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Unique common fixed point theorems on partial metric spaces

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Abstract

We prove the existence of the unique common fixed point theorems for self mappings which are weakly compatible satisfying some contractive conditions on partial metric spaces. Furthermore, we also prove the result on the continuity in the set of common fixed points for self mappings on partial metric spaces.

Keywords: Common fixed points, Weakly compatible mappings, Coincidence points, Partial metric spaces.
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1. Introduction and Preliminaries

The common fixed point theorems for mappings satisfying certain contractive conditions in metric spaces have been continually studied for decade (see [1, 3, 5, 6, 8, 9, 10, 11, 12, 14, 15] and references contained therein). In 1976, Jungck [7] proved the existence of common fixed point theorems for commuting mappings in metric spaces where the results require the continuity of one of two such mappings. In 1986, Jungck [8] introduced the concept of compatible mappings and proved that weakly commuting mappings are compatible mappings. After that, Jungck [10], generalized the notion of compatibility by introducing the weakly compatibility.

Recently, Abbas et al. [1] introduced the generalized condition (B) as the following:

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Definition 1.1. Let X be a metric space. A mapping $F : X \to X$ is said to satisfy a generalized condition (B) associated with a self mapping f on X if there exists $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(Fx, Fy) \leq \delta M(x, y) + L \min\{d(fx, Fx), d(fy, Fy), d(fx, Fy), d(fy, Fx)\}, \quad (1.1)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(fx, fy), d(fx, Fx), d(fy, Fy), \frac{1}{2}[d(fx, Fy) + d(fy, Fx)]\}.$$

Abbas et al. [1] established the existence of a unique common fixed point for two self mappings F and f on X where F satisfies a generalized condition (B) associated with f. In this work, we assure the analogous results proved by Abbas et al. [1] for four self mappings in partial metric spaces.

Mathews [13] introduced the notion of partial metric spaces. We now recall some definitions and lemmas that will be used in the sequel.

Definition 1.2. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A pair (X, p) is called a partial metric space and p is a partial metric on X.

If p is a partial metric on X, then p generates a T_0 topology τ_p on X whose base is the family of open $p-$balls

$$\{B_p(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\},$$

where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$. For each partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1.2)$$

is a usual metric on X.

Definition 1.3. Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.

(2) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).

(3) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Lemma 1.4. [13] Let (X, p) be a partial metric space. Then

(1) A sequence $\{x_n\}$ in a partial metric space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, p^s).

(2) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover,

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \text{ iff } \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

(3) A subset E of a partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Lemma 1.5. [2] Let (X,p) be a partial metric space. Then

(1) If $p(x,y)=0$, then $x=y$.

(2) If $x \neq y$, then $p(x,y)>0$.

Definition 1.6. Let (X,p) be a partial metric space. A mapping $f: X \to X$ is continuous at $x \in X$ if the sequence $\{fx_n\}$ converges to fx for every sequence $\{x_n\}$ in X converging to x.

Definition 1.7. Let f and g are self mappings on a set X. A point $x \in X$ is called a coincidence point of f and g if $fx = gx = w$ where w is called a point of coincidence of f and g.

Definition 1.8. Two self mappings f and g on a set X are said to be weakly compatible if f and g commute at their coincidence points. That is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

In this paper, we prove the uniqueness of a common fixed point of four self mappings on a partial metric space (X,p) satisfying the certain contractive condition and being the weak compatibility. Moreover, we also prove the result on the continuity in the set of common fixed points for self mappings.

2. Main results

We now prove the existence of the unique common fixed point theorems for four self mappings which are weakly compatible on a partial metric space (X,p). The proofs of the mentioned theorems have been taken from the technique used in [1] in the setting of metric spaces.

Theorem 2.1. *Let (X,p) be a complete partial metric space. Suppose that f,g,F and G are self mappings on X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$.

(b) There exist $\delta > 0$ and $L \geq 0$ with $\delta + 2L < 1$ such that

$$p(Fx, fy) \leq \delta M(x,y) + L \min\{p(gx,Fx), p(Gy,fy), p(gx,fy), p(Gy,Fx)\}, \tag{2.1}$$

for all $x,y \in X$, where

$$M(x,y) = \max\{p(gx,Gy), p(gx,Fx), p(Gy,fy), \frac{1}{2}[p(gx,fy)+p(Gy,Fx)]\}.$$

(c) $f(X)$ or $g(X)$ is closed.

