Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 205–217
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Nonlinear conservation law model for production network considering yield loss

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Communicated by R. Saadati

Abstract

A mathematical model describing yield loss in a production network has been introduced. Mathematical properties of the continuum model are discussed. Existence, uniqueness and stability of the solution are demonstrated through weak formulation and entropy criteria. Front tracking method is implemented to construct approximate solutions. Estimates of the solutions are also provided.

Keywords: Production system, conservation laws, yield loss, front tracking.
2010 MSC: 35L65, 35Q35, 97M10.

1. Introduction and Preliminaries

Understanding the behavior of large production system under disparate scenarios is a substantial issue for many business today. Assorted mathematical approaches have been proposed for production network modeling. One side there are discrete event simulation (DES) models (based on the individual parts), other side continuum models using partial differential equations have been introduced and analyzed during recent years [1, 5, 6, 7, 8, 9, 12, 19].

This paper is concerned with the evolution and analysis of continuum model for production system. We consider a chain of M suppliers. Every supplier m receives a certain good (measured in units of parts) from supplier $m-1$, processes the material and passes to the next supplier $m+1$. Supplier m is characterized by its throughput time $T(m)$ and its maximal capacity $\mu(m)$. To compute the time evolution of each part in the production system, the modeling of the queues are essential. By assuming FIFO policy, the state of the queue will be either empty or non-empty. Whenever the queue is non-empty, the parts have to wait and

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Received 2013-9-24

the waiting time is inverse of the processing rate. If $\tau(m,n)$ denotes the arrival time of part n to supplier m, then it leads to the following (see [1] for more details):

$$\tau(m+1,n) = \max\left\{\tau(m,n)+T(m), \tau(m+1,n-1)+\frac{1}{\mu(m,n-1)}\right\}. \tag{1.1}$$

The above time recursion is analyzed using the Newell-curves (N-curves) as following (see [15]):

$$U(m,t) = \sum_{n=0}^{\infty} H(t-\tau(m,n)), \quad m = 1,...,M, \quad t > 0. \tag{1.2}$$

Mapping each supplier onto one gridpoint in space, performing the asymptotic analysis (taking $M \to \infty$) and using the concept of virtual processors (decomposed each supplier into many virtual suppliers to validate in finite number of suppliers case) the time-recursion (1.1) can be approximated through the conservation law :

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} = 0, \quad \forall\ x \in (0,1], \quad t > 0, \tag{1.3}$$

$$f(x,t) := \min\{\mu(x,t), v(x,t)\rho(x,t)\}, \tag{1.4}$$

where $\rho(x,t)$ and $f(x,t)$ denote the parts density and flux respectively. Velocity $v(=\frac{1}{T})$ can be function of space and time. Maximal capacity μ can be function of x. In real life problems, these functions are mainly piecewise constants as the number of suppliers is finite. The above hyperbolic equation is non-linear due to the 'min' function. The equation (1.3) and (1.4) represent two cases:

- the queues are empty: the density of parts solves advection equation with velocity v.
- flux function $v\rho$ exceeds the capacity i.e. the queue starts to build up. Consequently, the flux saturates to the maximal capacity.

Our goal is to make the model adaptable to real situations like yield loss happening inside the production network. In order to do that, instead of going back to discrete model, we use the continuum model as a starting point. Let us concentrate on the production system consisting of essential commodities. An important aspect one can realize yield loss during the process. Yield loss can occur due to the following reasons:

- Increase in inventories
- Delay in Processing
- Demand variability
- Communication gap i.e., lack of information sharing between the customers and suppliers.

This paper is organized as follows. In section 2, we formulate a continuum model for yield loss in production system. First we consider a single node in a production network to establish the model and then extend this formulation to a production network in section 3. In section 4, we introduce the notion of weak formulation and its admissibility. Consequently, we study the existence and uniqueness of the solution in section 5. Solution procedure and estimates of solution are discussed in section 6. We end up with the concluding remark and future scopes in section 7.

2. Modeling Equation

Now let us concentrate on a single node (supplier) in the production system. Each supplier consists of a processor and a queue in front of it. If the queue is empty, the material will directly go to the processor; else the material has to wait. Let x be a continuous variable representing the completion of the product within the supplier. Let us consider raw materials entered into the supplier at $x = 0$. Parts at $x = 1$ are the finished products going out of the supplier. Let $\rho(x,t)$, $\mu(x,t)$ and $v(x,t)$ represent part density, maximal processing rate and velocity of parts moving in the supplier respectively. Let $y_l(x,t)$ denote the yield loss in the production system at stage x and time t, which can be considered as a function of parts density: $y_l = y_l(\rho)$.

