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On a new class of abstract impulsive functional differential equations of fractional order

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Abstract

In this paper, we prove the existence and uniqueness of mild solutions for the impulsive fractional differential equations for which the impulses are not instantaneous in a Banach space H. The results are obtained by using the analytic semigroup theory and the fixed points theorems.

Keywords: Impulsive fractional differential equations, Analytic semigroup, Fixed point theorems.
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1. Introduction

Fractional calculus refers to integration or differentiation of any (i.e., non-integer) order. The field has a history as old as calculus itself, which did not attract enough attention for a long time. In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences. For a nice introduction, we refer to the reader to [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and references cited therein.

Impulsive effects are common phenomena due to short-term perturbations whose duration is negligible in comparison with the total duration of the original process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The governing equations of such phenomena may be modeled as impulsive differential equations. In recent years, there has been a growing interest in the study of impulsive differential equations as these equations provide a natural frame

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work for mathematical modelling of many real world phenomena, namely in the control theory, physics, chemistry, population dynamics, biotechnology, economics and medical fields.

Due to the great development in the theory of fractional calculus and impulsive differential equations as well as having wide applications in several fields. Recently, the study of fractional differential equations with impulses has been studied by many authors (see [17, 18, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38]).

In [16], the authors introduced a new class of abstract differential equations for which the impulses are not instantaneous and investigated the existence of mild and classical solutions for the following system:

$$u'(t) = Au(t) + f(t, u(t)), \qquad t \in (s_i, t_{i+1}], \quad i = 0, 1, \cdots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)), \qquad t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \tag{1.2}$$

$$u(0) = u_0, \tag{1.3}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators, $\{S(t), t \geq 0\}$ on a Banach space $(H, \|.\|)$, the functions $g_i \in C((t_i, s_i] \times H; H)$ for each $i = 1, 2, \cdots, N$ and $f : [0, T_0] \times H \to H$ is suitable function.

Motivated by the work [16], In this article, we consider the following impulsive fractional differential equations in a Banach space $(H, \|.\|)$ for which impulses are not instantaneous:

$$\begin{aligned} {}^C D_t^{\beta} u(t) + Au(t) &= f(t, u(t), u(g(t))), \ t \in (s_i, t_{i+1}], i = 0, 1, \cdots, N, && (1.4) \\ u(t) &= h_i(t, u(t)), \ \ t \in (t_i, s_i], \ \ i = 1, 2, \cdots, N, && (1.5) \\ u(0) &= u_0 \in H, && (1.6) \end{aligned}$$

where ${}^C D_t^{\beta}$ is the Caputo fractional derivative of order β, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{S(t), t \geq 0\}$ on a Banach space H, the impulses start suddenly at the points t_i and their action continues on the interval $[t_i, s_i]$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 <, ..., < t_N \leq s_N \leq < t_{N+1} = T_0$, the functions $h_i \in C((t_i, s_i] \times H; H)$ for each $i = 1, 2, \cdots, N$, $g : [0, T_0] \to [0, T_0]$ and $f : [0, T_0] \times H \times H \to H$ are suitable functions.

The paper is organized as follows. In "Preliminaries and Assumptions" section, we provide some basic definitions, notations, lemmas and proposition which are used throughout the paper. In "Existence of mild solutions" section, we will prove some existence and uniqueness results concerning the $\mathcal{PC}$-mild solutions. In the last (i.e., In "Application") section, we give an example to demonstrate the application of the main results.

2. Preliminaries and assumptions

In this section, we will introduce some basic definitions, notations, lemmas and proposition which are used throughout this paper.

It is assume that $-A$ generates an analytic semigroup of bounded operators, denoted by $S(t)$, $t \geq 0$. It is known that there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq \tilde{M} e^{\omega t}, \quad t \geq 0.$$

If necessary, we may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ for $t \geq 0$, and that $0 \in \rho(-A)$, implies $-A$ is invertible. In this case, it is possible to define the fractional power A^{α} for $0 \leq \alpha \leq 1$ as closed linear operator with domain $D(A^{\alpha}) \subseteq H$. Furthermore, $D(A^{\alpha})$ is dense in H and the expression

$$\|x\|_{\alpha} = \|A^{\alpha} x\|,$$

defines a norm on $D(A^{\alpha})$. Henceforth, we denote the space $D(A^{\alpha})$ by H_{α} endowed with the norm $\| \cdot \|_{\alpha}$. Also, for each $\alpha > 0$, we define $H_{-\alpha} = (H_{\alpha})^*$, the dual space of H_{α}, is a $\|x\|_{-\alpha} = \|A^{-\alpha} x\|$. For more details, we refer to the reader to the book by Pazy [1, pp. 69].

Lemma 2.1. [1, pp. 72,74,195-196] *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following:*

(i) H_α is a Banach space for $0 \leq \alpha \leq 1$;
(ii) For any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $H_\alpha \hookrightarrow H_\delta$ is continuous;
(iii) The operator $A^\alpha S(t)$ is bounded, i.e., there exists a constant N such that

$$\|A^\alpha S(t)\| \leq N$$

and

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}$$

for each $t > 0$.

