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Heisenberg type uncertainty principle for continuous shearlet transform

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Abstract

We prove a Heisenberg type uncertainty principle for the continuous shearlet transform, and study two generalizations of it. Our work extends the shearlet theory.

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1. Introduction and Preliminaries

The uncertainty principle is a collection of mathematical statements expressing a fundamental property of the Fourier transform, namely, that a function f and its Fourier transform $\hat{f}$ cannot be simultaneously small. There are various kinds of formulation of uncertainty principles. For example, see V. Havin and B. Jöricke [9], D. Donoho and P. Strak [5], G. B. Folland and A. Sitaram [6], M. G. Cowling and J. F. Price [2].

S. Dahlke et al. in [4] employ the general uncertainty principle in order to derive mother wavelet functions that minimize the uncertainty relations derived for the infinitesimal generators of the wavelet group: scaling and translations. E. Wilczok [12] employ a Heisenberg type uncertainty principle in order to describe strict limits to maximal time-frequency resolution for the continuous wavelet transform and the continuous Gabor transform.

For the continuous shearlet transform, S. Dahlke et al. [3] show the minimizers of the uncertainty relations associated with the infinitesimal generators of the shearlet group: scaling, shear and translations. But, a description of the strict limits to maximal time-frequency resolution for shearlet is not given so far. Similar to the classical Gabor and wavelet cases, a very natural question arises: do there exist the strict

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limits to maximal time-frequency resolution related to the continuous shearlet transform? The answer is yes, since, K. Gröchenig has given a very important meta-uncertainty principle for time-frequency representation in [7].

Metatheorem: Every time-frequency representation comes with its own version of the uncertainty principle.

The shearlet transform [1, 8, 10, 11] has an advance over the classical wavelet transform as it provides information about the directionality within the image. One of the main problems in shearlet theory is the Heisenberg type uncertainty principle for the continuous shearlet transform.

In this paper, we present a recipe to derive a Heisenberg type uncertainty principle for the continuous shearlet transform.

1. Define a continuous shearlet transform, and choose a Heisenberg type uncertainty principle.

2. Replace the function f and $\hat{f}$ in the uncertainty principle by the continuous shearlet transform, and formulate a new uncertainty principle.

3. Prove the resulting inequality.

Many ideas in this paper are inspired by S. Dahlke et al. [3], E. Wilczok [12], M. G. Cowling et al. [2] and K. Gröchenig [7]. Our work extends shearlet theory.

The paper is organized as follows: in Section 2, we give some notation and definitions. Then in Section 3, we prove some lemmas on shearlet. In Section 4, we prove a Heisenberg type uncertainty principle for the continuous shearlet transform, and then we study two generalizations of it.

2. Preliminary

We will use the following conventions throughout the paper. For a function $f \in L^1(R^2) \cap L^2(R^2)$, the *Fourier transform* of f is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int\limits_{R^2} f(x)e^{-2\pi i\xi\cdot x}dx.$$

Set $A_a = \begin{pmatrix} a & o \\ 0 & \sqrt{a} \end{pmatrix}$ and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. We introduce the following fundamental operators of shearlet: the dilation operator

$$[D_{A_a}\psi](x) = |\det A_a|^{-\frac{1}{2}}\psi(A_a^{-1}x) \qquad a \in R^+, \;\; \psi \in L^2(R^2),$$

the shear operator

$$[D_{S_s}\psi](x) = \psi(S_s^{-1}x) \qquad s \in R, \;\; \psi \in L^2(R^2)$$

and the translation operator

$$[T_t\psi](x) = \psi(x - t) \qquad t \in R^2, \;\; \psi \in L^2(R^2).$$

Definition 2.1. Let $\psi \in L^2(R^2)$; shearlet for $a \in R^+$, $s \in R$, $t \in R^2$ is defined by

$$\psi_{a,s,t}(x) = \det|A_a|^{-\frac{1}{2}}\psi(A_a^{-1}S_s^{-1}(x - t)).$$

The shearlet system generated by ψ is defined by $\{\psi_{a,s,t}(x) : a \in R^+, s \in R, t \in R^2\}$.