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial x} + y_l(\rho) = 0, \quad \forall\, x \in (0,1], \quad t > 0, \tag{2.1}$$

$$\text{where } f(x,t) := \min\{\mu(x,t), v(x,t)\rho(x,t)\}.$$

$$\text{Initial condition: } \rho(x,0) = \rho_0(x),\ x \in [0,1]. \tag{2.2}$$

$$\text{Influx condition: } f(0,t) = \lambda(t),\ t > 0. \tag{2.3}$$

In the above model, $\rho_0(x)$ represents the initial situation in the supplier and $\lambda(t)$ describes the influx profile of the supplier. If there is 100% yield, then the term $y_l(\rho) = 0$. In rest of the paper, we consider the yield to be less than 100%. The form of $y_l(\rho)$ can be linear, non-linear even discontinuous in ρ. Therefore, we have the following:

- Linear: $y_l(\rho) = \alpha\rho$. Suppose there are 80% yield in the production network. So, the amount of yield loss is 20%. The value of α is considered as 0.2.
- Non-linear: $y_l(\rho) = \alpha\rho^2$. The term α represents same meaning as linear case i.e., $0 \le \alpha \le 1$.
- Discontinuous:

$$y_l(\rho) = \begin{cases} \alpha_1\rho & \text{for } \rho \le \rho_M \\ \alpha_2\rho & \text{for } \rho > \rho_M. \end{cases}$$

  Here α_1 and α_2 are constants. $0 \le \alpha_i \le 1,\ i = 1,2$. This case occurs quite often in production system. Let us consider the situation: there are no yield loss until the part density reaches to ρ_M ($\alpha_1 = 0$) and the loss starts afterwards ($\alpha_2 > 0$).

It is seen that due to the variability in the processing rate, some suppliers have heavy WIP (Work In Progress). It is quite natural that the loss may be taken place there and for other suppliers have complete yielding. We can interpret this situation through the above cases. For supplier $m = M_1$, let $y_l(\rho)$ be one of the above three cases and $y_l(\rho) = 0$ for $m \ne M_1$. Same way we can elucidate the case when yield loss is happening at the nodes of production network.

3. Extension to Network

In this section, we extend our model of single node to a serial production system network. Let us consider a chain of M suppliers. Every supplier m receives parts from supplier $m-1$ then it gets processed and passes it to the next supplier $m+1$. We assume that the materials flow from node $m-1$ to m, and then m to $m+1$ and so on. Finally the product exits from production system at node M. Let $\rho_j(x,t),\ j = 1,2,...,M$ denote the density of parts at node j, at stage x and at time t. If $y_l(\rho_j)$ be the yield loss at supplier j, then conservation of materials can be presented as follows:

$$\frac{\partial \rho_j}{\partial t} + \frac{\partial f_j}{\partial x} + y_l(\rho_j) = 0, \quad t \in (0,\infty), \quad x \in (0,1], \qquad j = 1,2,...,M, \tag{3.1}$$

$$\text{where } f_j(x,t,\rho) := \min\{\mu_j(t), v_j\rho\}.$$

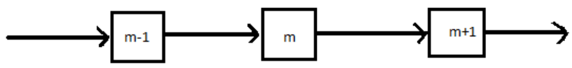

Figure 1: Serial production system network

Initial condition:

$$\rho_j(x,0) = \rho_{j,0}(x), \qquad j = 1,2,...,M. \tag{3.2}$$

Influx condition:

$$f_1(0,t,\rho(0,t)) = \lambda(t),\ \ t > 0. \tag{3.3}$$

The model will not make sense until we find out the interconnection between the suppliers. This basically implies that we need to define the flux function of the suppliers (except the first supplier due to influx condition) and the equation of queue in front of the suppliers. We define both as following:

- flux equation: for j=2, 3, ..., M

$$f_j(0,t) = \begin{cases} \min\{f_{j-1}(1,t) - (y_l)_j(0,t), \mu_j(t)\} & \text{if } q_j(t) = 0 \\ \mu_j(t) & \text{if } q_j(t) \neq 0. \end{cases}$$

- queue equation: Queue can be empty or non-empty. Both the cases are incorporated. For $j-$th supplier the queue equation will be

$$\frac{dq_j}{dt} = f_{j-1}(\rho_{j-1}(1,t)) - (y_l)_j(0,t) - f_j(\rho_j(0,t)).$$

The presented model consists of nonlinear hyperbolic conservation law. Several related work has been accomplished in the framework of homogeneous conservation law with smooth flux function (see [14] for more details). In the direction of nonhomogeneous conservation law with convex flux function, related works have been incorporated by Dafermos et al. [4] and Kan et al. [11]. For nonhomogeneous conservation law with the non-convex continuous function needs particular attention. Therefore, we will discuss in this direction in the following sections. Here we analyze the model of single supplier. The analysis of serial production network can be carried out in the same way.