Lemma 2.2. [16, Lemma 1.1] *A set $B \subseteq \mathcal{PC}(H_\alpha)$ is relatively compact in $\mathcal{PC}(H_\alpha)$ if and only if set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; H_\alpha])$, where $\mathcal{PC}(H_\alpha)$ is the space of piecewise continuous functions from $[0, T_0]$ into H_α to be specified later.*

Definition 2.3. [37, Def. 2.7] By the mild solution of the following system

$$\begin{aligned} {}^C D_t^\beta u(t) + Au(t) &= h(t), \quad t \in [t_0, T_0], &(2.1)\\ u(t_0) &= u_0, &(2.2)\end{aligned}$$

we mean a continuous function $u : [t_0, T_0] \to H$ which satisfies the following integral equation

$$u(t) = \mathfrak{T}(t - t_0)u_0 + \int_{t_0}^{t} (t - s)^{\beta-1}\mathfrak{P}(t - s)h(s)ds, t \in [t_0, T_0],$$

where

$$\mathfrak{T}(t) = \int_0^\infty \xi_\beta(\theta)S(t^\beta\theta)d\theta, \quad \mathfrak{P}(t) = \beta \int_0^\infty \theta\xi_\beta(\theta)S(t^\beta\theta)d\theta,$$

$$\xi_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\rho_\beta(\theta^{-\frac{1}{\beta}}) \geq 0,$$

$$\rho_\beta(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\beta-1}\frac{\Gamma(n\beta + 1)}{n!}\sin(n\pi\beta), \quad \theta \in (0, \infty),$$

ξ_β is a probability density function defined on $(0, \infty)$, that is

$$\xi_\beta(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \int_0^\infty \xi_\beta(\theta) = 1,$$

and

$$\int_0^\infty \theta^\gamma\xi_\beta(\theta) = \int_0^\infty \frac{1}{\theta^{\gamma\beta}}\rho_\beta(\theta) = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \gamma\beta)}, \quad \text{for any} \quad \gamma \in [0, 1].$$

Lemma 2.4. *The operators $\mathfrak{T}(.)$ and $\mathfrak{P}(.)$ have the following properties:*

(i). $\{\mathfrak{T}(t), t \geq 0\}$ and $\{\mathfrak{P}(t), t \geq 0\}$ are strongly continuous.

(ii). If $\{S(t), t > 0\}$ is compact, then $\mathfrak{T}(t)$ and $\mathfrak{P}(t)$ are also compact operators for every $t > 0$.

(iii). For any fixed $t \geq 0$, $\mathfrak{T}(t)$ and $\mathfrak{P}(t)$ are linear and bounded operators, i.e., for any $x \in H$,

$$\|\mathfrak{T}(t)x\| \leq M\|x\| \qquad \text{and} \quad \|\mathfrak{P}(t)x\| \leq \frac{\beta M}{\Gamma(1 + \beta)}\|x\|.$$

Remark 2.5. Since $\mathfrak{T}(.)$ and $\mathfrak{P}(.)$ are associated with the β, there are no analogue of the semigroup property, i.e., $\mathfrak{T}(t + s) \neq \mathfrak{T}(t)\mathfrak{T}(s)$, $\mathfrak{P}(t + s) \neq \mathfrak{P}(t)\mathfrak{P}(s)$ for $t, s > 0$.

3. Existence of mild solutions

In this section, we prove the existence of mild solutions for the impulsive system (1.4)-(1.6). To begin, we use the following definition.

Definition 3.1. *[16, Def. 2.1]* A function $u \in \mathcal{PC}(H)$ is said to be a mild solution of the problem (1.4)-(1.6) if $u(0) = u_0,\ u(t) = h_i(t, u(t))$ for all $t \in (t_i, s_i]$, for each $i = 1, \cdots, N$ and

$$\begin{aligned} u(t) \quad &= \quad \int_0^\infty \xi_\beta(\theta) S(t^\beta \theta) u_0 d\theta \\ &+ \quad \beta \int_0^t \int_0^\infty \theta \xi_\beta(\theta)(t-s)^{\beta-1} S((t-s)^\beta \theta) f(s, u(s), u(g(s))) d\theta ds, \end{aligned}$$

for all $t \in [0, t_1]$ and

$$\begin{aligned} u(t) \quad &= \quad \int_0^\infty \xi_\beta(\theta) S((t-s_i)^\beta \theta) h_i(s_i, u(s_i)) d\theta \\ &+ \quad \beta \int_{s_i}^t \int_0^\infty \theta \xi_\beta(\theta)(t-s)^{\beta-1} S((t-s)^\beta \theta) f(s, u(s), u(g(s))) d\theta ds, \end{aligned}$$

for all $t \in [s_i, t_{i+1}]$, for each $i = 1, \cdots, N$.

We define the set of functions as follows

$$\begin{aligned} \mathcal{PC}(H_\alpha) = \{u : [0, T_0] \to H_\alpha : u(.) \ \text{ is continuous at } \ t \neq t_i, u(t_k^-) = u(t_k), u(t_k^+) \\ \text{exists } \ \text{for all } \ i = 1, 2, \cdots, N\}. \end{aligned}$$

$\mathcal{PC}(H_\alpha)$ is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC}} := \sup_{t \in I} \|u(t)\|_\alpha.$$

Now, we define the functions $\widetilde{u}_i \in C([t_i, t_{i+1}]; H_\alpha)$ given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i \end{cases}$$

Let $B \subseteq \mathcal{PC}(H_\alpha)$, we define

$$\widetilde{B}_i = \{\widetilde{u}_i : u \in B\}.$$

We shall use the following conditions on f and h_i in its arguments:

(H1) Let $W \subset \mathrm{Dom}(f)$ be an open subset of $\mathbb{R}_+ \times H_\alpha \times H_\alpha$, where $0 \le \alpha < 1$. For each $(t, u, v) \in W$, there is a neighborhood $V_1 \subset W$ of (t, u, v), such that the nonlinear map $f : \mathbb{R}_+ \times H_\alpha \times H_\alpha \to H$ satisfies the following condition,

$$\|f(t, u, v) - f(t, u_1, v_1)\| \le L_f\{\|u - u_1\|_\alpha + \|v - v_1\|_\alpha\}$$

for all $(t, u, v),\ (t, u_1, v_1) \in V_1,\ L_f = L_f(t, u, v, V_1) > 0$ is a constant.

