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Fixed point theorems for compatible and subsequentially continuous mappings in Menger spaces

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Abstract

In the present paper, we prove common fixed point theorems using the notions of compatibility and subsequentially continuity (alternately subcompatibility and reciprocally continuity) in Menger spaces. We also give a common fixed point theorem satisfying an integral analogue. As applications to our results, we obtain the corresponding fixed point theorems in metric spaces. Some illustrative examples are also given which demonstrate the validity of our results.

Keywords: Menger space, compatible mappings, reciprocally continuity, occasionally weakly compatible mappings, subcompatible mappings, subsequentially continuity.
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1. Introduction

The notion of probabilistic metric spaces (briefly, PM-spaces) as a generalization of metric spaces was introduced by Menger [23]. In Menger's theory, the notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points x and y by a function $F_{x,y}$ whose value $F_{x,y}(t)$ at the real number t is interpreted as the probability that the distance between x and y is less than t. In fact the study of such spaces received an impetus with the pioneering works

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of Schweizer and Sklar [28]. Fixed point theory is one of the most fruitful and effective tools in mathematics which has many applications within as well as outside mathematics (see [9, 10]). The theory of fixed points in PM-spaces is a part of probabilistic analysis and presently a hot area of mathematical research.

In 1986, Jungck [18] introduced the notion of compatible maps for a pair of self maps in metric space. Most of the common fixed point theorems for contraction mappings invariably require a compatibility condition besides assuming continuity of at least one of the mappings. Pant [26] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in metric spaces. He also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. Later on, Jungck and Rhoades [19] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. In 2008, Al-Thagafi and Shahzad [3] introduced the concept of occasionally weakly compatible (briefly, OWC) mappings in metric spaces which is the most general concept among all the commutativity concepts. In an interesting note, Doric et al. [13] have shown that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence (or a unique common fixed point) of the given pair of maps. Thus, no generalization can be obtained by replacing weak compatibility with owc property. Recently, Bouhadjera and Godet-Thobie [6] introduced two new notions namely subsequential continuity and subcompatibility which are weaker than reciprocal continuity and compatibility respectively (see also [5, 7]). Further, Imdad et al. [17] improved the results of Bouhadjera and Godet-Thobie [6] and showed that these results can easily recovered by replacing subcompatibility with compatibility or subsequential continuity with reciprocally continuity. Several interesting and elegant results have been obtained by various authors in different settings (e.g. [7, 11, 12, 16, 17, 21, 25, 30]). Many authors [1, 2, 14, 15, 27] proved several fixed point theorems in Menger spaces and showed the applications of corresponding results in metric spaces. Most recently, Altun et al. [4] proved common fixed point theorems of integral type in Menger as well as in metric spaces satisfying an integral analogue due to Branciari [8].

The aim of this paper is to prove some common fixed point theorems using the notions of compatibility and subsequentially continuity (alternately subcompatibility and reciprocally continuity) in Menger spaces. Our results never require the conditions on completeness (or closedness) of the underlying space (or subspaces) together with conditions on continuity in respect of any one of the involved mappings.

2. Preliminaries

Definition 2.1. [28] A mapping $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (shortly, t-norm) if the following conditions are satisfied: for all $a, b, c, d \in [0,1]$

1. $\triangle(a,1) = a$ for all $a \in [0,1]$,
2. $\triangle(a,b) = \triangle(b,a)$,
3. $\triangle(a,b) \leq \triangle(c,d)$ for $a \leq c$, $b \leq d$,
4. $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Examples of t-norms are $\triangle(a,b) = \min\{a,b\}$, $\triangle(a,b) = ab$ and $\triangle(a,b) = \max\{a+b-1,0\}$.

Definition 2.2. [28] A mapping $F : R \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

We shall denote by $\Im$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3. [28] The ordered pair $(X, \mathcal{F})$ is called a PM-space if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

1. $F_{x,y}(t) = H(t) \Leftrightarrow x = y$,
2. $F_{x,y}(t) = F_{y,x}(t)$,
3. if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$.

