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Some fixed point theorems for G-rational Geraghty contractive mappings in ordered generalized b-metric spaces

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Abstract

In this paper, we introduce the notion of G-rational Geraghty contractive mappings in the setup of ordered generalized b-metric spaces and investigate the existence of fixed points for such mappings. We also provide an example to illustrate the presented results and show that they are more general then some existing ones.

Keywords: Geraghty-type condition, rational contractive condition, ordered generalized b-metric space, fixed point.
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1. Introduction and preliminaries

There is a great number of generalizations of Banach contraction principle by using different forms of contractive conditions in various spaces. Some of such generalizations are obtained by contraction conditions containing rational expressions.

Ran and Reurings initiated the studying of fixed point results on partially ordered sets in [18], where they gave many useful results on matrix equations. Recently, many researchers have focused on different

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contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results, their applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the reader to [16, 17].

In [5], the authors proved some unique fixed point results for an operator satisfying certain rational contraction condition in a partially ordered metric space. In fact, their results generalize the main result of Jaggi [12].

Czerwik introduced in [7] the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces.

Definition 1.1 ([7])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to \mathbb{R}^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

The concept of generalized metric space, or a G-metric space, was introduced by Mustafa and Sims.

Definition 1.2 ([14])**.** Let X be a nonempty set and $G : X \times X \times X \to R^+$ be a mapping satisfying the following properties:
(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).
Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

References to the results in these two kinds of spaces can be found in [4]. Recently, Aghajani *et al.* in [1] combined these two concepts and introduced the concept of generalized b-metric spaces (G_b-metric spaces) and presented some of their basic properties.

Definition 1.3 ([1])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s[G(x, a, a) + G(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or a G_b-metric space.

Each G-metric space is a $\mathrm{G_b}$-metric space with $s = 1$.

Example 1.4 ([1])**.** Let (X, G) be a G-metric space and $G_*(x, y, z) = G^p(x, y, z)$, where $p > 1$ is a real number. Then G_* is a G_b-metric with $s = 2^{p-1}$.

Example 1.5 ([15]). Let $X = \mathbb{R}$ and $d(x,y) = |x-y|^2$. We know that (X,d) is a b-metric space with $s = 2$. Let $G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. It is easy to see that (X,G) is not a $\mathrm{G_b}$-metric space. Indeed, (G_b3) is not true for $x = 0$, $y = 2$ and $z = 1$. However, $G(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$ is a $\mathrm{G_b}$-metric on $\mathbb{R}$ with $s = 2$.

Various fixed point results in G_b-metric spaces were subsequently obtained in [11, 13, 15, 19]. See also [2, 6, 20].

Definition 1.6 ([1, 14]). A G_b-metric G is said to be symmetric if $G(x,y,y) = G(y,x,x)$, for all $x, y \in X$.

Proposition 1.7 ([1]). *Let X be a G_b-metric space. Then for each $x, y, z, a \in X$ it follows that:*

(1) if $G(x,y,z) = 0$ then $x = y = z$,

(2) $G(x,y,z) \leq s(G(x,x,y) + G(x,x,z))$,

(3) $G(x,y,y) \leq 2sG(y,x,x)$,

(4) $G(x,y,z) \leq s(G(x,a,z) + G(a,y,z))$.

Definition 1.8 ([1]). Let (X,G) be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G_b-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(2) G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

The space (X,G) is said to be $\mathrm{G_b}$-complete if every G_b-Cauchy sequence is G_b-convergent in X.

Proposition 1.9 ([1]). *Let (X,G) be a G_b-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(1) the sequence $\{x_n\}$ is G_b-Cauchy.

(2) for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq n_0$.

Also, the following are equivalent:

(3) $\{x_n\}$ is G_b-convergent to x.

(4) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

(5) $G(x_n, x, x) \to 0$, as $n \to \infty$.

Definition 1.10. Let (X,G) be a G_b-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G_b-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n, y_n)\}$ is G_b-convergent to $F(x,y)$.

Proposition 1.11 ([1]). *Let (X,G) and (X',G') be two G_b-metric spaces. Then a function $f : X \to X'$ is G_b-continuous at a point $x \in X$ if and only if it is G_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is G_b-convergent to x, $\{f(x_n)\}$ is G'_b-convergent to $f(x)$.*

In general, a G_b-metric function $G(x,y,z)$ for $s > 1$ is not jointly continuous in all of its variables. The following is an example of a discontinuous G_b-metric.

Example 1.12 ([10, 15]). Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}$ be defined by

$$D(m,n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m,p) \leq \frac{5}{2}(D(m,n) + D(n,p)).$$

Thus, (X, D) is a b-metric space with $s = \frac{5}{2}$ (see [10]).

Let $G(x, y, z) = \max\{D(x, y), D(y, z), D(z, x)\}$. It is easy to see that G is a $\mathrm{G_b}$-metric with $s = \frac{5}{2}$ which is not a continuous function.

We shall need the following simple lemma about the G_{b}-convergent sequences in the proof of our main result.

Lemma 1.13 ([15])**.** *Let (X, G) be a G_b-metric space with $s > 1$ and suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are G_b-convergent to x, y and z, respectively. Then we have*

$$\frac{1}{s^3}G(x, y, z) \leq \liminf_{n\to\infty} G(x_n, y_n, z_n) \leq \limsup_{n\to\infty} G(x_n, y_n, z_n) \leq s^3 G(x, y, z).$$

In particular, if $x = y = z$, then we have $\lim_{n\to\infty} G(x_n, y_n, z_n) = 0$.