If $\{f,G\}$ and $\{g,F\}$ are weakly compatible, then f,g,F and G have a unique common fixed point in X.

Proof. Suppose that x_0 is an arbitrary point in X. Since $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$, we can construct a sequence $\{y_n\}$ in X satisfying

$$y_n = Fx_n = Gx_{n+1} \text{ and } y_{n+1} = fx_{n+1} = gx_{n+2} \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

By applying (2.1), we have

$$\begin{aligned} p(Fx_n, fx_{n+1}) &\leq \delta M(x_n, x_{n+1}) + L \min\{p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}), \\ &\quad p(gx_n, fx_{n+1}), p(Gx_{n+1}, Fx_n)\}. \end{aligned}$$

Since

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{p(gx_n, Gx_{n+1}), p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}), \\ &\quad \frac{1}{2}[p(gx_n, fx_{n+1}) + p(Gx_{n+1}, Fx_n)]\} \\ &= \max\{p(y_{n-1}, y_n), p(y_{n-1}, y_n), p(y_n, y_{n+1}), \\ &\quad \frac{1}{2}[p(y_{n-1}, y_{n+1}) + p(y_n, y_n)]\} \\ &\leq \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1}), \\ &\quad \frac{1}{2}[p(y_{n-1}, y_n) + p(y_n, y_{n+1}) - p(y_n, y_n) + p(y_n, y_n)]\} \\ &\leq \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}, \end{aligned}$$

and

$$\begin{aligned} &\min\{p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}), p(gx_n, fx_{n+1}) + p(Gx_{n+1}, Fx_n)\} \\ &\quad = \min\{p(y_{n-1}, y_n), p(y_n, y_{n+1}), p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} \\ &\quad = \min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\}, \end{aligned}$$

we obtain that

$$\begin{aligned} p(y_n, y_{n+1}) &= p(Fx_n, fx_{n+1}) \\ &\leq \delta \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} + L \min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\}. \end{aligned}$$

We separate the proof into the following cases.
Case I : If $\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} = p(y_{n-1}, y_n)$ and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_{n-1}, y_{n+1})$, then

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \delta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_{n+1}) \\ &\leq \delta p(y_{n-1}, y_n) + L(p(y_{n-1}, y_n) + p(y_n, y_{n+1}) - p(y_n, y_n)) \\ &\leq \delta p(y_{n-1}, y_n) + Lp(y_{n-1}, y_n) + Lp(y_n, y_{n+1}). \end{aligned}$$

This implies that

$$p(y_n, y_{n+1}) \leq \frac{\delta + L}{1 - L} p(y_{n-1}, y_n).$$

Let $k_1 = \frac{\delta+L}{1-L}$. Since $\delta + 2L < 1$, we have $k_1 < 1$. Therefore

$$p(y_n, y_{n+1}) \leq k_1 p(y_{n-1}, y_n).$$

Case II : If $\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} = p(y_{n-1}, y_n)$ and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_n, y_n)$, then

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \delta p(y_{n-1}, y_n) + Lp(y_n, y_n) \\ &\leq \delta p(y_{n-1}, y_n) + Lp(y_n, y_{n+1}). \end{aligned}$$

This implies that

$$p(y_n, y_{n+1}) \leq \frac{\delta}{1 - L} p(y_{n-1}, y_n).$$

Let $k_2 = \frac{\delta}{1-L}$. Since $\delta + 2L < 1$, we have $k_2 < 1$. Therefore

$$p(y_n, y_{n+1}) \leq k_2 p(y_{n-1}, y_n).$$

Case III : If $\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} = p(y_n, y_{n+1})$ and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_{n-1}, y_{n+1})$, then

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \delta p(y_n, y_{n+1}) + Lp(y_{n-1}, y_{n+1}) \\ &\leq \delta p(y_n, y_{n+1}) + L(p(y_{n-1}, y_n) + p(y_n, y_{n+1}) - p(y_n, y_n)) \\ &\leq \delta p(y_n, y_{n+1}) + Lp(y_{n-1}, y_n) + Lp(y_n, y_{n+1}). \end{aligned}$$