(H2) Let $g : [0, T_0] \to [0, T_0]$ is continuous and satisfies the delay property $g(t) \le t$ for $t \in [0, T_0]$.

(H3) The functions $h_i : [t_i, s_i] \times H_\alpha \to H_\alpha$ are continuous and there are positive constants L_{h_i} such that

$$\|h_i(t,x) - h_i(t,y)\|_\alpha \leq L_{h_i}\|x-y\|_\alpha,$$

for all $x, y \in H_\alpha,\ t \in [t_i, s_i]$ and each $i = 0, 1, \cdots, N$.

(H4) For $u, v \in H_\alpha$, the function $f(., u, v)$ is strongly measurable on $[0, T_0]$ and $f(t, ., .) \in C(H_\alpha \times H_\alpha, H) \in$ for $t \in [0, T_0]$. There exists a constant $\beta_1 \in [0, \beta)$ and $m_f \in L^{\frac{1}{\beta_1}}([0, T_0], \mathbb{R}^+)$ such that $\|f(t, u, v)\| \leq m_f(t)$ for all $(t, u, v) \in [0, T_0] \times H_\alpha \times H_\alpha$.

Theorem 3.2. *Suppose the assumptions* (H1)-(H3) *hold and*

$$L = \max\Big\{ML_{h_i} + \frac{2C_\alpha L_f \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\beta(1-\alpha))}T_0^{\beta(1-\alpha)} :\ i = 1, \cdots, N\Big\} < 1. \tag{3.1}$$

Then there exists a unique mild solution $u \in \mathcal{PC}(H_\alpha)$ of the problem (1.4)-(1.6).

Proof. Let us define a map $\Upsilon : \mathcal{PC}(H_\alpha) \to \mathcal{PC}(H_\alpha)$, given by $\Upsilon u(0) = u_0,\ \Upsilon u(t) = h_i(t, u(t))$ for $t \in (t_i, s_i],\ i = 1, 2, \cdots, N$ and

$$\begin{aligned}\Upsilon u(t) &= \int_0^\infty \xi_\beta(\theta) S(t^\beta\theta) u_0 d\theta \\ &+ \beta \int_0^t \int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1} S((t-s)^\beta\theta) f(s, u(s), u(g(s))) d\theta ds,\end{aligned}$$

for all $t \in [0, t_1]$ and

$$\begin{aligned}\Upsilon u(t) &= \int_0^\infty \xi_\beta(\theta) S((t-s_i)^\beta\theta) h_i(s_i, u(s_i)) d\theta \\ &+ \beta \int_{s_i}^t \int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1} S((t-s)^\beta\theta) f(s, u(s), u(g(s))) d\theta ds, \\ & t \in [s_i, t_{i+1}],\ i = 1, 2, \cdots, N.\end{aligned} \tag{3.2}$$

Clearly, Υ is well defined.

Next we show that Υ is contraction on $\mathcal{PC}(H_\alpha)$.

Let $\phi, \psi \in \mathcal{PC}(H_\alpha),\ i \in \{1, \cdots, N\}$ and $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned}\|\Upsilon\phi(t) - \Upsilon\psi(t)\| &\leq \| \int_0^\infty \xi_\beta(\theta)\|S((t-s_i)^\beta\theta)\|\ \|h_i(s_i, \phi(s_i)) - h_i(s_i, \psi(s_i))\|_\alpha ds \\ &+ \beta \int_{s_i}^t \int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}\|A^\alpha S((t-s)^\beta\theta)\| \\ &\quad \times\|f(s, \phi(s), \phi(g(s))) - f(s, \psi(s), \psi(g(s)))\| d\theta ds \\ &\leq ML_{h_i}\|\phi-\psi\|_{\mathcal{PC}} + \frac{C_\alpha L_f \beta\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_{s_i}^t (t-s)^{\beta(1-\alpha)-1}[\|\phi(s) - \psi(s)\|_\alpha \\ &\quad +\|\phi(g(s)) - \psi(g(s))\|_\alpha] ds \\ &\leq [ML_{h_i} + \frac{2C_\alpha L_f\Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\beta(1-\alpha))}T_0^{\beta(1-\alpha)}]\|\phi-\psi\|_{\mathcal{PC}}.\end{aligned}$$

Thus, we have

$$\|\Upsilon\phi - \Upsilon\psi\|_{C([s_i,t_{i+1}];H_\alpha)} \leq [ML_{h_i} + \frac{2C_\alpha L_f \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\beta(1-\alpha))} T_0^{\beta(1-\alpha)}]\|\phi - \psi\|_{\mathcal{PC}}. \tag{3.3}$$