4. Weak Formulation and Entropy Condition

In general, the initial-value problem (2.1)-(2.3) does not have global smooth solution even if the input data are smooth functions. In order to generalize the notion of the solution, we go for the weak formulation (obtained solution will be in distributional sense). To get the admissible weak solution, we present the entropy criteria in Kruzhkov [13] sense.

Definition 4.1. A bounded measurable function $\rho(x,t)$ is called a weak solution of the initial-value problem (2.1)-(2.3) with bounded, measurable initial data $\rho_0(x)$ and boundary data $u(t)$ (taking $\rho(0,t) = u(t)$), provided the following holds for every $\phi \in C^1([0,1] \times [0,T])$ with $\phi(x,T) = 0$, $\forall x \in [0,1]$ and $\phi(1,t) = 0$, $\forall t \in [0,T]$,

$$\int_0^T \int_0^1 [\rho\phi_t + f\phi_x + g\phi]dxdt + + \int_0^T \rho(0,t)\phi(0,t)dt + \int_0^1 \rho_0(x)\phi(x,0)dx = 0, \tag{4.1}$$

where $g(\rho) = -y_l(\rho)$.

Though the weak solution is a true generalization of classical solution, every discontinuity is not permissible. In fact, the condition (4.1) places the restriction on the curve of discontinuity. Let $\Gamma : x = x(t)$ be a curve along which ρ has the discontinuity. It is not hard to find the speed of discontinuity (s). $s = \frac{[f(\rho)]}{[\rho]}$, where $[f(\rho)] = f(\rho(x(t) - 0, t)) - f(\rho(x(t) + 0, t))$ and $[\rho] = \rho(x(t) - 0, t) - \rho(x(t) + 0, t)$. This turns out to be same as Rankine-Hugoniot jump condition in scalar conservation law. For admissible weak solution, we derive the entropy condition in Kruzhkov sense as following:

$$\begin{aligned}\int_0^T \int_0^1 [|\rho - k|\phi_t + q(\rho, k)\phi_x]dtdx + \int_0^1 [(|\rho - k|\phi)|_{t=0} - (|\rho - k|\phi)|_{t=T}]dx \\ \geq \int_0^T \int_0^1 [sign(\rho - k)g(\rho)\phi(x, t)]dtdx,\end{aligned} \tag{4.2}$$

for all $k \in \mathbb{R}$ and all nonnegative test functions $\phi \in C_0^\infty([0, 1] \times [0, T])$, where $q(\rho, k) = sign(\rho - k)(f(\rho) - f(k))$.

Now we discuss the existence of the solution. Consequently we will show that the solution which satisfies the entropy inequality, is the only solution. We assume that $g(\rho)$ is a continuous function. We extend the work presented in [10] to nonhomogeneous case with non-convex continuous flux function.

5. Existence and Stability Analysis

It is observed that velocity in production network depends on work in progress [16, 17]. If we assume that maximal processing rate is constant in each supplier then the flux function f can be considered as functions of density only. Consequently, flux function will be Lipschitz continuous.

We discuss the existence of solution of the following non-homogeneous hyperbolic equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} + y_l(\rho) = 0, \quad -\infty < x < \infty, \quad t > 0 \tag{5.1}$$

with the following initial condition

$$\rho(x, 0) = \begin{cases} \rho_l & \text{for } x < 0 \\ \rho_r & \text{for } x > 0. \end{cases}$$

We assume that $g(\rho)$ is linear in ρ. Let $g(\rho) = \alpha\rho$, where α is a non-negative constant. We approximate the flux function as a piecewise linear function. Let us consider $\rho_1, \rho_2, ..., \rho_n$ be the breakpoints of the flux function. In the production network case, the approximate flux function f is piecewise linear and concave in nature.

