



# A novel solution for fractional chaotic Chen system

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## Abstract

A novel solution to the fraction chaotic Chen system is presented in this paper by using the step homotopy analysis method. This method yields a continuous solution in terms of a rapidly convergent infinite power series with easily computable terms. Moreover, the residual error of the SHAM solution is defined and computed for each time interval. Via the computing of the residual error we observe that the accuracy of the present method tends to  $10^{-11}$  which is very high. ©2015 All rights reserved.

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## 1. Introduction

Nature is intrinsically nonlinear. So, it is not surprising that most of the systems we encounter in the real world are nonlinear. And what is interesting is that some of these nonlinear systems can be described by fractional-order differential equation (FDE), which can display a variety of behaviors including chaos and hyperchaos. The purpose of this paper is to obtain a continuous solution for fractional chaotic Chen system [16, 32, 35].

$$D_t^{\alpha_1} x = a(y - x), \tag{1.1}$$

$$D_t^{\alpha_2} y = (c - a)x - xz + cy, \tag{1.2}$$

$$D_t^{\alpha_3} z = xy - bz, \tag{1.3}$$

$$x(0) = c_1, \quad y(0) = c_2, \quad z(0) = c_3, \tag{1.4}$$

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where  $D^{\alpha_i}, i = 1, 2, 3$  are Caputo fractional derivatives,  $a, b, c$  from  $\mathbb{R}$  and  $0 < \alpha_i \leq 1$ .

Finding accurate solution for FDEs has been an active research undertaking [20, 27, 30, 31, 33]. Exact solutions of most of the FDEs cannot be found easily, thus analytical and numerical methods must be used. For example, generalized Adams-Bashforth-Moulton method (GABMM) is one of the most used method to solve fractional differential equations [9, 10, 16, 34]. Some of the recent analytic methods for solving nonlinear problems include the Adomian decomposition method (ADM) [12, 15, 29], homotopy-perturbation method (HPM) [21, 24] and variational iteration method (VIM) [22, 25].

Recently, homotopy analysis method (HAM) becomes one of the most famous technique to solve such nonlinear problems. the method was proposed by Liao [17, 18]. Many researchers have applied this method for different class of differential equations [1, 2, 3, 6, 7, 11, 13, 14, 19, 28]. Alomari et al. [4] used the idea of time step in the algorithm of HAM to get multistage homotopy analysis method (MSHAM) and apply to Chen system. Recently, Alomari et al. [5] introduce new algorithm to obtain the solution of fractional chaotic system using HAM.

This paper investigates for the first time the applicability and effectiveness of HAM when we hybrid the numerical with analytical in a sequence of intervals (i.e. time step) for finding accurate approximate solutions to the fractional Chen system. To the best of our knowledge, this is also the first time that the residual error can be calculated for the analytical solution at each subinterval of fractional Chen system. Numerical results are presented graphically and are found to be in excellent agreement with the GABMM solution.

## 2. Preliminaries and notations

In this section, we give some definitions and properties of the fractional calculus and homotopy-derivative

### 2.1. Fractional calculus

The following properties can found in [26].

**Definition 2.1.** A real function  $f(t), t > 0$ , is said to be in the space  $C_\mu, \mu \in \mathbb{R}$ , if there exists a real number  $p > \mu$ , such that  $f(t) = t^p f_1(t)$ , where  $f_1(t) \in C(0, \infty)$ , and it is said to be in the space  $C_\mu^n$  if and only if  $h^{(n)} \in C_\mu, n \in \mathbb{N}$ .

**Definition 2.2.** The Riemann-Liouville fractional integral operator ( $J^\alpha$ ) of order  $\alpha \geq 0$ , of a function  $f \in C_\mu, \mu \geq -1$ , is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (\alpha > 0), \tag{2.1}$$

$$J^0 f(t) = f(t), \tag{2.2}$$

where  $\Gamma(\alpha)$  is the well-known gamma function.

Some of the properties of the operator  $J^\alpha$ , which we will need here, are as follows:

For  $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$  and  $\gamma \geq -1$ :

1.  $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$ ,
2.  $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$ ,
3.  $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$ .