The associated continuous shearlet transform of an $f \in L^2(R^2)$ is given by

$$SH_\psi f(a, s, t) = \langle f, \psi_{a,s,t}\rangle = \int\limits_{R^2} f(x)\overline{\psi_{a,s,t}(x)}dx. \tag{2.1}$$

We denote the norm on $L^p(R^2)$ by

$$\|f\|_p = \left(\int\limits_{R^2} |f(x)|^p dx\right)^{\frac{1}{p}}.$$

For an $f \in L^2(R^2)\backslash\{0\}$, the Heisenberg–Pauli–Weyl inequality is

$$\left(\int_{R^2} x^2|f(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{R^2} \xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \geq \frac{1}{2}\int_{R^2}|f(x)|^2dx,$$

which is generalized in [2] as

$$\left(\int_{R^2} x^p|f(x)|^pdx\right)^{\frac{1}{p}}\left(\int_{R^2} \xi^p|\hat{f}(\xi)|^pd\xi\right)^{\frac{1}{p}} \geq \frac{1}{2}\int_{R^2}|f(x)|^2dx.$$

3. Notation

Before entering derivation of uncertainty principle for continuous shearlet transform, we prove four lemmas about its properties.

Lemma 3.1. *Let $f, \psi \in L^2(R^2)$; then*

$$\hat{f}(\xi)\overline{\hat{\psi}(\xi S_s A_a)} = \det|A_a|^{-\frac{1}{2}}\mathcal{F}_t(SH_\psi f(a,s,t)), \tag{3.1}$$

where $a \in R^+$, $s \in R$, $t \in R^2$.

Proof. In fact, we have

$$\begin{aligned}\hat{\psi}_{a,s,t}(\xi) &= \det|A_a|^{-\frac{1}{2}}\int_{R^2}\psi(A_a^{-1}S_s^{-1}(x-t))e^{-2\pi i\xi x}dx\\ &= \det|A_a|^{\frac{1}{2}}\int_{R^2}\psi(y)e^{-2\pi i\xi(S_sA_ay+t)}dy, \quad \textit{Let } y = A_a^{-1}S_s^{-1}(x-t)\\ &= \det|A_a|^{\frac{1}{2}}\hat{\psi}(\xi S_sA_a)e^{-2\pi i\xi t}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}SH_\psi f(a,s,t) &= \int_{R^2} f(x)\overline{\psi_{a,s,t}(x)}dx\\ &= \int_{R^2}\hat{f}(\xi)\overline{\hat{\psi}_{a,s,t}(\xi)}d\xi \quad \textit{Planchel Theorem}\\ &= \int_{R^2}\hat{f}(\xi)\overline{\det|A_a|^{\frac{1}{2}}\hat{\psi}(\xi S_sA_a)e^{-2\pi i\xi t}}d\xi\\ &= \det|A_a|^{\frac{1}{2}}\int_{R^2}\hat{f}(\xi)\overline{\hat{\psi}(\xi S_sA_a)}e^{2\pi i\xi t}d\xi\\ &= \det|A_a|^{\frac{1}{2}}\mathcal{F}^{-1}(\hat{f}(\xi)\overline{\hat{\psi}(\xi S_sA_a)})(t).\end{aligned}$$

A direct calculation yields

$$\begin{aligned}\det|A_a|^{\frac{1}{2}}\mathcal{F}^{-1}(\hat{f}(\xi)\overline{\hat{\psi}(\xi S_sA_a)})(t) &= SH_\psi f(a,s,t)\\ \mathcal{F}^{-1}(\hat{f}(\xi)\overline{\hat{\psi}(\xi S_sA_a)})(t) &= \det|A_a|^{-\frac{1}{2}}SH_\psi f(a,s,t)\\ \hat{f}(\xi)\overline{\hat{\psi}(\xi S_sA_a)} &= \det|A_a|^{-\frac{1}{2}}\mathcal{F}_t(SH_\psi f(a,s,t)),\end{aligned}$$

where $\mathcal{F}_t$ denotes the Fourier transform with respect to the variable t. □

Lemma 3.2. *Let $f, \psi \in L^2(R^2)$; then*

$$\int\limits_{R^2} |\mathcal{F}_t(SH_\psi f(a,s,t))(\xi)|^2 d\xi = \int\limits_{R^2} |SH_\psi f(a,s,t))|^2 dt,$$

where $a \in R^+$, $s \in R$, $t \in R^2$.

By using the Plancherel theorem, Lemma 3.2 could be proved.

Definition 3.3. Let $\psi \in L^2(R^2)$ be such that

$$C_\psi = \int\limits_R \int\limits_R \frac{|\hat{\psi}(\xi_x, \xi_y)|^2}{\xi_x^2} d\xi_x d\xi_y < \infty$$

is satisfied. Then ψ is admissible.

We need the following form of the admissibility condition.

Lemma 3.4. *Let $\psi \in L^2(R^2)$ and ψ be admissible. Then*

$$\int\limits_R \int\limits_R \frac{|\hat{\psi}(\xi S_s A_a)|^2}{a^{\frac{3}{2}}} dads = C_\psi < \infty \tag{3.2}$$

is satisfied.