This implies that

$$p(y_n, y_{n+1}) \leq \frac{L}{1-(\delta+L)} p(y_{n-1}, y_n).$$

Let $k_3 = \frac{L}{1-(\delta+L)}$. Since $\delta + 2L < 1$, we have $k_3 < 1$. Therefore

$$p(y_n, y_{n+1}) \leq k_3 p(y_{n-1}, y_n).$$

Case IV : If $\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\} = p(y_n, y_{n+1})$ and $\min\{p(y_{n-1}, y_{n+1}), p(y_n, y_n)\} = p(y_n, y_n)$, then

$$\begin{aligned} p(y_n, y_{n+1}) &\leq \delta p(y_n, y_{n+1}) + Lp(y_n, y_n) \\ &\leq \delta p(y_n, y_{n+1}) + Lp(y_{n-1}, y_n). \end{aligned}$$

This implies that

$$p(y_n, y_{n+1}) \leq \frac{L}{1-\delta} p(y_{n-1}, y_n).$$

Let $k_4 = \frac{L}{1-\delta}$. Since $\delta + 2L < 1$, we have $k_4 < 1$. Therefore

$$p(y_n, y_{n+1}) \leq k_4 p(y_{n-1}, y_n).$$

Choose $k = \max\{k_1, k_2, k_3, k_4\}$. Therefore $0 < k < 1$. For each $n \in \mathbb{N}$, we obtain that

$$p(y_n, y_{n+1}) \leq k^n p(y_0, y_1). \tag{2.2}$$

We will prove that $\{y_n\}$ is a Cauchy sequence in (X, p^s). Let $m, n \in \mathbb{N}$ with $m > n$. By applying (2.2), we have

$$\begin{aligned} p(y_m, y_n) &\leq [p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m)] \\ &\quad -[p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) + p(y_{m-1}, y_{m-1})] \\ &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\ &\leq [k^n + k^{n+1} + \cdot + k^{m-1}] p(y_0, y_1) \\ &\leq \frac{k^n}{1-k} p(y_0, y_1). \end{aligned}$$

It follows that

$$\lim_{n,m\to\infty} p(y_m, y_n) = 0. \tag{2.3}$$

Using (1.2), we have

$$\begin{aligned} p^s(y_m, y_n) &= 2p(y_m, y_n) - p(y_m, y_m) - p(y_n, y_n) \\ &\leq 2p(y_m, y_n). \end{aligned}$$

Applying (2.3), we obtain that

$$\lim_{n,m\to\infty} p^s(y_m, y_n) = 0. \tag{2.4}$$

This implies that $\{y_n\}$ is a Cauchy sequence in (X, p^s). Since X is complete, we have

$$\lim_{n\to\infty} y_n = z \text{ for some } z \in X. \tag{2.5}$$

By Lemma 1.4 and (2.5), we obtain that

$$p(z,z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n,m\to\infty} p(y_m, y_n) \tag{2.6}$$

From (2.3) and (2.6), we can conclude that $p(z,z) = 0$. Assume that $g(X)$ is closed. Therefore there exists a point $u \in X$ such that $z = gu$. Using (2.1), this yields

$$\begin{aligned}
p(z, Fu) &\leq p(z, y_{n+1}) + p(y_{n+1}, Fu) - p(y_{n+1}, y_{n+1}) \\
&\leq p(z, y_{n+1}) + p(Fu, fx_{n+1}) \\
&\leq p(z, y_{n+1}) + \delta \max\{p(gu, Gx_{n+1}), p(gu, Fu), p(Gx_{n+1}, fx_{n+1}), \\
&\quad \frac{1}{2}[p(gu, fx_{n+1}) + p(Gx_{n+1}, Fu)]\} + L \min\{p(gu, Fu), p(Gx_{n+1}, fx_{n+1}), \\
&\quad p(gu, fx_{n+1}), p(Gx_{n+1}, Fu)\} \\
&= p(z, y_{n+1}) + \delta \max\{p(z, y_n), p(z, Fu), p(y_n, y_{n+1}), \\
&\quad \frac{1}{2}[p(z, y_{n+1}) + p(y_n, Fu)]\} + L \min\{p(z, Fu), p(y_n, y_{n+1}), p(z, y_{n+1}), p(y_n, Fu)\} \\
&\leq p(z, y_{n+1}) + \delta \max\{p(z, y_n), p(z, Fu), p(y_n, z) + p(z, y_{n+1}) - p(z, z), \\
&\quad \frac{1}{2}[p(z, y_{n+1}) + p(y_n, z) + p(z, Fu) - p(z, z)]\} + L \min\{p(z, Fu), p(y_n, z) + p(z, y_{n+1}) \\
&\quad -p(z, z), p(z, y_{n+1}), p(y_n, z) + p(z, Fu) - p(z, z)\} \\
&\leq p(z, y_{n+1}) + \delta \max\{p(z, y_n), p(z, Fu), p(y_n, z) + p(z, y_{n+1}), \\
&\quad \frac{1}{2}[p(z, y_{n+1}) + p(y_n, z) + p(z, Fu)]\} + L \min\{p(z, Fu), p(y_n, z) + p(z, y_{n+1}), \\
&\quad p(z, y_{n+1}), p(y_n, z) + p(z, Fu)\}.
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the fact that $p(z,z) = 0$, we have

