the corollaries 2.2 and 2.3 can also be directly obtained by choosing $L = 2^{p-2}$ and $L = 2^{p-\frac{2}{3}}$, respectively.

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists a positive constant $L < 1$ satisfying*

$$\varphi(x, y, z) \leq \frac{1}{4} L\varphi(2x, 2y, 2z) \tag{3.3}$$

for all $x, y, z \in X$. Assume that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies the inequality (2.2) for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping Q defined by $Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ such that

$$h(f(x), Q(x)) \leq \frac{L}{4(1-L)}\varphi(x, x, x) \tag{3.4}$$

for all $x \in X$.

Proof. Let us consider the set S and introduce the generalized metric d on S given as in Theorem 3.1.

Define a mapping $T : S \to S$ by

$$Tg(x) = 4g\left(\frac{x}{2}\right)$$

for all $x \in X$. By a similar argument as in Theorem 3.1, we can obtain that T is a strictly contractive mapping with the Lipschitz constant L. From (2.12) and the condition (3.3), we can infer that $d(Tf, f) \leq \frac{L}{4}$. According to Lemma 1.2, there exists a set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ such that the following results hold.

(i) Q is a fixed point of T, i.e., $Q(x) = 4Q\left(\frac{x}{2}\right)$ for all $x \in X$. Moreover, Q is the unique fixed point of T in the set $\{g \in S | d(g, f) < \infty\}$, which means that there exists an $\eta \in (0, \infty)$ such that

$$h(f(x), Q(x)) \leq \eta\varphi(x, x, x)$$

for all $x \in X$.

(ii) $d(T^n f, Q) \to 0$ as $n \to \infty$. Then we can obtain

$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$.

(iii) $d(f, Q) \leq \frac{1}{1-L} d(f, Tf)$. Then we get $d(f, Q) \leq \frac{L}{4(1-L)}$ and hence the inequality (3.4) holds.

Replacing x, y, z by $2^{-n}x, 2^{-n}y, 2^{-n}z$ in (2.2), respectively, and multiplying both sides by 4^n, we have

$$\begin{aligned}
&4^n h\Big(f\Big(2^{-n}\Big(x - \frac{y+z}{2}\Big)\Big) \oplus f\Big(2^{-n}\Big(x + \frac{y-z}{2}\Big)\Big) \oplus f(2^{-n}(x+z)),\\
&\quad 3f(2^{-n}x) \oplus \frac{1}{2} f(2^{-n}y) \oplus \frac{3}{2} f(2^{-n}z)\Big)\\
&\quad \leq 4^n \varphi(2^n x, 2^n y, 2^n z)\\
&\quad \leq 4^n \cdot \frac{1}{4^n} L^n \varphi(x, y, z)\\
&\quad = L^n \varphi(x, y, z).
\end{aligned}$$

Since $L < 1$, the last expression tends to zero as $n \to \infty$. By (ii), we conclude that Q is a quadratic set-valued mapping satisfying (1.1). □

Remark 3.4. In view of Theorem 3.3, the corollaries 2.5 and 2.6 can also be directly obtained by taking $L = 2^{2-p}$ and $L = 2^{\frac{2}{3}-p}$, respectively.

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66. 1
[2] C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, in: Lec. Notes in Math., vol. 580, Springer, Berlin, (1977). 1, 1.1
[3] H.Y. Chu, A. Kim and S.Y. Yoo, *On the stability of the generalized cubic set-valued functional equation*, Appl. Math. Lett., **37** (2014), 7–14. 1
[4] J. B. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., 74 (1968), 305–309. 1.2
[5] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequat. Math., **50** (1995), 143–190. 1
[6] P. Găvruţa, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431-436. 1
[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA, **27** (1941), 222–224. 1
[8] S. Y. Jang, C. Park, Y. Cho, *Hyers-Ulam stability of a generalized additive set-valued functional equation*, J. Inequal. Appl., **2013**, 2013:101. 1
[9] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, (2011). 1
[10] H. A. Kenary, H. Rezaei, *On the stability of set-valued functional equations with the fixed point alternative*, Fixed Point Theory Appl., **2012** 2012:81. 1, 3
[11] G. Lu, C. Park, *Hyers-Ulam stability of additive set-valued functional equations*, Appl. Math. Lett., **24** (2011), 1312–1316. 1
[12] C. Park, D. O'Regan, R. Saadati, *Stability of some set-valued functional equations*, Appl. Math. Lett., **24** (2011), 1910–1914. 1
[13] M. Piszczek, *The properties of functional inclusions and Hyers-Ulam stability*, Aequat. Math., **85** (2013), 111–118. 1
[14] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1
[15] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., **62** (2000), 23–130. 1
[16] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc., **3** (1952), 165–169. 1
[17] P. K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, Boca Raton, (2011). 1
[18] Y. H. Shen, Y. Y. Lan, *On the general solution of a quadratic functional equation and its Ulam stability in various abstract spaces*, J. Nonlinear Sci. Appl., **7** (2014), 368–378. 1
[19] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, (1960).