Proof. In fact, we have

$$\begin{aligned}
\int\limits_R \int\limits_R \frac{|\hat{\psi}(\xi S_s A_a)|^2}{a^{\frac{3}{2}}} dads &= \int\limits_R \int\limits_R \frac{|\hat{\psi}(a\xi_x, \sqrt{a}s\xi_x + \sqrt{a}\xi_y)|^2}{a^{\frac{3}{2}}} dads \\
&= \int\limits_R \int\limits_R \frac{|\hat{\psi}(\nu_x, \sqrt{\nu_x \xi_x}s + \sqrt{\frac{\nu_x}{\xi_x}}\xi_y)|^2}{(\frac{\nu_x}{\xi_x})^{\frac{3}{2}}} \frac{d\nu_x}{\xi_x} ds \\
&= \int\limits_R \int\limits_R \frac{|\hat{\psi}(\nu_x, \sqrt{\nu_x \xi_x}s + \sqrt{\frac{\nu_x}{\xi_x}}\xi_y)|^2 \xi_x^{\frac{1}{2}}}{{\nu_x}^{\frac{3}{2}}} d\nu_x ds \\
&= \int\limits_R \int\limits_R \frac{|\hat{\psi}(\nu_x, \nu_y)|^2 \xi_x^{\frac{1}{2}}}{{\nu_x}^{\frac{3}{2}}} d\nu_x \frac{d\nu_y}{\sqrt{\nu_x \xi_x}} \\
&= \int\limits_R \int\limits_R \frac{|\hat{\psi}(\nu_x, \nu_y)|^2}{{\nu_x}^2} d\nu_x d\nu_y = C_\psi.
\end{aligned}$$

Since ψ is admissible, we obtain

$$\int\limits_R \int\limits_R \frac{|\hat{\psi}(\xi S_s A_a)|^2}{a^{\frac{3}{2}}} dads = C_\psi < \infty. \qquad \square$$

The admissibility condition for the continuous shearlet transform is given by the following lemma.

Lemma 3.5 ([3])**.** *Let $f, \psi \in L^2(R^2)$; then*

$$\int\limits_{R^2} \int\limits_R \int\limits_R |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3} = C_\psi \|f\|^2_{L^2(R^2)}, \tag{3.3}$$

where $a \in R^+$, $s \in R$, $t \in R^2$.

4. Uncertainty principle of Heisenberg type for continuous shearlet transform

We will prove the main results of our paper in this section.

Theorem 4.1. *Let $\psi \in L^2(R^2)$; then for arbitrary $f \in L^2(R^2)\backslash\{0\}$, we have*

$$\left(\int_{R^2}\int_{R}\int_{R^+} t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\left(\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \geq \frac{\sqrt{C_\psi}}{2}\|f\|^2_{L^2(R^2)}. \tag{4.1}$$

Proof. We get from admissibility condition (3.2) for ψ

$$C_\psi\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi = \int_{R}\int_{R^+}|\hat{\psi}(\xi S_s A_a)|^2\frac{dads}{a^{\frac{3}{2}}}\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi = \int_{R^2}\int_{R}\int_{R^+}\xi^2|\hat{f}(\xi)|^2|\hat{\psi}(\xi S_s A_a)|^2\frac{dads}{a^{\frac{3}{2}}}d\xi.$$

Put formula (3.1) into above equation and get

$$\begin{aligned}C_\psi\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi &= \det|A_a|^{-1}\int_{R^2}\int_{R}\int_{R^+}\xi^2|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2\frac{dads}{a^{\frac{3}{2}}}d\xi\\ &= \int_{R^2}\int_{R}\int_{R^+}\xi^2|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2\frac{dads}{a^3}d\xi.\end{aligned}$$

The Heisenberg–Pauli–Weyl inequality leads to

$$\left(\int_{R^2}t^2|SH_\psi f(a,s,t)|^2dt\right)^{\frac{1}{2}}\left(\int_{R^2}\xi^2|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2d\xi\right)^{\frac{1}{2}} \geq \frac{1}{2}\int_{R^2}|SH_\psi f(a,s,t)|^2dt$$

for all $a \in R^+$. Integrating with respect to $\frac{dads}{a^3}$, we obtain

$$\begin{aligned}&\int_{R}\int_{R^+}\left[\left(\int_{R^2}t^2|SH_\psi f(a,s,t)|^2dt\right)^{\frac{1}{2}}\left(\int_{R^2}\xi^2|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2d\xi\right)^{\frac{1}{2}}\right]\frac{dads}{a^3}\\ &\geq \frac{1}{2}\int_{R^2}\int_{R}\int_{R^+}|SH_\psi f(a,s,t)|^2\frac{dadsst}{a^3}.\end{aligned}$$