**Definition 2.3.** The fractional derivative ( $D^\alpha$ ) of  $f(t)$ , in the Caputo sense is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2.3}$$

for  $n - 1 < \alpha < n, n \in \mathbb{N}, t > 0, f \in C_{-1}^n$ .

The following are two basic properties of the Caputo fractional derivative:

1. Let  $f \in C_{-1}^n, n \in \mathbb{N}$ , then  $D^\alpha f, 0 \leq \alpha \leq n$  is well defined and  $D^\alpha f \in C_{-1}$ .
2. Let  $n - 1 \leq \alpha \leq n, n \in \mathbb{N}$  and  $f \in C_\mu^n, \mu \geq -1$ . Then

$$(J^\alpha D^\alpha)f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{2.4}$$

### 2.2. homotopy-derivative

The following properties can found in [19]

**Definition 2.4.** Let  $\phi$  be a function of the homotopy-parameter  $q$ , then

$$D_m(\phi) = \left. \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m} \right|_{q=0}. \tag{2.5}$$

is called the  $m$ th-order homotopy-derivative of  $\phi$ , where  $m \geq 0$  is an integer.

For homotopy-series the below

$$\phi_1 = \sum_{i=0}^{+\infty} u_i q^i, \quad \phi_2 = \sum_{i=0}^{+\infty} v_i q^i$$

hold and

1.  $D_m(\phi_1) = u_m$ .
2.  $D_m(q\phi_1) = D_{m-1}(\phi_1)$
3. If  $L$  be a linear operator independent of the homotopy-parameter  $q$ . For homotopy-series, then  $D_m(L\phi_1) = LD_m(\phi_1)$ .
4. If  $f$  and  $g$  be functions independent of the homotopy-parameter  $q$ , then

$$D_m(f\phi_1 \pm g\phi_2) = fD_m(\phi_1) \pm gD_m(\phi_2).$$

5.  $D_m(\phi_1\phi_2) = \sum_{i=0}^m \phi_{1,i}\phi_{2,m-i}$

### 3. Solution approaches

To solve (1.1)–(1.3), we choose the base function as

$$\{t^{n\alpha_1+m\alpha_2+k\alpha_3} | n \geq 0, m \geq 0, k \geq 0\}, \tag{3.1}$$

so the solutions are in the form

$$x(t) = a_{0,0,0} + \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} a_{n,m,k} t^{n\alpha_1+m\alpha_2+k\alpha_3}, \tag{3.2}$$

$$y(t) = b_{0,0,0} + \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} b_{n,m,k} t^{n\alpha_1+m\alpha_2+k\alpha_3}, \tag{3.3}$$

$$z(t) = c_{0,0,0} + \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{k=0}^{+\infty} c_{n,m,k} t^{n\alpha_1+m\alpha_2+k\alpha_3}, \tag{3.4}$$

where  $a_{n,m,k}$ ,  $b_{n,m,k}$  and  $c_{n,m,k}$  are the coefficients. It is straightforward to choose

$$x_0(t) = c_1, \quad y_0(t) = c_2, \quad z_0(t) = c_3, \tag{3.5}$$

as our initial approximations of  $x(t)$ ,  $y(t)$  and  $z(t)$ , and the linear operator should be

$$L_{\alpha_1}[\hat{x}] = D^{\alpha_1} \hat{x}, \tag{3.6}$$

$$L_{\alpha_2}[\hat{y}] = D^{\alpha_2} \hat{y}, \tag{3.7}$$

$$L_{\alpha_3}[\hat{z}] = D^{\alpha_3} \hat{z}, \tag{3.8}$$

since we used Caputo fractional derivative then we have the property

$$L_{\alpha_1}[A_1] = L_{\alpha_2}[A_2] = L_{\alpha_3}[A_3] = 0, \tag{3.9}$$

where  $A_i \quad i = 1, 2, 3$  are the integration constants that will be determined by the initial conditions.