Let $\mathfrak{F}$ denote the class of all real functions $\beta : [0, \infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies that } \ t_n \to 0, \quad \text{as} \ \ n \to \infty.$$

In order to generalize the Banach contraction principle, in 1973, Geraghty proved the following

Theorem 1.14 ([9])**.** *Let (X, d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{F}$ such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In 2010, Amini-Harandi and Emami extended the result of Geraghty to the framework of partially ordered complete metric spaces in the following way:

Theorem 1.15 ([3])**.** *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $f : X \to X$ be an increasing self-map such that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that there exists $\beta \in \mathfrak{F}$ such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$ with $y \preceq x$. Assume that either f is continuous or X is such that if an increasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n. Then f has a fixed point in X. Moreover, if for each $x, y \in X$ there exists $z \in X$ comparable with x and y, then the fixed point of f is unique.

In [8], Đukić *et al.* proved some fixed point theorems for mappings satisfying Geraghty-type contractive conditions in various generalized metric spaces. As in [8], we will consider the class of functions $\mathfrak{F}_s$, where $\beta \in \mathfrak{F}_s$ if $\beta : [0, \infty) \to [0, 1/s)$ and has the property

$$\beta(t_n) \to \frac{1}{s} \text{ implies that } \ t_n \to 0, \quad \text{as} \ \ n \to \infty.$$

Theorem 1.16 ([8])**.** *Let (X, d) be a complete b-metric space with parameter $s \geq 1$. Suppose that a mapping $f : X \to X$ satisfies the condition*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for some $\beta \in \mathfrak{F}_s$ and all $x, y \in X$. Then f has a unique fixed point $z \in X$, and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z in (X, d).

By unification of the recent results by Zabihi and Razani there is the following result.

Theorem 1.17 ([21]). *Let $(X, d, \preceq)$ be a partially ordered b-complete b-metric space (with parameter $s > 1$). Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in \mathfrak{F}_s$ such that*

$$sd(fx, fy) \le \beta(d(x, y))M(x, y) + L\,N(x, y)$$

for all comparable elements $x, y \in X$, where $L \ge 0$,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$

If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

The aim of this paper is to present some fixed point theorems for mappings in partially ordered G_b-metric spaces satisfying several versions of rational Geraghty-type contractive conditions. Our results extend some existing results in the literature. An example is presented showing the usefulness of these results and they are indeed more general than some known ones.

2. Main results

Further, let $\mathfrak{F}_s$ denote the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies that } \ t_n \to 0, \quad \text{as} \quad n \to \infty.$$

In the rest of the paper we shall always assume that the parameter $s > 1$. The case $s = 1$ (i.e., when we deal with a G-metric space) can be handled easily.

Definition 2.1. Let $(X, G, \preceq)$ be an ordered G_b-metric space. A mapping $f : X \to X$ is called a G-rational Geraghty contraction of type A if there exists $\beta \in \mathfrak{F}_s$ such that

$$G(fx, fy, fz) \le \beta(M_A(x, y, z))M_A(x, y, z) \tag{2.1}$$

for all comparable elements $x, y, z \in X$, where

$$\begin{aligned} M_A(x, y, z) = \max\Big\{&G(x, y, z), \frac{G(x, fx, fy)G(y, fy, fz)G(z, fz, fx)}{1 + G(x, y, z)G(fx, fy, fz)}, \\ &\frac{G(x, fx, fy)G(y, fy, fz)G(z, fz, fx)}{1 + G^2(fx, fy, fz)}\Big\}. \end{aligned} \tag{2.2}$$

Theorem 2.2. *Let $(X, G, \preceq)$ be an ordered G_b-complete G_b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that f is a G-rational Geraghty contraction of type A. If*
(I) f is continuous, or
(II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n x_0$. Since $x_0 \preceq fx_0$ and f is increasing, we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^n x_0 \preceq f^{n+1}x_0 \preceq \cdots .$$

We will do the proof in the following steps.

Step 1. We will show that $\lim\limits_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.1) we have

$$G(x_n, x_{n+1}, x_{n+2}) = G(fx_{n-1}, fx_n, fx_{n+1}) \leq \beta(M_A(x_{n-1}, x_n, x_{n+1}))M_A(x_{n-1}, x_n, x_{n+1}), \tag{2.3}$$

where

$$\begin{aligned}
&M_A(x_{n-1}, x_n, x_{n+1})\\
&= \max\left\{G(x_{n-1}, x_n, x_{n+1}), \frac{G(x_{n-1}, fx_{n-1}, fx_n)G(x_n, fx_n, fx_{n+1})G(x_{n+1}, fx_{n+1}, fx_{n-1})}{1 + G(x_{n-1}, x_n, x_{n+1})G(fx_{n-1}, fx_n, fx_{n+1})},\right.\\
&\quad\left.\frac{G(x_{n-1}, fx_{n-1}, fx_n)G(x_n, fx_n, fx_{n+1})G(x_{n+1}, fx_{n+1}, fx_{n-1})}{1 + G^2(fx_{n-1}, fx_n, fx_{n+1})}\right\}\\
&= \max\left\{G(x_{n-1}, x_n, x_{n+1}), \frac{G(x_{n-1}, x_n, x_{n+1})G(x_n, x_{n+1}, x_{n+2})G(x_{n+1}, x_{n+2}, x_n)}{1 + G(x_{n-1}, x_n, x_{n+1})G(x_n, x_{n+1}, x_{n+2})},\right.\\
&\quad\left.\frac{G(x_{n-1}, x_n, x_{n+1})G(x_n, x_{n+1}, x_{n+2})G(x_{n+1}, x_{n+2}, x_n)}{1 + G^2(x_n, x_{n+1}, x_{n+2})}\right\}\\
&\leq \max\left\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\right\}.
\end{aligned}$$