Continuing in this fashion, we get

$$\|\Upsilon\phi - \Upsilon\psi\|_{C([0,t_1];H_\alpha)} \leq \frac{2C_\alpha L_f \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\beta(1-\alpha))} T_0^{\beta(1-\alpha)} \|\phi - \psi\|_{\mathcal{PC}}, \tag{3.4}$$

$$\|\Upsilon\phi - \Upsilon\psi\|_{C([s_i,t_i];H_\alpha)} \leq L_{h_i}\|\phi - \psi\|_{\mathcal{PC}}, \quad i = 1, \cdots, N. \tag{3.5}$$

Hence, from (3.3)-(3.5), we have

$$\|\Upsilon\phi - \Upsilon\psi\|_{\mathcal{PC}} \leq L\|\phi - \psi\|_{\mathcal{PC}},$$

i.e., $\Upsilon(.)$ is a contraction and there exists a unique mild solution of (1.4)-(1.6). □

The next result concerning the existence and uniqueness of mild solutions for the impulsive system (1.4)-(1.6) under the assumptions (H3) and (H4).

Theorem 3.3. *Suppose the assumptions* (H3) *and* (H4) *hold. The semigroup* $\{S(t); t \geq 0\}$ *is compact, the functions* $h_i(.,0)$ *are bounded, (3.1) holds and for each* $u_0 \in H_\alpha$*, let* $r > 1$ *and* $0 < \delta < 1$ *be such that*

$$\begin{cases} M\|u_0\|_\alpha + (1+M)\max_{i=1,\cdots,N}\|h_i(.,0)\|_\alpha & \leq & (1-\delta)r, \\ \max_{i=1,\cdots,N}\left\{L_{h_i}(1+M)\|u\|_{\mathcal{PC}} + C_\alpha \frac{\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\frac{T_0^{\beta(1-\alpha)}}{1-\alpha}\aleph\right\} & \leq & \delta r, \\ \frac{C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\frac{T_0^{\beta(1-\alpha)}}{1-\alpha}\sup_{s\in[0,t_1],v\in B_r(0,\mathcal{PC}(H_\alpha))}\|f(s,v(s),v(g(s)))\| & \leq & \delta r, \end{cases} \tag{3.6}$$

where $\aleph = \sup_{s\in[s_i,t_{i+1}],v\in B_r(0,\mathcal{PC}(H_\alpha))}\|f(s,v(s),v(g(s)))\|$.
Then there exists a mild solution $u \in \mathcal{PC}(H_\alpha)$ of the impulsive problem (1.4)-(1.6).

Proof. Let

$$\Upsilon = \sum_{i=0}^{N}\Upsilon_i^1 + \sum_{i=0}^{N}\Upsilon_i^2, \quad \text{where} \quad \Upsilon_i^j : \mathcal{PC}(H_\alpha) \to \mathcal{PC}(H_\alpha), \ i = 0, 1, \cdots, N, \quad j = 1, 2,$$

are given by

$$\Upsilon_i^1 u(t) = \begin{cases} h_i(t, u(t)), & \text{for } t \in (t_i, s_i], i \geq 1, \\ \int_0^\infty \xi_\beta(\theta)S((t-s_i)^\beta\theta)h_i(s_i, u(s_i))ds, & \text{for } t \in (s_i, t_{i+1}], i \geq 1, \\ 0, & \text{for } t \notin [t_i, t_{i+1}], i \geq 0, \\ \int_0^\infty \xi_\beta(\theta)S(t^\beta\theta)u_0 d\theta, & \text{for } t \in [0, t_1], i = 1, \end{cases}$$

$$\Upsilon_i^2 u(t) = \begin{cases} \beta\int_{s_i}^t\int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}S((t-s)^\beta\theta) & \\ \times f(s, u(s), u(g(s)))d\theta ds, & \text{for } t \in (s_i, t_{i+1}], i \geq 0, \\ 0, & \text{for } t \notin (s_i, t_{i+1}], i \geq 0. \end{cases}$$

Also, let $\lambda = \frac{\beta-1}{1-\beta_1} \in (-1,0)$, $b = \frac{\beta(1-\alpha)-1}{1-\beta_1} > 0$, $M_1 = \|m_f\|_{L^{\frac{1}{\beta_1}}[0,T_0]}$, where $\beta_1 \in [0,\beta)$, $0 < \alpha \le 1$.

It is easy to see that $(t-s)^{\beta-1} \in L^{\frac{1}{1-\beta_1}}[0,t]$, for $t \in [0,T_0]$.

By using Hölder inequality and (H4), for $t \in [t_1,t_2] \subseteq [0,T_0]$, we get

$$\begin{aligned}\int_{t_1}^{t} |(t-s)^{\beta-1} f(s,u(s),u(g(s)))|ds &\le \Big(\int_{t_1}^{t} (t-s)^{\lambda} ds\Big)^{1-\beta_1} \|m_f\|_{L^{\frac{1}{\beta_1}}[t_1,t]} \\ &\le \frac{M_1}{(1+\lambda)^{1-\beta_1}} T_0^{(1+\lambda)(1-\beta_1)}.\end{aligned}$$