$$p(z, Fu) \leq \delta p(z, Fu) + Lp(z, Fu) = (\delta + L)p(z, Fu).$$

It follows that $p(z, Fu) = 0$ and so $Fu = z = gu$. Since F and g are weakly compatible, we obtain that $gFu = Fgu$. Therefore $gz = Fz$.

Since $F(X) \subseteq G(X)$, there exists a point $v \in X$ such that $z = Gv$. Applying (2.1), we have

$$\begin{aligned}
p(z, fv) &= p(Fu, fv) \\
&\leq \delta \max\{p(gu, Gv), p(gu, Fu), p(Gv, fv), \frac{1}{2}[p(gu, fv) + p(Gv, Fu)]\} + \\
&\quad L \min\{p(gu, Fu), p(Gv, fv), p(gu, fv), p(Gv, Fu)\} \\
&= \delta \max\{p(z, z), p(z, z), p(z, fv), \frac{1}{2}[p(z, fv) + p(z, z)]\} + \\
&\quad L \min\{p(z, z), p(z, fv), p(z, fv), p(z, z)\} \\
&\leq \delta p(z, fv).
\end{aligned}$$

This implies that $p(z, fv) = 0$ and so $fv = z = Gv$. Since G and f are weakly compatible, we obtain that $fGv = Gfv$. Therefore $fz = Gz$. We next prove that z is a common fixed point of f, g, F and G. Using

(2.1), this yields

$$\begin{aligned} p(Fz,z) &= p(Fz,fv) \\ &\leq \delta \max\{p(gz,Gv),p(gz,Fz),p(Gv,fv),\frac{1}{2}[p(gz,fv)+p(Gv,Fz)]\}+ \\ &\quad L\min\{p(gz,Fz),p(Gv,fv),p(gz,fv),p(Gv,Fz)\} \\ &= \delta \max\{p(Fz,z),p(Fz,Fz),p(z,z),\frac{1}{2}[p(Fz,z)+p(z,Fz)]\}+ \\ &\quad L\min\{p(Fz,Fz),p(z,z),p(Fz,z),p(z,Fz)\} \\ &\leq \delta \max\{p(Fz,z),p(Fz,z),p(z,z),\frac{1}{2}[p(Fz,z)+p(z,Fz)]\}+ \\ &\quad L\min\{p(Fz,Fz),p(z,z),p(Fz,z),p(z,Fz)\} \\ &\leq \delta p(Fz,z). \end{aligned}$$

This implies that $p(Fz,z)=0$ and so $gz=Fz=z$. Similarly, applying (2.1), we obtain that

$$\begin{aligned} p(z,fz) &= p(Fz,fz) \\ &\leq \delta \max\{p(gz,Gz),p(gz,Fz),p(Gz,fz),\frac{1}{2}[p(gz,fz)+p(Gz,Fz)]\}+ \\ &\quad L\min\{p(gz,Fz),p(Gz,fz),p(gz,fz),p(Gz,Fz)\} \\ &= \delta \max\{p(z,fz),p(z,z),p(fz,fz),\frac{1}{2}[p(z,fz)+p(fz,z)]\}+ \\ &\quad L\min\{p(z,z),p(fz,fz),p(z,fz),p(fz,z)\} \\ &\leq \delta \max\{p(z,fz),p(z,z),p(fz,z),\frac{1}{2}[p(z,fz)+p(fz,z)]\}+ \\ &\quad L\min\{p(z,z),p(fz,fz),p(z,fz),p(fz,z)\} \\ &\leq \delta p(z,fz). \end{aligned}$$