If $\max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\} = G(x_n, x_{n+1}, x_{n+2})$, then from (2.3) we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &\leq \beta(M_A(x_{n-1}, x_n, x_{n+1}))G(x_n, x_{n+1}, x_{n+2})\\
&< \frac{1}{s}G(x_n, x_{n+1}, x_{n+2})\\
&< G(x_n, x_{n+1}, x_{n+2}),
\end{aligned}$$

which is a contradiction. Hence, $\max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\} = G(x_{n-1}, x_n, x_{n+1})$. So, from (2.3),

$$G(x_n, x_{n+1}, x_{n+2}) \leq \beta(M_A(x_{n-1}, x_n, x_{n+1}))G(x_{n-1}, x_n, x_{n+1}) < \frac{1}{s}G(x_{n-1}, x_n, x_{n+1}).$$

Continuing by induction, we get that

$$G(x_n, x_{n+1}, x_{n+2}) < \frac{1}{s^n}G(x_0, x_1, x_2) \to 0 \text{ as } n \to \infty.$$

Hence, $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+2}) = 0$. Consequently, using ($G_b3$), we get that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.4}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a G_b-Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } G(x_{m_i}, x_{n_i}, x_{n_i}) \geq \varepsilon. \tag{2.5}$$

This means that

$$G(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon.$$

From the rectangular inequality, we get

$$\varepsilon \leq G(x_{m_i}, x_{n_i}, x_{n_i}) \leq sG(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + sG(x_{m_i+1}, x_{n_i}, x_{n_i}).$$

By taking the upper limit as $i \to \infty$ and by (2.4), we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i}, x_{n_i}).$$

From the definition of $M_A(x, y, z)$ and the above limits,

$$\begin{aligned}
\limsup_{i\to\infty} M_A(x_{m_i}, &x_{n_i-1}, x_{n_i-1}) \\
&= \limsup_{i\to\infty} \max\{G(x_{m_i}, x_{n_i-1}, x_{n_i-1}), \\
&\quad \frac{G(x_{m_i}, fx_{m_i}, fx_{n_i-1})G(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, fx_{n_i-1}, fx_{m_i})}{1+G(x_{m_i}, x_{n_i-1}, x_{n_i-1})G(fx_{m_i}, fx_{n_i-1}, fx_{n_i-1})}, \\
&\quad \frac{G(x_{m_i}, fx_{m_i}, fx_{n_i-1})G(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, fx_{n_i-1}, fx_{m_i})}{1+G^2(fx_{m_i}, fx_{n_i-1}, fx_{n_i-1})}\} \\
&= \limsup_{i\to\infty} \max\{G(x_{m_i}, x_{n_i-1}, x_{n_i-1}), \\
&\quad \frac{G(x_{m_i}, x_{m_i+1}, x_{n_i})G(x_{n_i-1}, x_{n_i}, x_{n_i})G(x_{n_i-1}, x_{n_i}, x_{m_i+1})}{1+G(x_{m_i}, x_{n_i-1}, x_{n_i-1})G(x_{m_i+1}, x_{n_i}, x_{n_i})}, \\
&\quad \frac{G(x_{m_i}, x_{m_i+1}, x_{n_i})G(x_{n_i-1}, x_{n_i}, x_{n_i})G(x_{n_i-1}, x_{n_i}, x_{m_i+1})}{1+G^2(x_{m_i+1}, x_{n_i}, x_{n_i})}\} \\
&\le \varepsilon.
\end{aligned}$$

Now, from (2.1) and the above inequalities, we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\le \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i}, x_{n_i}) \\
&\le \limsup_{i\to\infty} \beta(M_A(x_{m_i}, x_{n_i-1}, x_{n_i-1})) \limsup_{i\to\infty} M_A(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \\
&\le \varepsilon \limsup_{i\to\infty} \beta(M_A(x_{m_i}, x_{n_i-1}, x_{n_i-1}))
\end{aligned}$$

which implies that $\frac{1}{s} \le \limsup_{i\to\infty} \beta(M_A(x_{m_i}, x_{n_i-1}, x_{n_i-1}))$. Now, as $\beta \in \mathfrak{F}_s$ we conclude that $M_A(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$ which yields that $G(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$. Consequently,

$$G(x_{m_i}, x_{n_i}, x_{n_i}) \le sG(x_{m_i}, x_{n_i-1}, x_{n_i-1}) + sG(x_{n_i-1}, x_{n_i}, x_{n_i}) \to 0,$$

a contradiction to (2.5). Therefore, $\{x_n\}$ is a G_b-Cauchy sequence. G_b-completeness of X yields that $\{x_n\}$ G_b-converges to a point $u \in X$.

Step 3. u is a fixed point of f.

First of all, if f is continuous, then we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) hold. Using the assumption on X we have $x_n \preceq u$ for $n \in \mathbb{N}$. Now, we show that $u = fu$. By Lemma 1.13,

$$\begin{aligned}
\frac{1}{s^3} G(u, u, fu) &\le \limsup_{n\to\infty} G(x_{n+1}, x_{n+1}, fu) \\
&\le \limsup_{n\to\infty} \beta(M_A(x_n, x_n, u)) \limsup_{n\to\infty} M_A(x_n, x_n, u),
\end{aligned}$$

where

$$\begin{aligned}
\lim_{n\to\infty} M_A(x_n, x_n, u) = \lim_{n\to\infty} \max\{&G(x_n, x_n, u), \frac{G(x_n, fx_n, fx_n)G(x_n, fx_n, fu)G(u, fu, fx_n)}{1+G(x_n, x_n, u)G(fx_n, fx_n, fu)}, \\
&\frac{G(x_n, fx_n, fx_n)G(x_n, fx_n, fu)G(u, fu, fx_n)}{1+G^2(fx_n, fx_n, fu)}\} = 0.
\end{aligned}$$

Therefore, we deduce that $G(u, u, fu) = 0$, so $u = fu$.