The right-hand side of the above inequality can be rewritten as $C_\psi\|f\|^2_{L^2(R^2)}$. Therefore, we get

$$\begin{aligned}&\left(\int_{R^2}\int_{R}\int_{R^+}t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\sqrt{C_\psi}\left(\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}\\ &=\left(\int_{R^2}\int_{R}\int_{R^+}t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\left(C_\psi\int_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}\\ &=\left(\int_{R^2}\int_{R}\int_{R^+}t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\left(\int_{R^2}\int_{R}\int_{R^+}\xi^2|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2\frac{dadsd\xi}{a^3}\right)^{\frac{1}{2}}\\ &\geq\frac{1}{2}\int_{R^2}\int_{R}\int_{R^+}|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3} = \frac{1}{2}C_\psi\|f\|^2_{L^2(R^2)}.\end{aligned}$$

From the above inequality, we have

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\sqrt{C_\psi}\left(\int\limits_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \geq \frac{1}{2}C_\psi\|f\|^2_{L^2(R^2)},$$

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}\left(\int\limits_{R^2}\xi^2|\hat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}} \geq \frac{\sqrt{C_\psi}}{2}\|f\|^2_{L^2(R^2)}.$$

So, we obtain the Heisenberg type uncertainty principle for the continuous shearlet transform. □

Theorem 4.2. *Let $\psi \in L^2(R^2)$, arbitrary $f \in L^2(R^2)\backslash\{0\}$, and $1 \leq p \leq 2$, then*

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\left(\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^pd\xi\right)^{\frac{1}{p}} \geq \frac{a^{\frac{3}{p}}\sqrt{C_\psi}}{2a^{\frac{3}{2}}}\|f\|^2_{L^2(R^2)} \tag{4.2}$$

Proof. We prove this theorem as follows.

$$\begin{aligned}
C_\psi^{\frac{p}{2}}\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^pd\xi &= \left(\int\limits_{R}\int\limits_{R^+}|\hat{\psi}(\xi S_sA_a)|^2\frac{dads}{a^{\frac{3}{2}}}\right)^{\frac{p}{2}}\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^pd\xi\\
&\geq \int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}\xi^p|\hat{f}(\xi)|^p|\hat{\psi}(\xi S_sA_a)|^p\frac{dads}{a^{\frac{3p}{4}}}d\xi\\
&= \det|A_a|^{-\frac{p}{2}}\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}\xi^p|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^p\frac{dads}{a^{\frac{3p}{4}}}d\xi\\
&= \int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}\xi^p|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^p\frac{dads}{a^{\frac{3p}{2}}}d\xi.
\end{aligned}$$

The generalization of Heisenberg–Pauli–Weyl inequality in [2, Theorem 1.2] leads to

$$\left(\int\limits_{R^2} t^p|SH_\psi f(a,s,t)|^pdt\right)^{\frac{1}{p}}\left(\int\limits_{R^2}\xi^p|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^pd\xi\right)^{\frac{1}{p}} \geq \frac{1}{2}\int\limits_{R^2}|SH_\psi f(a,s,t)|^2dt.$$

The right-hand side of the above inequality can be rewritten as $C_\psi\|f\|^2_{L^2(R^2)}$, so that we have

$$\begin{aligned}
&\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\frac{\sqrt{C_\psi}}{a^{\frac{3}{p}-\frac{3}{2}}}\left(\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^pd\xi\right)^{\frac{1}{p}}\\
&= \left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\left(\frac{C_\psi^{\frac{p}{2}}}{a^{3-\frac{3p}{2}}}\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^pd\xi\right)^{\frac{1}{p}}\\
&\geq \left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}\xi^p|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^p\frac{dadsd\xi}{a^3}\right)^{\frac{1}{p}}\\
&\geq \frac{1}{2}\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3} = \frac{1}{2}C_\psi\|f\|^2_{L^2(R^2)}.
\end{aligned}$$

From the above inequality, we get

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\frac{\sqrt{C_\psi}}{a^{\frac{3}{p}-\frac{3}{2}}}\left(\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^p d\xi\right)^{\frac{1}{p}} \geq \frac{C_\psi}{2}\|f\|^2_{L^2(R^2)},$$

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^p|SH_\psi f(a,s,t)|^p\frac{dadsdt}{a^3}\right)^{\frac{1}{p}}\left(\int\limits_{R^2}\xi^p|\hat{f}(\xi)|^p d\xi\right)^{\frac{1}{p}} \geq \frac{a^{\frac{3}{p}}\sqrt{C_\psi}}{2a^{\frac{3}{2}}}\|f\|^2_{L^2(R^2)}. \qquad \square$$

By putting $p = 2$ in Theorem 4.2, we get Theorem 4.1. So, Theorem 4.1 is a special case of Theorem 4.2.