This implies that $p(z,fz)=0$ and so $Gz=fz=z$. Therefore z is a common fixed point of f,g,F and G. We will prove the uniqueness of a common fixed point of f,g,F and G. Let w be any common fixed point of f,g,F and G. By applying (2.1), it follows that

$$\begin{aligned} p(z,w) &= p(Fz,fw) \\ &\leq \delta \max\{p(gz,Gw),p(gz,Fz),p(Gw,fw),\frac{1}{2}[p(gz,fw)+p(Gw,Fz)]\}+ \\ &\quad L\min\{p(gz,Fz),p(Gw,fw),p(gz,fw),p(Gw,Fz)\} \\ &= \delta \max\{p(z,w),p(z,z),p(w,w),\frac{1}{2}[p(z,w)+p(w,z)]\}+ \\ &\quad L\min\{p(z,z),p(w,w),p(z,w),p(w,z)\} \\ &\leq \delta p(z,w). \end{aligned}$$

This implies that $p(z,w)=0$ and so $z=w$. Hence f,g,F and G have a unique common fixed point in X. □

Letting $F=f$ and $G=g$ in Theorem 2.1, we immediately obtain the following corollary:

Corollary 2.2. *Let (X,p) be a partial metric space. Suppose that f and g are self mappings on X satisfying the following conditions:*

(a) $f(X)\subseteq g(X)$.

(b) There exist $\delta>0$ and $L\geq 0$ with $\delta+2L<1$ such that

$$p(fx,fy)\leq \delta M(x,y)+L\min\{p(gx,fx),p(gy,fy),p(gx,fy),p(gy,fx)\}, \tag{2.7}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{1}{2}[p(gx,fy) + p(gy,fx)]\}.$$

(c) *$f(X)$ or $g(X)$ is complete.*

If $\{f,g\}$ is weakly compatible, then f and g have a unique common fixed point in X.

Theorem 2.3. *Let (X,p) be a complete partial metric space. Suppose that f, g, F and G are self mappings on X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$.*

(b) *There exist $\delta > 0$ and $L \geq 0$ with $\delta + 2L < 1$ such that*

$$p(Fx,fy) \leq \delta M(x,y) + L\min\{p(gx,Fx), p(Gy,fy), p(gx,fy), p(Gy,Fx)\}, \tag{2.8}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{p(gx,Gy), \frac{1}{2}[p(gx,Fx) + p(Gy,fy)], \frac{1}{2}[p(gx,fy) + p(Gy,Fx)]\}.$$

(c) *$f(X)$ or $g(X)$ is closed.*

If $\{f,G\}$ and $\{g,F\}$ are weakly compatible, then f, g, F and G have a unique common fixed point in X.

Proof. Since the inequality (2.8) implies the inequality (2.1), we have the result obtained from Theorem 2.1. □

Theorem 2.4. *Let (X,p) be a complete partial metric space. Suppose that f, g, F and G are self mappings on X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$.*

(b) *There exist $\delta > 0$ and $L \geq 0$ with $\delta + L < \frac{1}{2}$ such that*

$$p(Fx,fy) \leq \delta M(x,y) + L\min\{p(gx,Fx), p(Gy,fy), p(gx,fy), p(Gy,Fx)\}, \tag{2.9}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{p(gx,Gy), p(gx,Fx), p(Gy,fy), p(gx,fy), p(Gy,Fx)\}.$$

(c) *$f(X)$ or $g(X)$ is closed.*

If $\{f,G\}$ and $\{g,F\}$ are weakly compatible, then f, g, F and G have a unique common fixed point in X.