Finally, suppose that the set of fixed points of f is well ordered. Assume, to the contrary, that u and v are two distinct fixed points of f. Then by (2.1), we have

$$G(u,v,v)=G(fu,fv,fv)\le\beta(M_A(u,v,v))M_A(u,v,v)=\beta(G(u,v,v))G(u,v,v)<\frac{1}{s}G(u,v,v), \tag{2.6}$$

because

$$M_A(u,v,v)=\max\left\{G(u,v,v),\frac{G(u,u,v)G(v,v,v)G(v,v,u)}{1+G^2(u,v,v)}\right\}=G(u,v,v).$$

So, we get $G(u,v,v)<\frac{1}{s}G(u,v,v)$, a contradiction. Hence $u=v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is a singleton and so it is well ordered. □

Definition 2.3. Let $(X,G,\preceq)$ be an ordered G_{b}-metric space. A mapping $f:X\to X$ is called a G-rational Geraghty contraction of type B if, there exists $\beta\in\mathfrak{F}_s$ such that,

$$G(fx,fy,fz)\le\beta(M_B(x,y,z))M_B(x,y,z) \tag{2.7}$$

for all comparable elements $x,y,z\in X$, where

$$\begin{aligned}M_B&(x,y,z)\\ &=\max\Big\{G(x,y,z),\frac{G(x,x,fx)G(x,x,fy)+G(y,y,fy)G(y,y,fx)}{1+s[G(x,x,fx)+G(y,y,fy)]},\\ &\frac{G(y,y,fy)G(y,y,fz)+G(z,z,fz)G(z,z,fy)}{1+s[G(y,y,fy)+G(z,z,fz)]},\frac{G(x,x,fx)G(x,x,fy)+G(y,y,fy)G(y,y,fx)}{1+G(x,x,fy)+G(y,y,fx)},\\ &\frac{G(y,y,fy)G(y,y,fz)+G(z,z,fz)G(z,z,fy)}{1+G(y,y,fz)+G(z,z,fy)}\Big\}.\end{aligned} \tag{2.8}$$

Theorem 2.4. *Let $(X,G,\preceq)$ be an ordered G_b-complete G_b-metric space. Let $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq fx_0$. Suppose that f is a G-rational Geraghty contractive mapping of type B. If*
(I) f is continuous, or
(II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n\to u\in X$, one has $x_n\preceq u$ for all $n\in\mathbb{N}$, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n=f^nx_0$. Since $x_0\preceq fx_0$ and f is an increasing function, we obtain by induction that

$$x_0\preceq fx_0\preceq f^2x_0\preceq\cdots\preceq f^nx_0\preceq f^{n+1}x_0\preceq\cdots.$$

We will make the proof in the following steps.

Step I: We will show that $\lim_{n\to\infty}G(x_n,x_{n+1},x_{n+1})=0$. Without any loss of generality, we may assume that $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}$. Since $x_n\preceq x_{n+1}$ for each $n\in\mathbb{N}$, then by (2.7) we have

$$\begin{aligned}G(x_n,x_{n+1},x_{n+2})&=G(fx_{n-1},fx_n,fx_{n+1})\le\beta(M_B(x_{n-1},x_n,x_{n+1}))M_B(x_{n-1},x_n,x_{n+1})\\ &\le\beta(M_B(x_{n-1},x_n,x_{n+1}))G(x_{n-1},x_n,x_{n+1})<\frac{1}{s}G(x_{n-1},x_n,x_{n+1}),\end{aligned} \tag{2.9}$$

because

$$\begin{aligned}M_B(x_{n-1},x_n,x_{n+1})=\max\{&G(x_{n-1},x_n,x_{n+1}),\\ &\frac{G(x_{n-1},x_{n-1},fx_{n-1})G(x_{n-1},x_{n-1},fx_n)+G(x_n,x_n,fx_n)G(x_n,x_n,fx_{n-1})}{1+s[G(x_{n-1},x_{n-1},fx_{n-1})+G(x_n,x_n,fx_n)]},\end{aligned}$$

$$
\begin{aligned}
&\frac{G(x_n,x_n,fx_n)G(x_n,x_n,fx_{n+1})+G(x_{n+1},x_{n+1},fx_{n+1})G(x_{n+1},x_{n+1},fx_n)}{1+s[G(x_n,x_n,fx_n)+G(x_{n+1},x_{n+1},fx_{n+1})]},\\
&\frac{G(x_{n-1},x_{n-1},fx_{n-1})G(x_{n-1},x_{n-1},fx_n)+G(x_n,x_n,fx_n)G(x_n,x_n,fx_{n-1})}{1+G(x_{n-1},x_{n-1},fx_n)+G(x_n,x_n,fx_{n-1})},\\
&\frac{G(x_n,x_n,fx_n)G(x_n,x_n,fx_{n+1})+G(x_{n+1},x_{n+1},fx_{n+1})G(x_{n+1},x_{n+1},fx_n)}{1+G(x_n,x_n,fx_{n+1})+G(x_{n+1},x_{n+1},fx_n)}\}\\
&=\max\{G(x_{n-1},x_n,x_{n+1}),\\
&\frac{G(x_{n-1},x_{n-1},x_n)G(x_{n-1},x_{n-1},x_{n+1})+G(x_n,x_n,x_{n+1})G(x_n,x_n,x_n)}{1+s[G(x_{n-1},x_{n-1},x_n)+G(x_n,x_n,x_{n+1})]},\\
&\frac{G(x_n,x_n,x_{n+1})G(x_n,x_n,x_{n+2})+G(x_{n+1},x_{n+1},x_{n+2})G(x_{n+1},x_{n+1},x_{n+1})}{1+s[G(x_n,x_n,x_{n+1})+G(x_{n+1},x_{n+1},x_{n+2})]},\\
&\frac{G(x_{n-1},x_{n-1},x_n)G(x_{n-1},x_{n-1},x_{n+1})+G(x_n,x_n,x_{n+1})G(x_n,x_n,x_n)}{1+G(x_{n-1},x_{n-1},x_{n+1})+G(x_n,x_n,x_n)},\\
&\frac{G(x_n,x_n,x_{n+1})G(x_n,x_n,x_{n+2})+G(x_{n+1},x_{n+1},x_{n+2})G(x_{n+1},x_{n+1},x_{n+1})}{1+G(x_n,x_n,x_{n+2})+G(x_{n+1},x_{n+1},x_{n+1})}\}\\
&\le\max\{G(x_{n-1},x_n,x_{n+1}),\\
&\frac{G(x_{n-1},x_{n-1},x_n)s[G(x_{n-1},x_{n-1},x_n)+G(x_n,x_n,x_{n+1})]}{1+s[G(x_{n-1},x_{n-1},x_n)+G(x_n,x_n,x_{n+1})]},\\
&\frac{G(x_n,x_n,x_{n+1})s[G(x_n,x_n,x_{n+1})+G(x_{n+1},x_{n+1},x_{n+2})]}{1+s[G(x_n,x_n,x_{n+1})+G(x_{n+1},x_{n+1},x_{n+2})]},\\
&\frac{G(x_{n-1},x_{n-1},x_n)G(x_{n-1},x_{n-1},x_{n+1})}{1+G(x_{n-1},x_{n-1},x_{n+1})+G(x_n,x_n,x_n)},\\
&\frac{G(x_n,x_n,x_{n+1})G(x_n,x_n,x_{n+2})}{1+G(x_n,x_n,x_{n+2})+G(x_{n+1},x_{n+1},x_{n+1})}\}\\
&\le\max\{G(x_{n-1},x_n,x_{n+1}),G(x_{n-1},x_{n-1},x_n),G(x_n,x_n,x_{n+1})\}\\
&\le\max\{G(x_{n-1},x_n,x_{n+1}),G(x_n,x_{n+1},x_{n+2})\},
\end{aligned}
$$