Then, we have

$$\begin{aligned}&\beta \int_{t_1}^{t}\int_0^\infty \theta\xi_\beta(\theta)|(t-s)^{\beta-1} S((t-s)^\beta\theta) f(s,u(s),u(g(s)))|d\theta ds \\ &\qquad\le M\beta \int_{t_1}^{t}\int_0^\infty \theta\xi_\beta(\theta)|(t-s)^{\beta-1} f(s,u(s),u(g(s)))|d\theta ds \\ &\qquad\le \frac{M\beta}{\Gamma(1+\beta)} \int_{t_1}^{t} |(t-s)^{\beta-1} f(s,u(s),u(g(s)))|ds \\ &\qquad\le \frac{M\beta}{\Gamma(1+\beta)} \Big(\int_{t_1}^{t} (t-s)^{\lambda} ds\Big)^{1-\beta_1} \|m_f\|_{L^{\frac{1}{\beta_1}}[t_1,t]} \\ &\qquad\le \frac{\beta M M_1}{\Gamma(1+\beta)(1+\lambda)^{1-\beta_1}} T_0^{(1+\lambda)(1-\beta_1)}. \end{aligned} \tag{3.7}$$

Our aim is to prove that the map Υ is a condensing map from $B_r(0,\mathcal{PC}(H_\alpha))$ into $B_r(0,\mathcal{PC}(H_\alpha))$. For this we have divided our proof into four steps.

Step 1. First we show that $\Upsilon B_r(0,\mathcal{PC}(H_\alpha)) \subset B_r(0,\mathcal{PC}(H_\alpha))$, where

$$B_r(0,\mathcal{PC}(H_\alpha)) = \{u \in \mathcal{PC}(H_\alpha) : \|u\|_\alpha \le r\},$$

for $r > 0$.

Let $u \in B_r(0,\mathcal{PC}(H_\alpha))$. For $i \ge 1$ and $t \in (t_i,t_{i+1}]$, we get

$$\begin{aligned}\|\Upsilon u(t)\|_\alpha &\le \|h_i(t,u(t)) - h_i(t,0)\|_\alpha + \|h_i(t,0)\|_\alpha \\ &+ \int_0^\infty \xi_\beta(\theta)\|S((t-s_i)^\beta\theta)\| \; \|h_i(s_i,u(s_i)) - h_i(s_i,0)\|_\alpha ds \\ &+ \int_0^\infty \xi_\beta(\theta)\|S((t-s_i)^\beta\theta)\|\|h_i(s_i,0)\|_\alpha ds \\ &+ \beta \int_{s_i}^{t}\int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}\|A^\alpha S((t-s)^\beta\theta)\| \; \|f(s,u(s),u(g(s)))\|d\theta ds \\ &\le L_{h_i}\|u(t)\| + \|h_i(t,0)\|_\alpha + ML_{h_i}\|u(t)\| + M\|h_i(t,0)\|_\alpha \\ &+ C_\alpha\beta\Big(\int_{s_i}^{t} (t-s)^{\beta(1-\alpha)-1} \sup_{s\in[s_i,t_{i+1}],v\in B_r(0,\mathcal{PC}(H_\alpha))} \|f(s,v(s),v(g(s)))\|ds\Big) \\ &\times\Big(\int_0^\infty \theta^{1-\alpha}\xi_\beta(\theta)d\theta\Big) \\ &\le L_{h_i}(1+M)\|u\|_{\mathcal{PC}} + (1+M)\|h_i(t,0)\|_\alpha \\ &+ C_\alpha \frac{\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \frac{T_0^{\beta(1-\alpha)}}{1-\alpha} \sup_{s\in[s_i,t_{i+1}],v\in B_r(0,\mathcal{PC}(H_\alpha))} \|f(s,v(s),v(g(s)))\| \end{aligned} \tag{3.8}$$

which implies that $\|\Upsilon u\|_\alpha \leq r$ for all $i \geq 1$. Similarly, for each $t \in [0, t_1]$, we find that

$$\begin{aligned}
\|\Upsilon u(t)\|_\alpha &\leq \int_0^\infty \xi_\beta(\theta)\|S(t^\beta\theta)\| \, \|A^\alpha u_0\|d\theta \\
&+ \beta\int_0^t\int_0^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}\|A^\alpha S((t-s)^\beta\theta)\|\|f(s,u(s),u(g(s)))\|d\theta ds \\
&\leq M\|u_0\|_\alpha + \frac{C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\frac{T_0^{\beta(1-\alpha)}}{1-\alpha}\sup_{s\in[0,t_1],v\in B_r(0,\mathcal{PC}(H_\alpha))}\|f(s,v(s),v(g(s)))\|
\end{aligned}$$

from which we get $\|\Upsilon u\|_\alpha \leq r$ and Υ has values in $B_r(0,\mathcal{PC}(H_\alpha))$.

Step 2. The map $\Upsilon_1 = \sum_{i=0}^N \Upsilon_i^1$ is a contraction on $B_r(0,\mathcal{PC}(H_\alpha))$.

Let $t \in (t_i, t_{i+1}]$ and $u, v \in B_r(0,\mathcal{PC}(H_\alpha))$, $i = 1, \cdots, N$, we have

$$\|\Upsilon_i^1 u(t) - \Upsilon_i^1 v(t)\|_{\mathcal{PC}(H_\alpha)} \leq (1+M)L_{h_i}\|u-v\|_{C((t_i,t_{i+1}],H_\alpha)},$$

which implies that $\|\sum_{i=0}^N \Upsilon_i^1 u - \sum_{i=0}^N \Upsilon_i^1 v\|_{\mathcal{PC}} \leq L\|u-v\|_{\mathcal{PC}}$, i.e., Υ_1 is a contraction on $B_r(0,\mathcal{PC}(H_\alpha))$.

Let $\Upsilon_i^2 B_r(0,\mathcal{PC}(H_\alpha))(t) = \{\Upsilon_i^2 u(t) : u \in B_r(0,\mathcal{PC}(H_\alpha))\}$.