Proof. Suppose that x_0 is an arbitrary point in X. Since $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$, we can construct a sequence $\{y_n\}$ in X satisfying

$$y_n = Fx_n = Gx_{n+1} \text{ and } y_{n+1} = fx_{n+1} = gx_{n+2} \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Applying (2.9), this yields

$$\begin{aligned} p(Fx_n, fx_{n+1}) \leq \;& \delta M(x_n, x_{n+1}) + L\min\{p(gx_n, Fx_n), p(Gx_{n+1}, fx_{n+1}), \\ & p(gx_n, fx_{n+1}), p(Gx_{n+1}, Fx_n)\}. \end{aligned}$$

Since

$$\begin{aligned} M(x_n,x_{n+1}) &= \max\{p(gx_n,Gx_{n+1}),p(gx_n,Fx_n),p(Gx_{n+1},fx_{n+1}),\\ &\quad p(gx_n,fx_{n+1}),p(Gx_{n+1},Fx_n)\}\\ &= \max\{p(y_{n-1},y_n),p(y_{n-1},y_n),p(y_n,y_{n+1}),p(y_{n-1},y_{n+1}),p(y_n,y_n)\}\\ &= \max\{p(y_{n-1},y_n),p(y_n,y_{n+1}),p(y_{n-1},y_{n+1})\}\\ &\leq \max\{p(y_{n-1},y_n),p(y_n,y_{n+1}),p(y_{n-1},y_n)+p(y_n,y_{n+1})-p(y_n,y_n)\}\\ &\leq \max\{p(y_{n-1},y_n),p(y_n,y_{n+1}),p(y_{n-1},y_n)+p(y_n,y_{n+1})\}\\ &= p(y_{n-1},y_n)+p(y_n,y_{n+1}), \end{aligned}$$

and

$$\begin{aligned} &\min\{p(gx_n,Fx_n),p(Gx_{n+1},fx_{n+1}),p(gx_n,fx_{n+1})+p(Gx_{n+1},Fx_n)\}\\ &\quad= \min\{p(y_{n-1},y_n),p(y_n,y_{n+1}),p(y_{n-1},y_{n+1}),p(y_n,y_n)\}\\ &\quad= \min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}, \end{aligned}$$

we obtain that

$$\begin{aligned} p(y_n,y_{n+1}) &= p(Fx_n,fx_{n+1})\\ &\leq \delta(p(y_{n-1},y_n)+p(y_n,y_{n+1}))+L\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}. \end{aligned}$$

We separate the proof into the following cases.
Case I : If $\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}=p(y_{n-1},y_{n+1})$, then

$$\begin{aligned} p(y_n,y_{n+1}) &\leq \delta(p(y_{n-1},y_n)+p(y_n,y_{n+1}))+Lp(y_{n-1},y_{n+1})\\ &\leq \delta p(y_{n-1},y_n)+\delta p(y_n,y_{n+1})+L(p(y_{n-1},y_n)+p(y_n,y_{n+1})-p(y_n,y_n))\\ &\leq \delta p(y_{n-1},y_n)+\delta p(y_n,y_{n+1})+Lp(y_{n-1},y_n)+Lp(y_n,y_{n+1}). \end{aligned}$$

This implies that

$$p(y_n,y_{n+1}) \leq \frac{\delta+L}{1-(\delta+L)}p(y_{n-1},y_n).$$

Let $k_1=\frac{\delta+L}{1-(\delta+L)}$. Since $\delta+L<\frac{1}{2}$, we have $k_1<1$. Therefore

$$p(y_n,y_{n+1}) \leq k_1 p(y_{n-1},y_n).$$

Case II : If $\min\{p(y_{n-1},y_{n+1}),p(y_n,y_n)\}=p(y_n,y_n)$, then

$$\begin{aligned} p(y_n,y_{n+1}) &\leq \delta(p(y_{n-1},y_n)+p(y_n,y_{n+1}))+Lp(y_n,y_n)\\ &\leq \delta p(y_{n-1},y_n)+\delta p(y_n,y_{n+1})+Lp(y_{n-1},y_n). \end{aligned}$$

This implies that

$$p(y_n,y_{n+1}) \leq \frac{\delta+L}{1-\delta}p(y_{n-1},y_n).$$

Let $k_2=\frac{\delta+L}{1-\delta}$. Since $\delta+L<\frac{1}{2}$, we have $k_2<1$. Therefore

$$p(y_n,y_{n+1}) \leq k_2 p(y_{n-1},y_n).$$

Choose $k=\max\{k_1,k_2\}$. Therefore $0<k<1$. For each $n\in\mathbb{N}$, we obtain that

$$p(y_n,y_{n+1}) \leq k^n p(y_0,y_1). \tag{2.10}$$

We can complete the proof by the same arguments appeared in Theorem 2.1. □

Letting $F=f$ and $G=g$ in Theorem 2.4, we immediately have the following result:

Corollary 2.5. *Let (X, p) be a partial metric space. Suppose that f and g are self mappings on X satisfying the following conditions:*

(a) $f(X) \subseteq g(X)$.