and it is easy to see that $\max\{G(x_{n-1},x_n,x_{n+1}),G(x_n,x_{n+1},x_{n+2})\}=G(x_{n-1},x_n,x_{n+1})$. Hence, in the same way as in the proof of the preceding theorem, we obtain from (2.9) that

$$\lim_{n\to\infty}G(x_n,x_{n+1},x_{n+1})=0$$

is true.

Step 2. Now, we prove that the sequence $\{x_n\}$ is a G_b-Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \text{ and } G(x_{m_i},x_{n_i},x_{n_i})\ge\varepsilon.$$

This means that

$$G(x_{m_i},x_{n_i-1},x_{n_i-1})<\varepsilon.$$

As in the proof of Theorem 2.2, we have,

$$\frac{\varepsilon}{s}\le\limsup_{i\to\infty}G(x_{m_i+1},x_{n_i},x_{n_i}).$$

From the definition of $M_B(x,y,z)$ and the above limits,

$$
\begin{aligned}
&\limsup_{i\to\infty} M_B(x_{m_i}, x_{n_i-1}, x_{n_i-1})\\
&= \limsup_{i\to\infty} \max\{G(x_{m_i}, x_{n_i-1}, x_{n_i-1}),\\
&\quad \frac{G(x_{m_i}, x_{m_i}, fx_{m_i})G(x_{m_i}, x_{m_i}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{m_i})}{1 + s[G(x_{m_i}, x_{m_i}, fx_{m_i}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\
&\quad \frac{G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1 + s[G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\
&\quad \frac{G(x_{m_i}, x_{m_i}, fx_{m_i})G(x_{m_i}, x_{m_i}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{m_i})}{1 + G(x_{m_i}, x_{m_i}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{m_i})},\\
&\quad \left.\frac{G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1 + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}\right\}\\
&\le \varepsilon.
\end{aligned}
$$

Now, from (2.7) and the above inequalities, we have

$$
\begin{aligned}
\frac{\varepsilon}{s} &\le \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i}, x_{n_i})\\
&\le \limsup_{i\to\infty} \beta(M_B(x_{m_i}, x_{n_i-1}, x_{n_i-1})) \limsup_{i\to\infty} M_B(x_{m_i}, x_{n_i-1}, x_{n_i-1})\\
&\le \varepsilon \limsup_{i\to\infty} \beta(M_B(x_{m_i}, x_{n_i-1}, x_{n_i-1}))
\end{aligned}
$$

which implies that $\frac{1}{s} \le \limsup_{i\to\infty} \beta(M_B(x_{m_i}, x_{n_i-1}, x_{n_i-1}))$. Now, as $\beta \in \mathfrak{F}_s$ we conclude that $\{x_n\}$ is a G_b-Cauchy sequence. G_b-completeness of X yields that $\{x_n\}$ G_b-converges to a point $u \in X$.

Step 3. u is a fixed point of f. This step is proved in the same way as Step 3 of Theorem 2.2. □

Definition 2.5. Let $(X, G, \preceq)$ be an ordered G_b-metric space. A mapping $f : X \to X$ is called a G-rational Geraghty contraction of type C if there exists $\beta \in \mathfrak{F}_s$ such that,

$$2s^3 G(fx, fy, fz) \le \beta(M_C(x, y, z))M_C(x, y, z) \tag{2.10}$$

for all comparable elements $x, y, z \in X$, where

$$
\begin{aligned}
M_C(x, y, z) = \max\Bigg\{&G(x, y, z), \frac{G(x, x, fx)G(y, y, fy)}{1 + s[G(x, y, y) + G(x, x, fy) + G(y, y, fx)]},\\
&\frac{G(x, y, y)G(x, x, y)}{1 + G(x, fx, fx) + G(y, y, fx) + G(y, fy, fy)},\\
&\frac{G(y, y, fy)G(z, z, fz)}{1 + s[G(y, z, z) + G(y, y, fz) + G(z, z, fy)]},\\
&\frac{G(y, z, z)G(y, y, z)}{1 + G(y, fy, fy) + G(z, z, fy) + G(z, fz, fz)}\Bigg\}.
\end{aligned}
\tag{2.11}
$$

Theorem 2.6. *Let $(X, G, \preceq)$ be an ordered G_b-complete G_b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that f is a G-rational Geraghty contractive mapping of type C. If*

(I) f is continuous, or,

(II) whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n x_0$.