Definition 2.4. [28] A Menger space is a triplet $(X, \mathcal{F}, \triangle)$ where $(X, \mathcal{F})$ is a PM-space and t-norm $\triangle$ is such that the inequality

$$F_{x,z}(t+s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

holds for all $x, y, z \in X$ and all $t, s > 0$.

Every metric space (X, d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$.

Definition 2.5. [28] Let $(X, \mathcal{F}, \triangle)$ be a Menger space with continuous t-norm $\triangle$. A sequence $\{x_n\}$ in X is said to be

1. converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq N(\epsilon, \lambda)$.
2. Cauchy if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N(\epsilon, \lambda)$.

A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6. [24] Two self mappings A and S of a Menger space $(X, \mathcal{F}, \triangle)$ are said to be compatible if and only if $F_{ASx_n,SAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to z$ for some $z \in X$ as $n \to \infty$.

Definition 2.7. [29] Two self mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az = Sz$ some $z \in X$, then $ASz = SAz$.

Remark 2.8. Two compatible self mappings are weakly compatible, however the converse is not true in general (see [29, Example 1]).

Definition 2.9. [20] Two self mappings A and S of a non-empty set X are OWC iff there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute.

The notion of OWC is more general than weak compatibility (see [3]).

The following definitions (subcompatible and subsequentially continuous mappings) are on the lines of Bouhadjera and Godet-Thobie [6].

Definition 2.10. A pair of self mappings (A, S) defined on a Menger space $(X, \mathcal{F}, \triangle)$ is said to be subcompatible iff there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$, for all $t > 0$.

Remark 2.11. Two owc mappings are subcompatible, but the converse is not true in general (see [7, Example 1.2]).

Definition 2.12. [22] A pair of self mappings (A, S) defined on a Menger space $(X, \mathcal{F}, \triangle)$ is called reciprocally continuous if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$, whenever

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Remark 2.13. If two self mappings are continuous, then they are obviously reciprocally continuous but converse is not true. Moreover, in the setting of common fixed point theorems for compatible pair of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity but not conversely (see [26]).

Definition 2.14. A pair of self mappings (A, S) defined on a Menger space $(X, \mathcal{F}, \triangle)$ is called subsequentially continuous iff there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$.

Remark 2.15. If two self mappings are continuous or reciprocally continuous, then they are naturally subsequentially continuous. However, there exist subsequentially continuous pair of mappings which are neither continuous nor reciprocally continuous (see [7, Example 1.4]).

Lemma 2.16. *[24] Let $(X, \mathcal{F}, \triangle)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geq F_{x,y}(t),$$

for all $t > 0$ with fixed $x, y \in X$ then $x = y$.

3. Main results

Theorem 3.1. *Let A, B, S and T be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (B, T) are compatible and subsequentially continuous, then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *There exists a constant $k \in (0, 1)$ such that*

$$F_{Ax,By}(kt) \geq \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), F_{By,Sx}(t)\} \tag{3.1}$$

for all $x, y \in X$ and $t > 0$, then A, B, S and T have a unique common fixed point in X.

Proof. Since the pair (A, S) (also (B, T)) is subsequentially continuous and compatible maps, therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$,

and

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = F_{Az,Sz}(t) = 1,$$

for all $t > 0$ then $Az = Sz$, whereas in respect of the pair (B, T), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w,$$

for some $w \in X$,
and

$$\lim_{n\to\infty} F_{BTy_n,TBy_n}(t) = F_{Bw,Tw}(t) = 1,$$

for all $t > 0$ then $Bw = Tw$. Hence z is a coincidence point of the pair (A, S) whereas w is a coincidence point of the pair (B, T).