(b) There exist $\delta > 0$ and $L \geq 0$ with $\delta + L < \frac{1}{2}$ such that

$$p(fx, fy) \leq \delta M(x, y) + L \min\{p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}, \tag{2.11}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}.$$

(c) $f(X)$ or $g(X)$ is complete.

If $\{f, G\}$ is weakly compatible, then f and g have a unique common fixed point in X.

We finally prove the result on the continuity in the set of common fixed points for self mappings in partial metric spaces.

Theorem 2.6. *Let (X, p) be a partial metric space. Suppose that f, g and T are self mappings on X satisfying the following conditions:*

(a) There exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$p(Tx, fy) \leq \delta M(x, y) + L \min\{p(gx, Tx), p(gy, fy), p(gx, fy), p(gy, Tx)\}, \tag{2.12}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{p(gx, gy), p(gx, Tx), p(gy, fy), \frac{1}{2}[p(gx, fy) + p(gy, Tx)]\}.$$

(b) The set $F(f, g, T) = \{z \in X : fz = gz = Tz = z, p(z, z) = 0\}$ of all common fixed points of f, g and T is nonempty.

If g is continuous at $z \in F(f, g, T)$, then f and T are continuous at z.

Proof. Assume that $z \in F(f, g, T)$ and $\{x_n\}$ is a sequence in X converging to z. Using (2.12), we obtain that

$$p(Tz, fx_n) \leq \delta M(z, x_n) + L \min\{p(gz, Tz), p(gx_n, fx_n), p(gz, fx_n), p(gx_n, Tz)\},$$

where

$$M(z, x_n) = \max\{p(gz, gx_n), p(gz, Tz), p(gx_n, fx_n), \frac{1}{2}[p(gz, fx_n) + p(gx_n, Tz)]\}.$$

This implies that

$$\begin{aligned} p(Tz, fx_n) &\leq \delta \max\{p(gz, gx_n), p(z, z), p(gx_n, fx_n), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)]\} + \\ &\quad L \min\{p(z, z), p(gx_n, fx_n), p(fz, fx_n), p(gx_n, gz)\} \\ &\leq \delta \max\{p(gz, gx_n), p(gx_n, gz) + p(fz, fx_n) - p(z, z), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)]\} \\ &\leq \delta \max\{p(gz, gx_n), p(gx_n, gz) + p(fz, fx_n), \frac{1}{2}[p(fz, fx_n) + p(gx_n, gz)]\} \\ &= \delta(p(gx_n, gz) + p(fz, fx_n)). \end{aligned}$$

It follows that

$$p(fz, fx_n) \leq \delta(p(gx_n, gz) + p(fz, fx_n)).$$

Therefore

$$p(fz, fx_n) \leq \frac{\delta}{1-\delta} p(gx_n, gz). \tag{2.13}$$

By continuity of g, we obtain that

$$\lim_{n\to\infty} p(gx_n, gz) = p(gz, gz) = p(z, z) = 0.$$

Using (2.13), this yields

$$\lim_{n\to\infty} p(fz, fx_n) = 0.$$

This implies that f is continuous at z. Similarly, by applying (2.12), we have

$$p(Tx_n, fz) \leq \delta M(x_n, z) + L\min\{p(gx_n, Tx_n), p(gz, fz), p(gx_n, fz), p(gz, Tx_n)\},$$

where

$$M(x_n, z) = \max\{p(gx_n, gz), p(gx_n, Tx_n), p(gz, fz), \frac{1}{2}[p(gx_n, fz) + p(gz, Tx_n)]\}.$$