Step 1. We will show that $\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$. Without loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (2.10) we have

$$\begin{aligned}
G(x_n, x_{n+1}, x_{n+2}) &\leq 2s^3 G(x_n, x_{n+1}, x_{n+2}) = 2s^3 G(fx_{n-1}, fx_n, fx_{n+1}) \\
&\leq \beta(M_C(x_{n-1}, x_n, x_{n+1}))M_C(x_{n-1}, x_n, x_{n+1}) \\
&\leq \beta(M_C(x_{n-1}, x_n, x_{n+1}))G(x_{n-1}, x_n, x_{n+1}) \\
&< \frac{1}{s} G(x_{n-1}, x_n, x_{n+1}),
\end{aligned}$$

because

$$\begin{aligned}
&M_C(x_{n-1}, x_n, x_{n+1}) \\
&\quad = \max\{G(x_{n-1}, x_n, x_{n+1}), \\
&\qquad \frac{G(x_{n-1}, x_{n-1}, fx_{n-1})G(x_n, x_n, fx_n)}{1 + s[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n-1}, fx_n) + G(x_n, x_n, fx_{n-1})]}, \\
&\qquad \frac{G(x_{n-1}, x_n, x_n)G(x_{n-1}, x_{n-1}, x_n)}{1 + G(x_{n-1}, fx_{n-1}, fx_{n-1}) + G(x_n, x_n, fx_{n-1}) + G(x_n, fx_n, fx_n)}, \\
&\qquad \frac{G(x_n, x_n, fx_n)G(x_{n+1}, x_{n+1}, fx_{n+1})}{1 + s[G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, fx_{n+1}) + G(x_{n+1}, x_{n+1}, fx_n)]}, \\
&\qquad \frac{G(x_n, x_{n+1}, x_{n+1})G(x_n, x_n, x_{n+1})}{1 + G(x_n, fx_n, fx_n) + G(x_{n+1}, x_{n+1}, fx_n) + G(x_{n+1}, fx_{n+1}, fx_{n+1})}\Big\} \\
&\quad = \max\{G(x_{n-1}, x_n, x_{n+1}), \\
&\qquad \frac{G(x_{n-1}, x_{n-1}, x_n)G(x_n, x_n, x_{n+1})}{1 + s[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n-1}, x_{n+1}) + G(x_n, x_n, x_n)]}, \\
&\qquad \frac{G(x_{n-1}, x_n, x_n)G(x_{n-1}, x_{n-1}, x_n)}{1 + G(x_{n-1}, x_n, x_n) + G(x_n, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})}, \\
&\qquad \frac{G(x_n, x_n, x_{n+1})G(x_{n+1}, x_{n+1}, x_{n+2})}{1 + s[G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_{n+2}) + G(x_{n+1}, x_{n+1}, x_{n+1})]}, \\
&\qquad \frac{G(x_n, x_{n+1}, x_{n+1})G(x_n, x_n, x_{n+1})}{1 + G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})}\Big\} \\
&\quad \leq \max\{G(x_{n-1}, x_n, x_{n+1}), \\
&\qquad \frac{G(x_{n-1}, x_{n-1}, x_n)s[G(x_n, x_n, x_{n-1}) + G(x_{n-1}, x_{n-1}, x_{n+1})]}{1 + s[G(x_{n-1}, x_n, x_n) + G(x_{n-1}, x_{n-1}, x_{n+1})]}, \\
&\qquad \frac{G(x_{n-1}, x_n, x_n)G(x_{n-1}, x_{n-1}, x_n)}{1 + G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})}, \\
&\qquad \frac{G(x_n, x_n, x_{n+1})s[G(x_{n+1}, x_{n+1}, x_n) + G(x_n, x_n, x_{n+2})]}{1 + s[G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_{n+2})]}, \\
&\qquad \frac{G(x_n, x_{n+1}, x_{n+1})G(x_n, x_n, x_{n+1})}{1 + G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2})}\Big\} \\
&\quad \leq \max\{G(x_{n-1}, x_n, x_{n+1}), G(x_{n-1}, x_{n-1}, x_n), G(x_n, x_n, x_{n+1})\} \\
&\quad \leq \max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\}
\end{aligned}$$

and it is again easy to see that $\max\{G(x_{n-1}, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+2})\} = G(x_{n-1}, x_n, x_{n+1})$. Hence, in the same way as in the proof of Theorems 2.2 and 2.4, we obtain that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0 \tag{2.12}$$

is true.

Step 2. Now, we prove that the sequence $\{x_n\}$ is a G_b-Cauchy sequence. Suppose the contrary. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } G(x_{m_i}, x_{n_i}, x_{n_i}) \geq \varepsilon.$$

This means that

$$G(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon.$$

Hence,

$$G(x_{m_i}, x_{m_i}, x_{n_i-1}) < 2s\varepsilon.$$

From the rectangular inequality, we get

$$\varepsilon \leq G(x_{m_i}, x_{n_i}, x_{n_i}) \leq sG(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + sG(x_{m_i+1}, x_{n_i}, x_{n_i}).$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i}, x_{n_i}).$$

From (2.12) and using the rectangular inequality, we get

$$\begin{aligned}\varepsilon \leq G(x_{m_i}, x_{n_i}, x_{n_i}) &\leq sG(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + sG(x_{m_i+1}, x_{n_i}, x_{n_i})\\ &\leq sG(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + s^2G(x_{m_i+1}, x_{n_i-1}, x_{n_i-1}) + s^2G(x_{n_i-1}, x_{n_i}, x_{n_i}).\end{aligned}$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s^2} \leq \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i-1}, x_{n_i-1}).$$