Now we prove that $z = w$. By putting $x = x_n$ and $y = y_n$ in inequality (3.1) we have

$$F_{Ax_n,By_n}(kt) \geq \min\{F_{Sx_n,Ty_n}(t), F_{Ax_n,Sx_n}(t), F_{By_n,Ty_n}(t), F_{Ax_n,Ty_n}(t), F_{By_n,Sx_n}(t)\}.$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} F_{z,w}(kt) &\geq \min\{F_{z,w}(t), F_{z,z}(t), F_{w,w}(t), F_{z,w}(t), F_{w,z}(t)\} \\ &= \min\{F_{z,w}(t), 1, 1, F_{z,w}(t), F_{w,z}(t)\} \\ &= F_{z,w}(t). \end{aligned}$$

From Lemma 2.16, we have $z = w$. Now we prove that $Az = z$ then by putting $x = z$ and $y = y_n$ in inequality (3.1) we get

$$F_{Az,By_n}(kt) \geq \min\{F_{Sz,Ty_n}(t), F_{Az,Sz}(t), F_{By_n,Ty_n}(t), F_{Az,Ty_n}(t), F_{By_n,Sz}(t)\}.$$

Taking the limit as $n \to \infty$, we get

$$F_{Az,w}(kt) \geq \min\{F_{Az,w}(t), F_{Az,Az}(t), F_{w,w}(t), F_{Az,w}(t), F_{w,Az}(t)\},$$

and so

$$\begin{aligned} F_{Az,z}(kt) &\geq \min\{F_{Az,z}(t), 1, 1, F_{Az,z}(t), F_{z,Az}(t)\} \\ &= F_{Az,z}(t). \end{aligned}$$

From Lemma 2.16, we have $Az = z$. Therefore, $Az = Sz = z$. Now we assert that $Bz = z$, then by putting $x = x_n$ and $y = z$ in inequality (3.1) we have

$$F_{Ax_n,Bz}(kt) \geq \min\{F_{Sx_n,Tz}(t), F_{Ax_n,Sx_n}(t), F_{Bz,Tz}(t), F_{Ax_n,Tz}(t), F_{Bz,Sx_n}(t)\}.$$

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} F_{z,Bz}(kt) &\geq \min\{F_{z,Bz}(t), F_{z,z}(t), F_{Bz,Bz}(t), F_{z,Bz}(t), F_{Bz,z}(t)\} \\ &= \min\{F_{z,Bz}(t), 1, 1, F_{z,Bz}(t), F_{Bz,z}(t)\} \\ &= F_{z,Bz}(t). \end{aligned}$$

From Lemma 2.16, we have $Bz = z$. Thus $Bz = Sz = z$. Therefore in all, $z = Az = Sz = Bz = Tz$ i.e. z is the common fixed point of A, B, S and T. The uniqueness of common fixed point is an easy consequence of inequality (3.1). This completes the proof of the theorem. □

Theorem 3.2. *Let A, B, S and T be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (B, T) are subcompatible and reciprocally continuous, then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *Further, the maps A, B, S and T have a unique common fixed point in X provided the involved maps satisfy the inequality (3.1) of Theorem 3.1.*

Proof. Since the pair (A, S) (also (B, T)) is subcompatible and reciprocally continuous, therefore there exists a sequences $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$,

and

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = \lim_{n\to\infty} F_{Az,Sz}(t) = 1,$$

for all $t > 0$, whereas in respect of the pair (B, T), there exists a sequence $\{y_n\}$ in X with

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w,$$

for some $w \in X$,

and

$$\lim_{n\to\infty} F_{BTx_n,TBx_n}(t) = \lim_{n\to\infty} F_{Bz,Tz}(t) = 1,$$

for all $t > 0$. Therefore, $Az = Sz$ and $Bw = Tw$ i.e. z is a coincidence point of the pair (A, S) whereas w is a coincidence point of the pair (B, T).

The rest of the proof can be completed on the lines of Theorem 3.1. □

Remark 3.3. It is clear that the conclusion of Theorem 3.1 remains valid if we replace compatibility with subcompatibility and subsequential continuity with reciprocally continuity, besides retaining the rest of the hypothesis (see [17]).

By setting $A = B$ in Theorem 3.1, we can derive a corollary for three mappings which runs as follows.