If  $q \in [0, 1]$  and  $\hbar$  indicate the embedding and non-zero auxiliary parameters, respectively, then the *zeroth-order deformation* problems are of the following form:

$$(1 - q)L_{\alpha_1}[\hat{x}(t; q) - x_0(t)] = q\hbar N_x[\hat{x}(t; q), \hat{y}(t; q)], \tag{3.10}$$

$$(1 - q)L_{\alpha_2}[\hat{y}(t; q) - y_0(t)] = q\hbar N_y[\hat{x}(t; q), \hat{y}(t; q), \hat{z}(t; q)], \tag{3.11}$$

$$(1 - q)L_{\alpha_3}[\hat{z}(t; q) - z_0(t)] = q\hbar N_z[\hat{x}(t; q), \hat{y}(t; q), \hat{z}(t; q)], \tag{3.12}$$

subject to the initial conditions

$$\hat{x}(0; q) = c_1, \quad \hat{y}(0; q) = c_2, \quad \hat{z}(0; q) = c_3, \tag{3.13}$$

in which we define the nonlinear operators  $N_x$ ,  $N_y$  and  $N_z$  as

$$\begin{aligned} N_x[\hat{x}(t; q), \hat{y}(t; q)] &= \frac{\partial^{\alpha_1} \hat{x}(t; q)}{\partial t^{\alpha_1}} - a(\hat{y}(t; q) - \hat{x}(t; q)), \\ N_y[\hat{x}(t; q), \hat{y}(t; q), \hat{z}(t; q)] &= \frac{\partial^{\alpha_2} \hat{y}(t; q)}{\partial t^{\alpha_2}} - (c - a)\hat{x}(t; q) + \hat{x}(t; q)\hat{z}(t; q) - c\hat{y}(t; q), \\ N_z[\hat{x}(t; q), \hat{y}(t; q), \hat{z}(t; q)] &= \frac{\partial^{\alpha_3} \hat{z}(t; q)}{\partial t^{\alpha_3}} - \hat{x}(t; q)\hat{y}(t; q) + b\hat{z}(t; q). \end{aligned}$$

For  $q = 0$  and  $q = 1$ , the above zeroth-order equations (3.10)-(3.12) have the solutions

$$\hat{x}(t; 0) = x_0(t), \quad \hat{y}(t; 0) = y_0(t), \quad \hat{z}(t; 0) = z_0(t), \tag{3.14}$$

and

$$\hat{x}(t; 1) = x(t), \quad \hat{y}(t; 1) = y(t), \quad \hat{z}(t; 1) = z(t). \tag{3.15}$$

When  $q$  increases from 0 to 1, then  $\hat{x}(t; q), \hat{y}(t; q)$  and  $\hat{z}(t; q)$  vary from  $x_0(t), y_0(t)$  and  $z_0(t)$  to  $x(t), y(t)$  and  $z(t)$ , respectively. Expanding  $\hat{x}, \hat{y}$  and  $\hat{z}$  in Taylor series with respect to  $q$ , we have

$$\hat{x}(t; q) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)q^m, \tag{3.16}$$

$$\hat{y}(t; q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m, \tag{3.17}$$

$$\hat{z}(t; q) = z_0(t) + \sum_{m=1}^{\infty} z_m(t)q^m, \tag{3.18}$$

in which

$$x_m(t) = D_m(\hat{x}(t; q)), \quad y_m(t) = D_m(\hat{y}(t; q)), \quad z_m(t) = D_m(\hat{z}(t; q)), \tag{3.19}$$

where  $\hbar$  is chosen in such a way that these series are convergent at  $q = 1$ . Therefore, through Eqs. (3.14)–(3.19), we have

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t), \tag{3.20}$$

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t), \tag{3.21}$$

$$z(t) = z_0(t) + \sum_{m=1}^{\infty} z_m(t). \tag{3.22}$$

Take the  $m$ th-order homotopy-derivative of *zeroth-order* equations (3.10)–(3.12) and used the properties (1)–(4), then we have the *m*th-order deformation equations

$$L_{\alpha_1}[x_m(t) - \chi_m x_{m-1}(t)] = \hbar R_m^x(t), \tag{3.23}$$

$$L_{\alpha_2}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar R_m^y(t), \tag{3.24}$$

$$L_{\alpha_3}[z_m(t) - \chi_m z_{m-1}(t)] = \hbar R_m^z(t), \tag{3.25}$$

with the following initial conditions:

$$x_m(0) = 0, \quad y_m(0) = 0, \quad z_m(0) = 0, \tag{3.26}$$

where  $R_m^x(t), R_m^y(t)$  and  $R_m^z(t)$  can be found by used the properties (1),(4) and (5) as

$$R_m^x(t) = D_t^{\alpha_1} x_{m-1} - a(y_{m-1} - x_{m-1}), \tag{3.27}$$

$$R_m^y(t) = D_t^{\alpha_2} y_{m-1} - (c - a)x_{m-1} + \sum_{i=0}^{m-1} x_i(t)z_{m-1-i}(t) - cy_{m-1}(t), \tag{3.28}$$

$$R_m^z(t) = D_t^{\alpha_3} z_{m-1} - \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) + bz_{m-1}(t). \tag{3.29}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In this way, it is easy to solve the linear non-homogeneous Eqs. (3.23)–(3.25) at initial conditions (3.26) for all  $m \geq 1$ , and now we successfully obtain

$$\begin{aligned} x_1(t) &= \frac{\hbar t^{\alpha_1} a (-c_2 + c_1)}{\Gamma(\alpha_1 + 1)}, \\ y_1(t) &= \frac{\hbar t^{\alpha_2} (-c_1 c + c_1 a - c c_2 + c_3 c_1)}{\Gamma(\alpha_2 + 1)}, \\ z_1(t) &= \frac{\hbar t^{\alpha_3} c_2 (b - c_1)}{\Gamma(\alpha_3 + 1)}, \end{aligned}$$

etc. Then the 13-term of the approximate solutions of Eqs. (1.1)–(1.4) are

$$x(t) = x_0(t) + \sum_{m=1}^{12} x_m(t), \tag{3.30}$$

$$y(t) = y_0(t) + \sum_{m=1}^{12} y_m(t), \tag{3.31}$$

$$z(t) = z_0(t) + \sum_{m=1}^{12} z_m(t). \tag{3.32}$$

To determine the value of  $\hbar$  we plot the  $\hbar$ -curves for Eqs. (3.30)–(3.32) in Fig. (1).

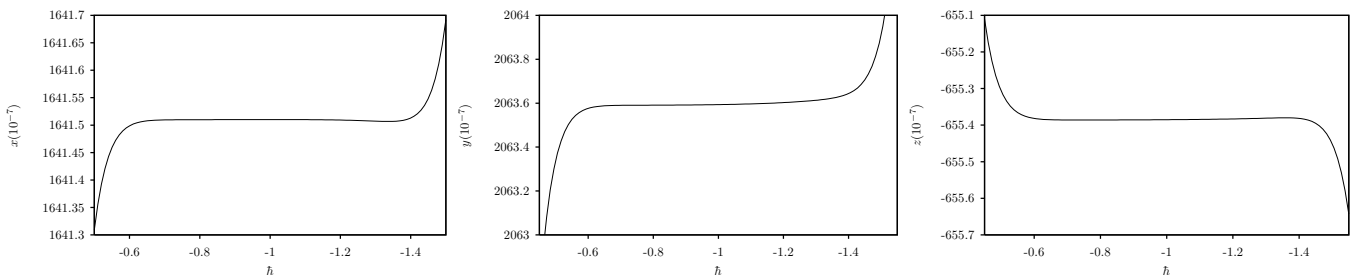


Figure 1:  $\hbar$ -curves of 13th-order approximation for (0.9,0.9,0.9)

From this figure, it is noted that the valid regions of  $\hbar$  correspond to the line segments nearly parallel to the horizontal axis.

If  $\hbar = -1$  we get the homotopy perturbation method (HPM) solution when  $\alpha_1 = \alpha_2 = \alpha_3 = 1$  which is not effective for large values of  $t$  (for more detail see [8]).