From the definition of $M_C(x, y, z)$ and the above limits,

$$\begin{aligned}&\limsup_{i\to\infty} M_C(x_{m_i}, x_{n_i-1}, x_{n_i-1})\\ &\quad= \limsup_{i\to\infty} \max\{G(x_{m_i}, x_{n_i-1}, x_{n_i-1}),\\ &\qquad \frac{G(x_{m_i}, x_{m_i}, fx_{m_i})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1 + s[G(x_{m_i}, x_{n_i-1}, x_{n_i-1}) + G(x_{m_i}, x_{m_i}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{m_i})]},\\ &\qquad \frac{G(x_{m_i}, x_{n_i-1}, x_{n_i-1})G(x_{m_i}, x_{m_i}, x_{n_i-1})}{1 + G(x_{m_i}, fx_{m_i}, fx_{m_i}) + G(x_{n_i-1}, x_{n_i-1}, fx_{m_i}) + G(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1})},\\ &\qquad \frac{G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1 + s[G(x_{n_i-1}, x_{n_i-1}, x_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\ &\qquad \frac{G(x_{n_i-1}, x_{n_i-1}, x_{n_i-1})G(x_{n_i-1}, x_{n_i-1}, x_{n_i-1})}{1 + G(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + G(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1})}\Bigg\}\\ &\quad\leq \varepsilon \cdot 2s\varepsilon \cdot \frac{s^2}{\varepsilon} = 2s^3\varepsilon.\end{aligned}$$

Now, from (2.10) and the above inequalities, we have

$$\begin{aligned}2s^3 \cdot \frac{\varepsilon}{s} &\leq 2s^3 \cdot \limsup_{i\to\infty} G(x_{m_i+1}, x_{n_i}, x_{n_i})\\ &\leq \limsup_{i\to\infty} \beta(M_C(x_{m_i}, x_{n_i-1}, x_{n_i})) \limsup_{i\to\infty} M_C(x_{m_i}, x_{n_i-1}, x_{n_i})\\ &\leq 2s^3\varepsilon \limsup_{i\to\infty} \beta(M_C(x_{m_i}, x_{n_i-1}, x_{n_i-1}))\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M_C(x_{m_i}, x_{n_i-1}, x_{n_i-1}))$. Now, as $\beta \in \mathfrak{F}_s$ we conclude that $\{x_n\}$ is a G_b-Cauchy sequence. G_b-completeness of X yields that $\{x_n\}$ G_b-converges to a point $u \in X$.

Step 3. u is a fixed point of f.

When f is continuous, the proof is straightforward. Now, let (II) hold. We omit the proof as it is similar to the proof of Step 3 of Theorem 2.2. □

If in the above theorems we take $\beta(t) = r$, where $0 \le r < \frac{1}{s}$, then we have the following corollary.

Corollary 2.7. *Let $(X, G, \preceq)$ be a partially ordered G_b-complete G_b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$G(fx, fy, fz) \le rM(x, y, z)$$

for all comparable elements $x, y, z \in X$, where

$$M(x, y, z) = M_A(x, y, z) \text{ or } M(x, y, z) = M_B(x, y, z)$$

(see (2.2), (2.8)), or

$$2s^3G(fx, fy, fz) \le rM_C(x, y, z)$$

(see (2.11)). If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

The following corollaries can also be easily deduced from the proved theorems.

Corollary 2.8. *Let $(X, G, \preceq)$ be a partially ordered G_b-complete G_b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\begin{aligned} G(fx, fy, fz) \le{}& \alpha G(x, y, z) + \beta \frac{G(x, fx, fy)G(y, fy, fz)G(z, fz, fx)}{1 + G(x, y, z)G(fx, fy, fz)} \\ &+ \gamma \frac{G(x, fx, fy)G(y, fy, fz)G(z, fz, fx)}{1 + G^2(fx, fy, fz)}, \end{aligned}$$

or

$$\begin{aligned} G(fx, fy, fz) \le{}& aG(x, y, z) + b\frac{G(x, x, fx)G(x, x, fy) + G(y, y, fy)G(y, y, fx)}{1 + s[G(x, x, fx) + G(y, y, fy)]} \\ &+ c\frac{G(y, y, fy)G(y, y, fz) + G(z, z, fz)G(z, z, fy)}{1 + s[G(y, y, fy) + G(z, z, fz)]} \\ &+ d\frac{G(x, x, fx)G(x, x, fy) + G(y, y, fy)G(y, y, fx)}{1 + G(x, x, fy) + G(y, y, fx)} \\ &+ e\frac{G(y, y, fy)G(y, y, fz) + G(z, z, fz)G(z, z, fy)}{1 + G(y, y, fz) + G(z, z, fy)}, \end{aligned}$$

or

$$\begin{aligned} 2s^3G(fx, fy, fz) \le{}& aG(x, y, z) + b\frac{G(x, x, fx)G(y, y, fy)}{1 + s[G(x, y, y) + G(x, x, fy) + G(y, y, fx)]} \\ &+ c\frac{G(x, y, y)G(x, x, y)}{1 + G(x, fx, fx) + G(y, y, fx) + G(y, fy, fy)} \\ &+ d\frac{G(y, y, fy)G(z, z, fz)}{1 + s[G(y, z, z) + G(y, y, fz) + G(z, z, fy)]} \\ &+ e\frac{G(y, z, z)G(y, y, z)}{1 + G(y, fy, fy) + G(z, z, fy) + G(z, fz, fz)}, \end{aligned}$$

for all comparable elements $x, y, z \in X$, where $\alpha, \beta, \gamma, a, b, c, d, e \ge 0$ and $0 \le \alpha + \beta + \gamma < \frac{1}{s}$ and $0 \le a + b + c + d + e < \frac{1}{s}$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