Corollary 3.4. *Let A, S and T be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (A, T) are compatible and subsequentially continuous, then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (A, T) has a coincidence point.*
3. *There exists a constant $k \in (0, 1)$ such that*

$$F_{Ax,Ay}(kt) \geq \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{Ay,Ty}(t), F_{Ax,Ty}(t), F_{Ay,Sx}(t)\} \tag{3.2}$$

for all $x, y \in X$ and $t > 0$, then A, S and T have a unique common fixed point in X.

Alternately, by setting $S = T$ in Theorem 3.1, we can also derive another corollary for three mappings which runs as follows.

Corollary 3.5. *Let A, B and S be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (B, S) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, S) has a coincidence point.*

3. *There exists a constant* $k \in (0,1)$ *such that*

$$F_{Ax,By}(kt) \geq \min\{F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{By,Sy}(t), F_{Ax,Sy}(t), F_{By,Sx}(t)\} \tag{3.3}$$

for all $x, y \in X$ *and* $t > 0$*, then* A, B *and* S *have a unique common fixed point in* X*.*

On taking $A = B$ and $S = T$ in Theorem 3.1, we get the interesting result.

Corollary 3.6. *Let* A *and* S *be self maps of a Menger space* $(X, \mathcal{F}, \triangle)$*, where* $\triangle$ *is a continuous t-norm. If the pair* (A, S) *is compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair* (A, S) *has a coincidence point.*
2. *There exists a constant* $k \in (0,1)$ *and* $\varphi \in \Phi$ *such that*

$$F_{Ax,Ay}(kt) \geq \min\{F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ax,Sy}(t), F_{Ay,Sx}(t)\} \tag{3.4}$$

for all $x, y \in X$ *and* $t > 0$*. Then* A *and* S *have a unique common fixed point in* X*.*

Example 3.7. Let $X = [0, \infty)$ and d be the usual metric on X and for each $t \in [0,1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space, where t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0,1]$. Now we define the self maps A and S on X by

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,1]; \\ 5x - 4, & \text{if } x \in (1, \infty). \end{cases} \quad S(X) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,1]; \\ 4x - 3, & \text{if } x \in (1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(\frac{1}{4n}\right) = 0 = \lim_{n\to\infty} \left(\frac{1}{5n}\right) = \lim_{n\to\infty} S(x_n).$$

Next,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{5n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = A(0),$$
$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{4n}\right) = \lim_{n\to\infty} \left(\frac{1}{20n}\right) = 0 = S(0),$$

and

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} \left(5 + \frac{5}{n} - 4\right) = 1 = \lim_{n\to\infty} \left(4 + \frac{4}{n} - 3\right) = \lim_{n\to\infty} S(x_n).$$

Also,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1 + \frac{4}{n}\right) = \lim_{n\to\infty} \left(5 + \frac{20}{n} - 4\right) = 1 \neq A(1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1+\frac{5}{n}\right) = \lim_{n\to\infty}\left(4+\frac{20}{n}-3\right) = 1 \neq S(1),$$

but $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$. Thus, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous. Therefore all the conditions of Corollary 3.6 are satisfied for some $k \in (0, 1)$. Here, 0 is a coincidence as well as unique common fixed point of the pair (A, S). It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both or by involving conditions on completeness (or closedness) of underlying space (or subspaces). Also, in this example neither X is complete nor any subspace $A(X) = \left[0, \frac{1}{4}\right] \cup (1, \infty)$ and $S(X) = \left[0, \frac{1}{5}\right] \cup (1, \infty)$ are closed. It is noted that this example cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both.