This implies that

$$\begin{aligned}
p(Tx_n, fz) &\leq \delta \max\{p(gx_n, gz), p(gx_n, Tx_n), p(z, z), \frac{1}{2}[p(gx_n, gz) + p(Tz, Tx_n)]\} + \\
&\quad L\min\{p(gx_n, Tx_n), p(z, z), p(gx_n, gz), p(Tz, Tx_n)\} \\
&\leq \delta \max\{p(gx_n, gz), p(gx_n, gz) + p(Tz, Tx_n) - p(z, z), \frac{1}{2}[p(gx_n, gz) + p(Tz, Tx_n)]\} \\
&\leq \delta \max\{p(gx_n, gz), p(gx_n, gz) + p(Tz, Tx_n), \frac{1}{2}[p(gx_n, gz) + p(Tz, Tx_n)]\} \\
&= \delta(p(gx_n, gz) + p(Tz, Tx_n)).
\end{aligned}$$

Therefore

$$p(Tx_n, Tz) \leq \frac{\delta}{1-\delta} p(gx_n, gz). \tag{2.14}$$

By continuity of g, we obtain that

$$\lim_{n\to\infty} p(Tx_n, Tz) = 0.$$

This implies that T is continuous at z. □

If $T = f$ in Theorem 2.6, then we obtain the following results:

Corollary 2.7. *Let (X, p) be a partial metric space. Suppose that f and g are self mappings on X satisfying the following conditions:*

(a) There exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$p(fx, fy) \leq \delta M(x, y) + L\min\{p(gx, fx), p(gy, fy), p(gx, fy), p(gy, fx)\}, \tag{2.15}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{p(gx, gy), p(gx, fx), p(gy, fy), \frac{1}{2}[p(gx, fy) + p(gy, fx)]\}.$$

(b) The set $F(f, g) = \{z \in X : fz = gz = z, p(z, z) = 0\}$ of all common fixed points of f and g is nonempty.

If g is continuous at $z \in F(f, g)$, then f is continuous at z.

Corollary 2.8. *(Theorem 2.7, [1]) Let (X, d) be a metric space. Suppose that f and g are self mappings on X satisfying the following conditions:*

(a) There exist $\delta \in (0, 1)$ and $L \geq 0$ such that

$$d(fx, fy) \leq \delta M(x, y) + L \min\{d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}, \tag{2.16}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}[d(gx, fy) + d(gy, fx)]\}.$$

(b) The set $F(f, g) = \{z \in X : fz = gz = z\}$ of all common fixed points of f and g is nonempty.

If g is continuous at $z \in F(f, g)$, then f is continuous at z.

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References

[1] M. Abbas, G.V.R. Babu and G.N. Alemayehu, *On common fixed points of weakly compatible mappings satisfying generalized condition*, Filomat. **25** (2011) 9–19.
[2] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric space*, J. Math. Anal. Appl. **341** (2008) 416–420. 1, 1, 2, 2.8
[3] M. Abbas, S. H. Khan and T. Nazir, *Common fixed points of R-weakly commuting maps in generalized metric spaces*, Fixed Point Theory Appl. **2011** (2011) 41pp. 1.5
[4] M. Abbas, T. Nazir and S. Radenović, *Some periodic point results in generalized metric spaces*, Appl. Math. Comput. **217** (2010) 195–202. 1
[5] M. Abbas and B.E. Rhoades, *Common fixed point results for noncommuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput. **215** (2009) 262–269. 1
[6] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. **2006** (2006) (Article ID 74503, 7 pages). 1
[7] G. Jungck, *Commuting maps and fixed points*, Amer. Math. Monthly **83** (1976) 261–263. 1
[8] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Sci. **9** (1986) 771–779. 1
[9] G. Jungck, *Common fixed points for commuting and compatible maps on compacta*, Proc. Amer. Math. Soc. **103** (1988) 977–983. 1
[10] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. **4** (1996) 199–215. 1
[11] G. Jungck and N. Hussain, *Compatible maps and invariant approximations*, J. Math. Model. Algorithms **325** (2007) 1003–1012. 1
[12] D. Mihet, *Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces*, J. Nonlinear Sci. Appl. **6** (2013), 35–40. 1
[13] S. G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci. **728** (1994), 183–197. 1, 1.4
[14] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. *188* (1994) 436–440. 1
[15] C. Vetro and F. Vetro, *Common fixed points of mappings satisfying implicit relations in partial metric spaces*, J. Nonlinear Sci. Appl. **6** (2013), 152–161. 1