Corollary 2.9. *Let $(X, G, \preceq)$ be a partially ordered G_b-complete G_b-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f^m x_0$ and*

$$G(f^m x, f^m y, f^m z) \leq \beta(M(x, y, z))M(x, y, z)$$

for all comparable elements $x, y, z \in X$, where

$$M(x,y,z) = \max\Big\{G(x,y,z), \frac{G(x,f^mx,f^my)G(y,f^my,f^mz)G(z,f^mz,f^mx)}{1+G(x,y,z)G(f^mx,f^my,f^mz)}, \frac{G(x,f^mx,f^my)G(y,f^my,f^mz)G(z,f^mz,f^mx)}{1+G^2(f^mx,f^my,f^mz)}\Big\},$$

or

$$\begin{aligned} &M(x,y,z) \\ &= \max\Big\{G(x,y,z), \frac{G(x,x,f^mx)G(x,x,f^my)+G(y,y,f^my)G(y,y,f^mx)}{1+s[G(x,x,f^mx)+G(y,y,f^my)]}, \\ &\frac{G(y,y,f^my)G(y,y,f^mz)+G(z,z,f^mz)G(z,z,f^my)}{1+s[G(y,y,f^my)+G(z,z,f^mz)]}, \\ &\frac{G(x,x,f^mx)G(x,x,f^my)+G(y,y,f^my)G(y,y,f^mx)}{1+G(x,x,f^my)+G(y,y,f^mx)}, \\ &\frac{G(y,y,f^my)G(y,y,f^mz)+G(z,z,f^mz)G(z,z,f^my)}{1+G(y,y,f^mz)+G(z,z,f^my)}\Big\}, \end{aligned}$$

or

$$2s^3 G(f^m x, f^m y, f^m z) \leq \beta(M(x, y, z))M(x, y, z),$$

where

$$\begin{aligned} M(x,y,z) = \max\Big\{&G(x,y,z), \frac{G(x,x,f^mx)G(y,y,f^my)}{1+s[G(x,y,y)+G(x,x,f^my)+G(y,y,f^mx)]}, \\ &\frac{G(x,y,y)G(x,x,y)}{1+G(x,f^mx,f^mx)+G(y,y,f^mx)+G(y,f^my,f^my)}, \\ &\frac{G(y,y,f^my)G(z,z,f^mz)}{1+s[G(y,z,z)+G(y,y,f^mz)+G(z,z,f^my)]}, \\ &\frac{G(y,z,z)G(y,y,z)}{1+G(y,f^my,f^my)+G(z,z,f^my)+G(z,f^mz,f^mz)}\Big\}. \end{aligned}$$

for some positive integer m.

If f^m is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in N$, then f has a fixed point.

We illustrate a possible usage of Theorem 2.2 by the following example. It is clear that similar examples can be constructed for Theorems 2.4 and 2.6.

Example 2.10. Let $X = \{a, b, c\}$ and $G : X^3 \to \mathbb{R}^+$ be given as

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z, \\ 1, & \text{if } (x,y,z) \in \{(a,a,b),(a,b,b),(a,c,c),(b,c,c)\}, \\ 4, & \text{if } (x,y,z) \in \{(a,a,c),(b,b,c),(a,b,c)\}, \end{cases}$$

and extended by symmetry. Then it is easy to check that X is a G_b-metric space, with $s = 2$, which is asymmetric since, e.g., $G(a, a, c) \neq G(a, c, c)$ (this is important, since it is well-known that the results in symmetric G-metric (G_b-metric) spaces can usually be easily reduced to their standard metric (b-metric) counterparts). Define a reflexive and transitive order relation $\preceq$ on X by $c \preceq b \preceq a$ and a mapping $f : X \to X$ by

$$f = \begin{pmatrix} a & b & c \\ a & a & b \end{pmatrix}.$$

Then f is increasing, $c \preceq fc$ and the space $(X, G, \preceq)$ is G_b-complete. We will show that the contractive condition (2.1) of Theorem 2.2 is fulfilled with $\beta \in \mathfrak{F}_s$ given by

$$\beta(t) = \frac{1}{2}e^{-t/8} \text{ for } t \in (0, \infty) \text{ and } \beta(0) \in [0, 1/2).$$

Consider the following possible cases.

1. If $x = y = z$ or $x, y, z \in \{a, b\}$, then $G(fx, fy, fz) = 0$ and inequality (2.1) is trivial.

In all other cases $G(fx, fy, fz) = 1$.

2. If $(x, y, z) \in \{(a, a, c), (b, b, c), (a, b, c), \dots\}$ ($\dots$ stays for permutations), then $M_A(x, y, z) = G(x, y, z) = 4$. Thus, (2.1) is satisfied since it reduces to

$$1 \leq \frac{1}{2}e^{-4/8} \cdot 4, \ \textit{i.e.}, \ e^{1/2} \leq 2.$$

3. If $(x, y, z) \in \{(a, c, c), (b, c, c), \dots\}$ ($\dots$ stays for permutations), then it is easy to check that $M_A(x, y, z) = 8$. Thus, (2.1) is again satisfied since it reduces to

$$1 \leq \frac{1}{2}e^{-8/8} \cdot 8, \ \textit{i.e.}, \ e \leq 4.$$

All the assumptions of Theorem 2.2 are fulfilled and f has a unique fixed point (equal to a).

Note that simple non-rational Geraghty-type condition

$$G(fx, fy, fz) \leq \beta(G(x, y, z))G(x, y, z)$$

is not satisfied. Indeed, *e.g.*, for $(x, y, z) = (a, c, c)$ it reduces to $1 \leq \frac{1}{2}e^{-1/8} \cdot 1$ and does not hold.

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