Example 3.8. Let $X = \mathbb{R}$ (set of real numbers) and d be the usual metric on X and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space, where t-norm $\triangle$ is defined by $\triangle(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Now we define the self maps A and S on X by

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in (-\infty, 1); \\ 5x-4, & \text{if } x \in [1, \infty). \end{cases} \quad S(X) = \begin{cases} x+3, & \text{if } x \in (-\infty, 1); \\ 4x-3, & \text{if } x \in [1, \infty). \end{cases}$$

Consider a sequence $\{x_n\} = \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty}\left(5+\frac{5}{n}-4\right) = 1 = \lim_{n\to\infty}\left(4+\frac{4}{n}-3\right) = \lim_{n\to\infty} S(x_n).$$

Also,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(1+\frac{4}{n}\right) = \lim_{n\to\infty}\left(5+\frac{20}{n}-4\right) = 1 = A(1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(1+\frac{5}{n}\right) = \lim_{n\to\infty}\left(4+\frac{20}{n}-3\right) = 1 = S(1),$$

and

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1,$$

for all $t > 0$. Consider another sequence $\{x_n\} = \left\{\frac{1}{n} - 4\right\}_{n\in\mathbb{N}}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty}\left(\frac{1}{4n}-1\right) = -1 = \lim_{n\to\infty}\left(\frac{1}{n}-4+3\right) = \lim_{n\to\infty} S(x_n).$$

Next,

$$\lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A\left(\frac{1}{n}-1\right) = \lim_{n\to\infty}\left(\frac{1}{4n}-\frac{1}{4}\right) = -\frac{1}{4} = A(-1),$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S\left(\frac{1}{4n}-1\right) = \lim_{n\to\infty}\left(\frac{1}{4n}-1+3\right) = 2 = S(-1),$$

and $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) \neq 1$. Thus, the pair (A, S) is reciprocally continuous as well as subcompatible but not compatible. Therefore all the conditions of Corollary 3.6 are satisfied for some $k \in (0, 1)$. Thus 1 is a coincidence as well as unique common fixed point of the pair (A, S). It is also noted that this example too cannot be covered by those fixed point theorems which involve compatibility and reciprocal continuity both.

Remark 3.9. The conclusions of Theorem 3.1 and Theorem 3.2 remain true if we replace inequality (3.1) by the following:

$$F_{Ax,By}(kt) \geq \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t)\} \tag{3.5}$$

$$F_{Ax,By}(kt) \geq F_{Sx,Ty}(t). \tag{3.6}$$

Remark 3.10. The results similar to Corollary 3.4, Corollary 3.5 and Corollary 3.6 can also be outlined in respect of Remark 3.9.

Now we state and prove a integral type common fixed point theorem in Menger spaces. First, we need the following lemma and remark due to Altun et al. [4].

Lemma 3.11. *Let $(X, \mathcal{F}, \triangle)$ be a Menger space. If there exists a constant $k \in (0,1)$ such that*

$$\int_0^{F_{x,y}(kt)} \psi(t)dt \geq \int_0^{F_{x,y}(t)} \psi(t)dt, \tag{3.7}$$

for all $t > 0$ with fixed $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a non-negative summable Lebesgue integrable function such that $\int_\epsilon^1 \psi(t)dt > 0$ for each $\epsilon \in [0,1)$, then $x = y$.

Remark 3.12. By setting $\psi(t) = 1$ for each $t > 0$ in inequality (3.7) in Lemma 3.11, we have

$$\int_0^{F_{x,y}(kt)} \psi(t)dt = F_{x,y}(kt) \geq F_{x,y}(t) = \int_0^{F_{x,y}(t)} \psi(t)dt,$$

which shows that Lemma 3.11 is a generalization of the Lemma 2.16.

Theorem 3.13. *Let A, B, S and T be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *For any $x, y \in X$ and for all $t > 0$,*

$$\int_0^{F_{Ax,By}(kt)} \psi(t)dt \geq \int_0^{m(x,y)} \psi(t)dt, \tag{3.8}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a non-negative summable Lebesgue integrable function such that $\int_\epsilon^1 \psi(t)dt > 0$ for each $\epsilon \in [0,1)$, where $0 < k < 1$ and

$$m(x,y) = \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), F_{By,Sx}(t)\}.$$

Then A, B, S and T have a unique common fixed point in X.