Theorem 4.3. *Let $\psi \in L^2(R^2)$, $f \in L^2(R^2)\backslash\{0\}$, and $q \geq 2$. Then*

$$\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^q|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{q}}\left(\int\limits_{R^2}\xi^q|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}} \geq \frac{C_\psi^{\frac{1}{q}}}{2}\|f\|^{\frac{4}{q}}_{L^2(R^2).}$$

Proof. By using the Hölder inequality

$$\begin{aligned}
&\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^q|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{2}{q}}\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} |SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{1-\frac{2}{q}}\\
&=\left(\iiint\limits_{R^2RR^+}\left(t^2|SH_\psi f(a,s,t)|^{\frac{4}{q}}\right)^{\frac{q}{2}}\frac{dadsdt}{a^3}\right)^{\frac{2}{q}}\left(\iiint\limits_{R^2RR^+}\left(|SH_\psi f(a,s,t)|^{2-\frac{4}{q}}\right)^{\frac{1}{1-\frac{2}{q}}}\frac{dadsdt}{a^3}\right)^{1-\frac{2}{q}}\\
&\geq\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}|(t^2SH_\psi f(a,s,t)^{\frac{4}{q}})(SH_\psi f(a,s,t)^{2-\frac{4}{q}})|\frac{dadsdt}{a^3}\\
&=\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3},
\end{aligned}$$

we obtain

$$\left(\iiint\limits_{R^2RR^+} t^q|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{q}} \geq \frac{\left(\iiint\limits_{R^2RR^+} t^2|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{1}{2}}}{\left(\iiint\limits_{R^2RR^+}|SH_\psi f(a,s,t)|^2\frac{dadsdt}{a^3}\right)^{\frac{q-2}{2q}}}. \tag{4.3}$$

Similarly, we infer from formula (3.1) that

$$\begin{aligned}
\left(\int\limits_{R^2}\xi^q|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}} &\geq \frac{\left(\int\limits_{R^2}\xi^2|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}}{\left(\int\limits_{R^2}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}-\frac{1}{q}}} = \frac{\left(\int\limits_{R^2}\xi^2|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}}{\left(C_\psi\int\limits_{R^2}|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}-\frac{1}{q}}}C_\psi^{\frac{q-2}{2q}}\\
&=\frac{\left(\int\limits_{R^2}\xi^2|\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}}{\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+}|\mathcal{F}_t[SH_\psi f(a,s,t)](\xi)|^2\frac{dadsd\xi}{a^3}\right)^{\frac{1}{2}-\frac{1}{q}}}C_\psi^{\frac{q-2}{2q}}.
\end{aligned}$$

Applying Lemma 3.2 to the above inequality, we get

$$\left(\int\limits_{R^2} \xi^q |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}} \geq \frac{\left(\int\limits_{R^2} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}}{\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3}\right)^{\frac{1}{2}-\frac{1}{q}}} C_\psi^{\frac{q-2}{2q}}. \tag{4.4}$$

Multiplying (4.3) and (4.4), then by using formula (4.1) and (3.3), we obtain

$$\begin{aligned}
&\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^q |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3}\right)^{\frac{1}{q}} \left(\int\limits_{R^2} \xi^q |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{q}} \\
&\geq \frac{\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} t^2 |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3}\right)^{\frac{1}{2}} \left(\int\limits_{R^2} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}}{\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3}\right)^{1-\frac{2}{q}}} C_\psi^{\frac{q-2}{2q}} \\
&\geq \frac{\sqrt{C_\psi}\|f\|^2_{L^2(R^2)}}{2\left(\int\limits_{R^2}\int\limits_{R}\int\limits_{R^+} |SH_\psi f(a,s,t)|^2 \frac{dadsdt}{a^3}\right)^{1-\frac{2}{q}}} C_\psi^{\frac{q-2}{2q}} \\
&= \frac{C_\psi^{\frac{1}{2}} C_\psi^{\frac{q-2}{2q}} \|f\|^2_{L^2(R^2)}}{2\left(C_\psi \|f\|^2_{L^2(R^2)}\right)^{1-\frac{2}{q}}} = \frac{C_\psi^{\frac{1}{q}}}{2} \|f\|^{\frac{4}{q}}_{L^2(R^2)}.
\end{aligned}$$

□

Theorem 4.1 is a special case of Theorem 4.3 obtained by putting $q = 2$ in it.

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