Step 3. Next, we prove that the set $\bigcup \Upsilon_i^2 B_r(0,\mathcal{PC}(H_\alpha))(t)$ is relatively compact in H_α.

Let $t \in (s_i, t_{i+1}]$, for $i = 0, 1, \cdots, N$, then for each $\epsilon \in (s_i, s)$ and for each $\delta > 0$, we define an operator $(\Upsilon_i^2)_{\epsilon,\delta}$ on $B_r(0,\mathcal{PC}(H_\alpha))$ by

$$\begin{aligned}
((\Upsilon_i^2)_{\epsilon,\delta}u)(t) &= S(\epsilon^\beta\delta)\int_{s_i}^{t-\epsilon}(t-s)^{\beta-1}\Big\{\beta\int_\delta^\infty \theta\xi_\beta(\theta)S((t-s)^\beta\theta-\epsilon^\beta\delta)d\theta\Big\} \\
&\times f(s,u(s),u(g(s)))ds,
\end{aligned}$$

where $u \in B_r(0,\mathcal{PC}(H_\alpha))$. The set

$$(B_r)_{\epsilon,\delta}(0,\mathcal{PC}(H_\alpha))(t) = \{((\Upsilon_i^2)_{\epsilon,\delta}u)(t) : u \in B_r(0,\mathcal{PC}(H_\alpha))\}$$

is relatively compact in H_α. Since the operator $S(\epsilon^\beta\delta)$, is compact.

Also, for each $u \in B_r(0,\mathcal{PC}(H_\alpha))$, we have

$$\begin{aligned}
&\|(\Upsilon_i^2 u)(t) - ((\Upsilon_i^2)_{\epsilon,\delta}u)(t)\|_\alpha \\
\leq\ &\beta\|\int_{s_i}^t\int_0^\delta \theta\xi_\beta(\theta)(t-s)^{\beta-1}S((t-s)^\beta\theta)f(s,u(s),u(g(s)))d\theta ds \\
+\ &\int_{s_i}^t\int_\delta^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}S((t-s)^\beta\theta)f(s,u(s),u(g(s)))d\theta ds \\
-\ &\int_{s_i}^{t-\epsilon}\int_\delta^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}S((t-s)^\beta\theta)f(s,u(s),u(g(s)))d\theta ds\|_\alpha \\
\leq\ &\beta\int_{s_i}^t\int_0^\delta \theta\xi_\beta(\theta)(t-s)^{\beta-1}\|A^\alpha S((t-s)^\beta\theta)\|\|f(s,u(s),u(g(s)))\|d\theta ds \\
+\ &\beta\int_{t-\epsilon}^t\int_\delta^\infty \theta\xi_\beta(\theta)(t-s)^{\beta-1}\|A^\alpha S((t-s)^\beta\theta)\|\|f(s,u(s),u(g(s)))\|d\theta ds \\
\leq\ &C_\alpha\beta\Big(\int_{s_i}^t(t-s)^b ds\Big)^{1-\beta_1}\|m_f\|_{L^{\frac{1}{\beta_1}}[s_i,t]}\int_0^\delta \theta^{1-\alpha}\xi_\beta(\theta)d\theta \\
&+C_\alpha\beta\Big(\int_{t-\epsilon}^t(t-s)^b ds\Big)^{1-\beta_1}\|m_f\|_{L^{\frac{1}{\beta_1}}[t-\epsilon,t]}\int_0^\infty \theta^{1-\alpha}\xi_\beta(\theta)d\theta \\
\leq\ &\frac{\beta M_1 C_\alpha T_0^{(1+b)(1-\beta_1)}}{(1+b)^{(1-\beta_1)}}\int_0^\delta \theta\xi_\beta(\theta)d\theta + \frac{\beta M_1 C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))(1+b)^{(1-\beta_1)}}\epsilon^{(1+b)(1-\beta_1)}.
\end{aligned}$$

Therefore, there exists relatively compact sets arbitrarily close to the set

$\bigcup \Upsilon_i^2 B_r(0, \mathcal{PC}(H_\alpha))(t)$. Thus, the set $\bigcup \Upsilon_i^2 B_r(0, \mathcal{PC}(H_\alpha))(t)$ is relatively compact in H_α.

Step 4. In this step, our aim is to prove that the set of functions $[\Upsilon_i^2 B_r(0, \widetilde{\mathcal{PC}(H_\alpha)})]_i$, $i = 0, 1, \cdots, N$ is an equicontinuous subset of $C([t_i, t_{i+1}], H_\alpha)$.
Let $t_1, t_2 \in [s_i, t_{i+1}]$, $t_1 < t_2$ and $u \in B_r(0, \mathcal{PC}(H_\alpha))$, we get

$$\begin{aligned}
& \|(\Upsilon_i^2 u)(t_2) - (\Upsilon_i^2 u)(t_1)\|_\alpha \\
\leq \quad & \beta \int_{t_1}^{t_2} \int_0^\infty \theta \xi_\beta(\theta)(t_2 - s)^{\beta-1} \|A^\alpha S((t_2 - s)^\beta \theta)\| \|f(s, u(s), u(g(s)))\| d\theta ds \\
+ \quad & \beta \int_{s_i}^{t_1} \int_0^\infty \theta \xi_\beta(\theta)[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \|A^\alpha S((t_2 - s)^\beta \theta)\| \\
& \quad \times \|f(s, u(s), u(g(s)))\| d\theta ds \\
+ \quad & \beta \int_{s_i}^{t_1} \int_0^\infty \theta \xi_\beta(\theta)(t_1 - s)^{\beta-1} \|A^\alpha [S((t_2 - s)^\beta \theta) - S((t_1 - s)^\beta \theta)]\| \\
\times \quad & \|f(s, u(s), u(g(s)))\| d\theta ds.
\end{aligned} \tag{3.9}$$