Proof. It is easy to see that inequality (3.8) is a special case of inequality (3.1). Then the result follows immediately from Theorem 3.1 and Theorem 3.2 using the Lemma 3.11. □

Remark 3.14. Similarly, we can also obtain several integral type common fixed point theorems for a pair or triod of mappings as showed earlier.

The following result involves a lower semi-continuous function $\phi : [0,1] \to [0,1]$ such that $\phi(t) > t$ for all $t \in (0,1)$, $\phi(0) = 0$ and $\phi(1) = 1$.

Theorem 3.15. *Let A, B, S and T be self maps of a Menger space $(X, \mathcal{F}, \triangle)$, where $\triangle$ is a continuous t-norm. If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *For all $x, y \in X$ and $t > 0$*

$$F_{Ax,By}(t) \geq \phi\left(\min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), F_{By,Sx}(t)\}\right). \tag{3.9}$$

Then A, B, S and T have a unique common fixed point in X.

Remark 3.16. The results similar to Corollary 3.4, Corollary 3.5 and Corollary 3.6 can also be obtained in respect of Theorem 3.15 and Remark 3.9.

4. Related Result in Metric Spaces

In this section, we utilize Theorem 3.1, Theorem 3.13 and Theorem 3.15 to derive corresponding common fixed point theorem in metric spaces.

Theorem 4.1. *Let A, B, S and T be self maps of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *There exists a constant $k \in (0, 1)$ such that*

$$d(Ax, By) \leq k \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\} \tag{4.1}$$

for all $x, y \in X$, then A, B, S and T have a unique common fixed point in X.

Proof. Define $F_{x,y}(t) = H(t - d(x, y))$ and $\triangle(a, b) = \min\{a, b\}$. Then metric space (X, d) can be realized as a Menger space $(X, \mathcal{F}, \triangle)$. It is straightforward to notice that compatibility and subsequentially continuity (alternately subcompatibility and reciprocally continuity) of the pairs (A, S) and (B, T) and the conditions (1) and (2) of Theorem 4.1 imply corresponding conditions of Theorem 3.1 (or Theorem 3.2). Also inequality (4.1) of Theorem 4.1 implies inequality (3.1) of Theorem 3.1. For any $x, y \in X$ and $t > 0$, $F_{Ax,By}(kt) = 1$ if $kt > d(Ax, By)$ which confirms the verification of inequality (3.1). Otherwise, if $kt \leq d(Ax, By)$, then

$$t \leq \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\},$$

which shows that inequality (3.1) of Theorem 3.1 is satisfied. Thus, all the conditions of Theorem 3.1 (or Theorem 3.2) are completely satisfied and hence conclusions follow immediately from Theorem 3.1 (or Theorem 3.2). □

Theorem 4.2. *Let A, B, S and T be self maps of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *There exists a constant $k \in (0, 1)$ such that*

$$\int_0^{d(Ax,By)} \psi(t)dt \leq k \int_0^{m(x,y)} \psi(t)dt, \tag{4.2}$$

for all $x, y \in X$, where

$$m(x, y) = \max\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), F_{By,Sx}(t)\}.$$

Then A, B, S and T have a unique common fixed point in X.

Theorem 4.3. *Let A, B, S and T be self maps of a metric space (X, d). If the pairs (A, S) and (B, T) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), then*

1. *the pair (A, S) has a coincidence point,*
2. *the pair (B, T) has a coincidence point.*
3. *For all $x, y \in X$,*

$$\phi(d(Ax, By)) \leq \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\} \tag{4.3}$$

where ϕ is a lower semi-continuous function $\phi : [0, 1] \to [0, 1]$ such that $\phi(t) > t$ for all $t \in (0, 1)$, $\phi(0) = 0$ and $\phi(1) = 1$. Then A, B, S and T have a unique common fixed point in X.

Remark 4.4. We can obtain the metrical versions of corresponding results (that is, Corollary 3.4, Corollary 3.5 and Corollary 3.6) in respect of Remark 3.9 due to Theorem 4.1, Theorem 4.2 and Theorem 4.3.

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