### 3.1. SHAM

The HAM solution for Eqs. (3.30)–(3.32) is not effective for larger  $t$ . In case if we need the solution for  $[0, 7]$ , then the simple idea is to divide the interval  $[0, 7]$  to subintervals with time step  $\Delta t$  and we get the solution at each subinterval. So in this case we have to satisfy the initial condition at each of the subinterval. Accordingly, the initial values  $x_0, y_0, z_0$  will be changed for each subinterval, i.e.  $x(t^*) = c_1^* = x_0, y(t^*) = c_2^* = y_0$  and  $z(t^*) = c_3^* = z_0$  and we should satisfy the initial conditions  $x_m(t^*) = 0, y_m(t^*) = 0$  and  $z_m(t^*) = 0$  for all  $m \geq 1$  so

$$\begin{aligned} x_1(t) &= \frac{\hbar(t - t^*)^{\alpha_1} a (-c_2 + c_1)}{\Gamma(\alpha_1 + 1)}, \\ y_1(t) &= \frac{\hbar(t - t^*)^{\alpha_2} (-c_1 c + c_1 a - c c_2 + c_3 c_1)}{\Gamma(\alpha_2 + 1)}, \end{aligned}$$

$$z_1(t) = \frac{\hbar(t - t^*)^{\alpha_3} c_2 (b - c_1)}{\Gamma(\alpha_3 + 1)},$$

⋮

So, the solution will be as follows:

$$x(t) = c_1^* + \sum_{m=1}^{12} x_m(t - t^*), \tag{3.33}$$

$$y(t) = c_2^* + \sum_{m=1}^{12} y_m(t - t^*), \tag{3.34}$$

$$z(t) = c_3^* + \sum_{m=1}^{12} z_m(t - t^*), \tag{3.35}$$

where  $t^*$  starting from  $t_0 = 0$  until  $t_n = T = 7$ . To carry out the solution on every subinterval of equal length  $\Delta t$ , we need to know the values of the following initial conditions:

$$c_1 = x(t^*), \quad c_2 = y(t^*), \quad c_3 = z(t^*).$$

In general, we do not have these information at our clearance except at the initial point  $t^* = t_0 = 0$ , but we can obtain these values by assuming that the new initial condition is the solution in the previous interval. (i.e. If we need the solution in interval  $[t_i, t_{i+1}]$ , then the initial conditions of this interval will be as

$$c_1 = x(t_i) = \sum_{m=0}^{12} x_m(t_i - t_{i-1}), \tag{3.36}$$

$$c_2 = y(t_i) = \sum_{m=0}^{12} y_m(t_i - t_{i-1}), \tag{3.37}$$

$$c_3 = z(t_i) = \sum_{m=0}^{12} z_m(t_i - t_{i-1}), \tag{3.38}$$

where  $c_1, c_2$  and  $c_3$  are the initial conditions in the interval  $[t_i, t_{i+1}]$ . By this way we get modified homotopy perturbation method (MHPM) solution as a special case when  $\hbar = -1$  and  $\alpha_1 = \alpha_2 = \alpha_3 = 1$  [8].

### 3.2. Error analysis for SHAM

The different between the exact solution and the given solution which we will so-call residual error can be define as

$$E_x = D_t^{\alpha_1} X - a(Y - X), \tag{3.39}$$

$$E_y = D_t^{\alpha_2} Y - (c - a)X + XZ - cY, \tag{3.40}$$

$$E_z = D_t^{\alpha_3} Z - XY + bZ. \tag{3.41}$$

where  $X, Y$  and  $Z$  are the HAM solution for the equations (1.1)–(1.3) respectively. Since the SHAM solution is analytic at each time step then it is easy to obtain the residual error at each time step. According Eqs. (3.39)–(3.41), we can find the residual error on each time step by applying the that equations and using  $c_1, c_2$  and  $c_3$  which is defined as in SHAM solution. We noted that the orders of magnitude of the errors in SHAM solution depend on the order of approximation and the length of the subintervals.