First we consider the case when $\rho_l < \rho_r$. We can represent the weak solution of (5.1) with the above initial condition as following:

$$\rho(x, t) = \begin{cases} \rho_l \exp(-\alpha t), & \text{for } -\infty < \dfrac{x}{t} \leq \dfrac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} \\ \rho_r \exp(-\alpha t), & \text{for } \dfrac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} < \dfrac{x}{t} < \infty. \end{cases}$$

If $\rho_l > \rho_r$, we need to check the interval $[\rho_r, \rho_l]$ contains any breakpoints of f. Suppose that $[\rho_r, \rho_l]$ contains $\rho_1, \rho_2, ..., \rho_n$. Then the weak solution of (5.1) with the above initial condition can be represented as following:

$$\rho(x,t) = \begin{cases} \rho_l \exp(-\alpha t), & \text{for } -\infty < \dfrac{x}{t} \leq \dfrac{f(\rho_1) - f(\rho_l)}{\rho_1 - \rho_l}, \\ \rho_1 \exp(-\alpha t), & \text{for } \dfrac{f(\rho_1) - f(\rho_l)}{\rho_1 - \rho_l} < \dfrac{x}{t} \leq \dfrac{f(\rho_2) - f(\rho_1)}{\rho_2 - \rho_1}, \\ . \\ . \\ . \\ \rho_n \exp(-\alpha t), & \text{for } \dfrac{f(\rho_n) - f(\rho_{n-1})}{\rho_n - \rho_{n-1}} < \dfrac{x}{t} \leq \dfrac{f(\rho_r) - f(\rho_n)}{\rho_r - \rho_n}, \\ \rho_r \exp(-\alpha t), & \text{for } \dfrac{f(\rho_r) - f(\rho_n)}{\rho_r - \rho_n} < \dfrac{x}{t} < \infty. \end{cases}$$

For more details about the construction of the solution one can refer [3].

Before we proceed to the stability analysis, we need to prove a lemma. We establish certain results using mollifiers or approximate delta functions. This is a sequence of smooth functions η_ϵ such that the corresponding distributions tend to the δ_0 distribution i.e., $\eta_\epsilon \to \delta_0$ as $\epsilon \to 0$. There are several ways to define these distributions. We use the following:
Let $\eta(\sigma)$ be a C^∞ function having property that

- $0 \leq \eta(\sigma) \leq 1$,
- $supp(\eta) \subseteq [-1, 1]$,
- $\eta(-\sigma) = \eta(\sigma)$,
- $\displaystyle\int_{-1}^{1} \eta(\sigma) d\sigma = 1$.

We define

$$\eta_\epsilon(\sigma) := \frac{1}{\epsilon}\eta\left(\frac{\sigma}{\epsilon}\right) \tag{5.2}$$

Using the above properties, one can easily show that $\eta_\epsilon \to \delta_0$ as $\epsilon \to 0$ as a distribution. Now, we have the following result.

Lemma 5.1. *Let $p(x, y) \in L^\infty(\mathbb{R}^2)$ with compact support. Assume that for almost all $x_0 \in \mathbb{R}$, the function $p(x, y)$ is continuous at (x_0, x_0). Then*

$$\lim_{\epsilon \to 0} \int \int p(x, y)\eta_\epsilon(x - y) dy dx = \int p(x, x) dx$$

(Using Lebesgue's bounded convergence theorem, one can prove the Lemma. We refer [10] for details). □

Now our aim is to show stability with respect to the initial value function and boundary data. Consequently, uniqueness follows by using the choice of test function ϕ and entropy formulation.

Theorem 5.2. *Let $\rho = \rho(x, t)$ and $\rho_1 = \rho_1(x, t)$ are two weak solutions of*

$$\partial_t \rho(x, t) + \partial_x f(\rho(x, t)) + y_l(\rho(x, t)) = 0,$$

with the initial data

$$\rho(x, 0) = \rho^0(x), \qquad \rho_1(x, 0) = \rho_1^0(x),$$

and boundary data

$$\rho(0, t) = u(t), \qquad \rho_1(0, t) = u^0(t),$$

respectively in domain $(x,t) \in [0,1] \times [0,T]$, satisfying the entropy condition (4.2). Assume ρ^0, ρ_1^0, u and u^0 are integrable functions. If the flux function f is Lipschitz continuous and g is continuous and linear in ρ, where $g(\rho) = -y_l(\rho)$, then

$$\|\rho(.,t) - \rho_1(.,t)\|_1 \leq \exp(T\|g\|_{C^0})(\|\rho^0 - \rho_1^0\|_1 + \|u - u^0\|_1).$$