For the first term on the right hand side of (3.9), we have

$$\begin{aligned}
& \beta \int_{t_1}^{t_2} \int_0^\infty \theta \xi_\beta(\theta)(t_2 - s)^{\beta-1} \|A^\alpha S((t_2 - s)^\beta \theta)\| \|f(s, u(s), u(g(s)))\| d\theta ds \\
& \leq \frac{\beta C_\alpha M_1 \Gamma(2 - \alpha)}{\Gamma(1 + \beta(1 - \alpha))} \Big(\int_{t_1}^{t_2} (t_2 - s)^b ds \Big)^{1-\beta_1} \\
& \leq \frac{\beta C_\alpha M_1 \Gamma(2 - \alpha)}{(1 + b)^{(1-\beta_1)} \Gamma(1 + \beta(1 - \alpha))} (t_2 - t_1)^{(1+b)(1-\beta_1)}
\end{aligned} \tag{3.10}$$

For the second term on the right hand side of (3.9), we have

$$\begin{aligned}
& \beta \int_{s_i}^{t_1} \int_0^\infty \theta \xi_\beta(\theta)[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] \|A^\alpha S((t_2 - s)^\beta \theta)\| \\
& \qquad \times \|f(s, u(s), u(g(s)))\| d\theta ds \\
& \leq \frac{\beta N M_1}{\Gamma(1 + \beta)} \Big(\int_{s_i}^{t_1} [(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}]^{\frac{1}{1-\beta_1}} ds \Big)^{1-\beta_1} \\
& \leq \frac{\beta N M_1}{\Gamma(1 + \beta)} \Big(\int_{s_i}^{t_1} [(t_1 - s)^\lambda - (t_2 - s)^\lambda] ds \Big)^{1-\beta_1} \\
& \leq \frac{\beta N M_1}{\Gamma(1 + \beta)(1 + \lambda)^{1-\beta_1}} \Big((t_2 - t_1)^{1+\lambda} - ((t_2 - s_i)^{1+\lambda} - (t_1 - s_i)^{1+\lambda}) \Big)^{1-\beta_1} \\
& \leq \frac{\beta N M_1}{\Gamma(1 + \beta)(1 + \lambda)^{1-\beta_1}} (t_2 - t_1)^{(1+\lambda)(1-\beta_1)}
\end{aligned} \tag{3.11}$$

For $t_1 = s_i$, it is easy to see that the third term on the right hand side of (3.9) will be zero. For $t_1 > s_i$ and $\epsilon > 0$ be sufficiently small, we have

$$
\begin{aligned}
& \beta\int_{s_i}^{t_1-\epsilon}\int_0^\infty \theta\xi_\beta(\theta)(t_1-s)^{\beta-1}\|A^\alpha[S((t_2-s)^\beta\theta)-S((t_1-s)^\beta\theta)]\| \\
&\qquad \times\|f(s,u(s),u(g(s)))\|d\theta ds \\
& +\beta\int_{t_1-\epsilon}^{t_1}\int_0^\infty \theta\xi_\beta(\theta)(t_1-s)^{\beta-1}\|A^\alpha[S((t_2-s)^\beta\theta)-S((t_1-s)^\beta\theta)]\| \\
&\qquad \times\|f(s,u(s),u(g(s)))\|d\theta ds \\
\leq\quad & \frac{M_1(t_1^{1+\lambda}-\epsilon^{1+\lambda})^{(1-\beta_1)}}{\Gamma(1+\beta)(1+\lambda)^{1-\beta_1}}\sup_{s\in[s_i,t_1-\epsilon]}\|A^\alpha[S((t_2-s)^\beta\theta)-S((t_1-s)^\beta\theta)]\| \\
+\quad & \frac{2\beta NM_1}{\Gamma(1+\beta)(1+\lambda)^{1-\beta_1}}\epsilon^{(1+\lambda)(1-\beta_1)}
\end{aligned}
\tag{3.12}
$$

Thus, from (3.10)-(3.12) we see that $\|\Upsilon_i^2u(t_2)-\Upsilon_i^2u(t_1)\|_\alpha$ tends to zero as $t_2\to t_1$ for any $u\in B_r(0,\mathcal{PC}(H_\alpha))$, which means that $[\Upsilon_i^2B_r(0,\mathcal{PC}(H_\alpha))]_i$ is equicontinuous.

Lemma (2.2) and the above steps shows that Υ_1 is a contraction, Υ_2 is completely continuous and $\Upsilon=\Upsilon_1+\Upsilon_2$ is a condensing map on $B_r(0,\mathcal{PC}(H_\alpha))$. Then Krasnoselskii's fixed point theorem ensures that Υ has a fixed point, which gives rise to a mild solution. □

4. Application

Consider the following impulsive system of fractional partial differential equations

$$
\left.
\begin{aligned}
{}^CD_t^\beta u(t,x) &= \frac{\partial^2u}{\partial x^2}+F(t,x,u(t,x),u(g(t),x)),\quad (t,x)\in\bigcup_{i=1}^N[s_i,t_{i+1}]\times[0,\pi],\\
u(t,0) &= u(t,\pi)=0,\qquad t\in[0,T_0]\\
u(0,x) &= u_0(x),\qquad x\in[0,\pi],\\
u(t,x) &= H_i(t,u(t,x)),\qquad x\in[0,\pi],t\in(t_i,s_i]
\end{aligned}
\right\}
\tag{4.1}
$$

where $0=t_0=s_0<t_1\leq s_1<\cdots<t_N\leq s_N<t_{N+1}=T_0$ are fixed real numbers, $u_0\in H$, $F\in([0,T_0]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $H_i\in C((t_i,s_i]\times\mathbb{R},\mathbb{R})$ for all $i=1,\cdots,N$.