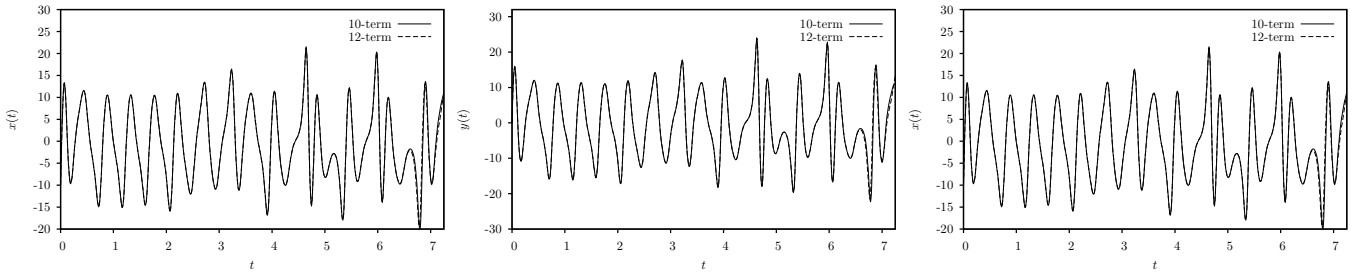


Figure 2: Time series of the SHAM solution Using 10 and 12 order of approximation for (0.95,0.95,0.95)

#### 4. Results and discussion

In this part, we set  $a = 35, c = 28, b = 3$  and we take the initial conditions  $x(0) = -10, y(0) = 0$  and  $z(0) = 37$  as in [4] after the standard case  $(\alpha_1, \alpha_2, \alpha_3)$  as  $(1, 1, 1)$ . To observe the convergent of the solution, we plot the 10-term and 12-term of SHAM solution with  $\Delta t = 0.005$  in Fig. 2. It is clear that the solution of 10-term like the solution of 12-term then we can consider 12-term as good approximate solution. The phase portraits of the SHAM solution and GABMM solution are given in Fig. 3 and Fig. 4 at different fractional derivative. The figure gives that SHAM solution have good agreement with GABMM solution.

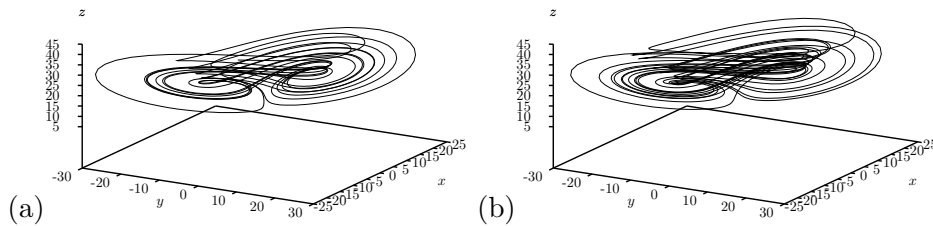


Figure 3: Comparison the phase portraits of  $x - y - z$  using 12-term SHAM in (b) with GABMM in (a) when  $(\alpha_1, \alpha_2, \alpha_3)$  as (0.95,0.95,0.95)

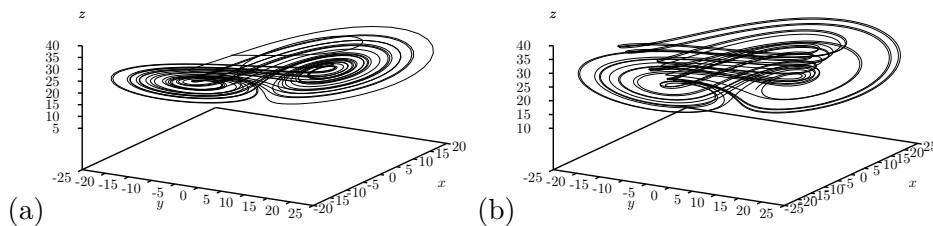


Figure 4: Comparison the phase portraits of  $x - y - z$  using 12-term SHAM in (b) with GABMM in (a) when  $(\alpha_1, \alpha_2, \alpha_3)$  as (0.9,0.9,0.9)

The residual error of the SHAM solution is presented in Fig.5 and 6 for

$$(0.95, 0.95, 0.95)$$

and

$$(0.99, 0.99, 0.99)$$

respectively. Table (1) and (2) give the residual error for the given solution at several points. We observe that a higher accuracy of the given solution is cited which is not extended than  $10^{-10}$ . On the other hand the GABMM usually used accuracy with  $10^{-6}$ .