(A1). Let $H=L^2([0,\pi])$ and $Au=-\frac{\partial^2}{\partial x^2}u$ with

$$D(A)=\{u\in H:\frac{\partial u}{\partial x},\frac{\partial^2u}{\partial x^2}\in H,u(0)=u(\pi)=0\}=H^2(0,\pi)\cap H_0^1(0,\pi),$$

clearly, the operator A is the infinitesimal generator of a compact analytic semigroup $S(t)$. Taking $\alpha=1/2$, we have $D(A^{1/2})$ is Banach space endowed with norm

$$\|u\|_{1/2}=\|A^{1/2}u\|,\quad u\in D(A^{1/2}).$$

We can formulate the impulsive system (4.1) in the abstract form (1.4)-(1.6), where $u(t)=u(t,.)$, i.e., $u(t)(x)=u(t,x)$ and the functions $f:[0,T_0]\times H_{1/2}\times H_{1/2}\to H$ and $h_i:(t_i,s_i]\times H_{1/2}\to H$ are given by

$$
\begin{aligned}
f(t,u(t),u(g(t)))(x) &= F(t,x,u(t,x),u(g(t),x)),\\
h_i(t,u(t))(x) &= H_i(t,u(t,x)).
\end{aligned}
$$

Case 1. Define

$$
\begin{aligned}
(A2).\qquad & f(t,u(t),u(g(t)))(x)=\frac{e^{-t}\{|u(t,x)|+|u(g(t),x)|]\}}{(\gamma+e^t)\{1+|u(t,x)|+|u(g(t),x)|]\}},\quad \gamma>-1,\\
& t\in[0,T_0],\quad u\in H_{1/2},\ x\in(0,\pi).
\end{aligned}
$$

Clearly, $f : [0, T_0] \times H_{1/2} \times H_{1/2} \to H$ is continuous function, such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L_f\{\|u_1 - u_2\|_{1/2} + \|v_1 - v_2\|_{1/2}\},$$

with $L_f = \frac{1}{\gamma+1}$.

$$(A3). \qquad h_i(t, u(t))(x) = \frac{\cos t|u(t,x)|}{2(1 + |u(t,x)|)}, \ t \in (t_i, s_i], \quad i = 1, 2, \cdots, N, \\ u \in H_{1/2}, \ x \in (0, \pi). \tag{4.2}$$

Clearly, $h_i : (t_i, s_i] \times H_{1/2} \to H$ are continuous functions, such that

$$\|h_i(t, u_1) - f(t, u_2)\| \leq L_{h_i}\|u_1 - u_2\|_{1/2},$$

with $L_{h_i} = \frac{1}{2}$.

(A4). We can choose g as follows

(i). $g(t) = kt$ for $t \in [0, T_0], \quad k \in [0, 1]$,

(ii). $g(t) = k \sin t$ for $t \in [0, \pi/2], \quad k \in [0, 1]$.

Hence, (A1)+(A2)+(A3)+(A4) implies that the assumptions in Theorem (3.2) are satisfied. For more details, we refer to [18].

Case 2. Define

$$(A5). \qquad f(t, u(t), u(g(t)))(x) = \frac{e^{-t}(sin(u(t,x)) + cos(u(g(t), x)))}{(1+t)(e^t + e^{-t})} + e^{-t}, \\ t \in [0, t_1] \cup (s_1, t_2] \cdots \cup (s_N, T_0], \quad u \in H, \ x \in (0, \pi).$$

Satisfies,

$$\|f(t, u)\| \leq \frac{2e^{-t}}{e^t + e^{-t}} + e^{-t} = m_f(t), \text{ with } m_f(t) \in L^\infty([0, T_0], \mathbb{R}_+).$$

Hence, (A1)+(A3)+(A4)+(A5) implies that the assumptions in Theorem (3.3) are also satisfied.

System (4.1) has a mild solution $u \in \mathcal{PC}(H)$ if $u(.)$ is a mild solution of the associated abstract form (1.4)-(1.6).

The following theorem follows immediately from Theorem 3.2 and Theorem 3.3.

Theorem 4.1. *If any of the following assumption is hold. Then there exists a mild solution $u \in \mathcal{PC}(H)$ of* (4.1)

(i). The functions F and H_i are Lipschitz with Lipschitz constant L_F and L_{H_i} respectively and $\max\{ML_{H_i} + \frac{2C_\alpha L_F \Gamma(2-\alpha)}{(1-\alpha)\Gamma(1+\beta(1-\alpha))} T_0^{\beta(1-\alpha)} : \ i = 1, \cdots, N\} < 1$.

(ii). The functions $H_i(.)$ are Lipschitz with Lipschitz constant L_{H_i}, the function $F(.)$ is bounded with $L_{H_i} < 1$ for all $i = 1, \cdots, N$.

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