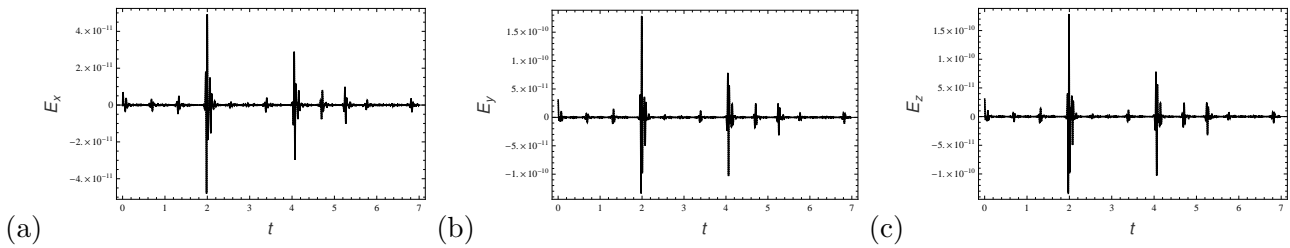


Figure 5: Residual error for SHAM solution using 8-terms with  $\Delta t = 0.001$  when  $(\alpha_1, \alpha_2, \alpha_3)$  as  $(0.95, 0.95, 0.95)$

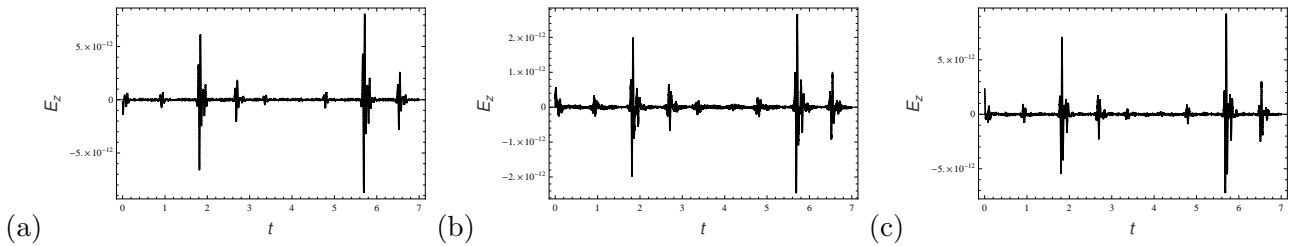


Figure 6: Residual error for SHAM solution using 8-terms with  $\Delta t = 0.001$  when  $(\alpha_1, \alpha_2, \alpha_3)$  as  $(0.99, 0.99, 0.99)$

Table 1: The Residual error for  $\alpha_i = 0.95$  with 8-terms and  $\Delta t = 0.001$  using 20 Digit

$t$	$E_x$	$E_y$	$E_z$
1	5.68434E-14	-7.81597E-14	-2.84217E-14
2	1.13687E-13	-1.27898E-13	-1.56319E-13
3	-7.10543E-15	0	-1.42109E-14
4	0	5.68434E-14	-3.55271E-15
5	-8.52651E-14	0	-3.01981E-14
6	-8.52651E-14	-2.84217E-14	-6.12843E-14
7	-7.10543E-15	-5.68434E-14	-7.10543E-15

Table 2: The Residual error for  $\alpha_i = 0.99$  with 8-terms and  $\Delta t = 0.001$  using 20 Digit

$t$	$E_x$	$E_y$	$E_z$
1	-2.27374E-13	-5.68434E-14	3.97904E-13
2	-8.9706E-14	-1.06581E-13	-7.10543E-14
3	2.84217E-14	1.42109E-13	1.42109E-14
4	-2.8777E-13	-4.05009E-13	-5.40012E-13
5	0	-5.68434E-14	0
6	-1.42109E-14	-2.84217E-14	1.42109E-14
7	8.52651E-14	3.97904E-13	6.53699E-13

### 5. Conclusions

In this present work continuous solution for fractional Chen system is obtained by SHAM. The modified method has the advantage of giving an analytical form of the solution within each time interval which is not possible in purely numerical techniques like fourth-order Runge–Kutta method RK4 or ABMM. The residual error for subintervals solution is defined and calculated. We also note that the SHAM solutions

were computed via a simple algorithm without any need for perturbation techniques, special transformations, linearization or discretization. The SHAM solutions are in excellent agreement with the GABMM solution. Moreover The HPM and MHPM solution is a special case when  $\hbar = -1$  and  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ .

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