Proof. Since f is Lipschitz continuous function and g is continuous in ρ, there exist constants L_1 and L_2 such that

$$\|f\|_{Lip} := \sup_{\rho \neq \rho_1} \left| \frac{f(\rho) - f(\rho_1)}{\rho - \rho_1} \right| \leq L_1, \tag{5.3}$$

$$\|g\|_{C^0} := L_2, \tag{5.4}$$

where $\|.\|_{Lip}$ denotes seminorm of the corresponding function.
If ϕ is compactly supported in $t > 0$, then the entropy condition (4.2) reduces to

$$\int\int \big[|\rho - k|\phi_t(x,t) + sign(\rho - k)(f(\rho) - f(k))\phi_x(x,t) + sign(\rho - k)y_l(\rho)\phi(x,t)\big] dtdx \geq 0$$

using $y_l(\rho) = -g(\rho)$. We use the following notations:

$$\begin{aligned} q(\rho,k) &:= sign(\rho - k)(f(\rho) - f(k)), \\ h(\rho,k) &:= sign(\rho - k)(y_l(\rho) - y_l(k)). \end{aligned}$$

Further we define

$$\|q\|_{Lip} := \sup_{(\rho,\rho_1) \neq (\rho',\rho_1')} \frac{|q(\rho,\rho_1) - q(\rho',\rho_1')|}{|\rho - \rho'| + |\rho_1 - \rho_1'|}.$$

Since $q_\rho(\rho,k) = sign(\rho - k)Df(\rho)$ and $q_k(\rho,k) = -sign(\rho - k)Df(k)$ (D represents the weak derivative), it follows that if $\|f\|_{Lip} \leq L_1$, then $\|q\|_{Lip} \leq L_1$. Clearly, we can also observe that $\|h\|_{Lip} \leq L_2$.

Let us consider $\phi = \phi(x,t,y,s)$ be a nonnegative test function both in (x,t) and (y,s) with compact support in $t > 0$ and $s > 0$. It is given that ρ and ρ_1 satisfy the above entropy condition. We set $k = \rho_1(y,s)$ in the equation for ρ and $k = \rho(x,t)$ in the equation for ρ_1. We integrate the equation for $\rho(x,t)$ with respect to y and s and the equation for ρ_1 with respect to x and t. Adding the two resulting equations, we obtain

$$\int\int\int\int \big[|\rho(x,t) - \rho_1(y,s)|(\phi_t + \phi_s) + q(\rho,\rho_1)(\phi_x + \phi_y) + h(\rho,\rho_1)\phi(x,t)\big] dxdtdyds \geq 0.$$

We choose test function $\phi(x,y,t,s)$ such that

$$\phi(x,t,y,s) = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\eta_{\epsilon_0}(t-s)\eta_\epsilon(x-y),$$

where $\psi(x,t)$ is a test function that has support in $t > \epsilon_0$. Here ϵ_0 and ϵ are small positive numbers. Now we have

$$\phi_t + \phi_s = \frac{\partial \psi}{\partial t}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\eta_{\epsilon_0}(t-s)\eta_\epsilon(x-y),$$

$$\phi_x + \phi_y = \frac{\partial \psi}{\partial x}\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\eta_{\epsilon_0}(t-s)\eta_\epsilon(x-y),$$

where η_ϵ is defined in (5.2). Let us consider

$$p_1(x,y) = |\rho(x) - \rho_1(y)|\eta_{\epsilon_0}\frac{\partial \psi}{\partial t}, \qquad p_2(x,y) = q(x,y)\eta_{\epsilon_0}\frac{\partial \psi}{\partial x}.$$

Letting ϵ_0 and ϵ tend to zero, we use the Lemma 5.1 by taking $p(x,y)$ as $p_1(x,y)$ and $p_2(x,y)$ respectively. Using the resultant equations in (5.5), we get the following inequality

$$\int\int \left[|\rho(x,t)-\rho_1(x,t)|\psi_t(x,t)+q(\rho,\rho_1)\psi_x(x,t)+h(\rho,\rho_1)\psi(x,t)\right]dxdt \geq 0, \tag{5.5}$$

for any non-negative test function ψ with support in $t>\epsilon$.

Since in production system we basically work with finite time and finite space, we consider the time $t\in[0,T]$ and $x\in[0,1]$. Consequently, we take the test function whose support includes 0 and T. The entropy formulation would imply

$$\begin{aligned}\int\int\int\int \Big[|\rho(x,t)-\rho_1(y,s)|(\phi_t+\phi_s)+q(\rho,\rho_1)(\phi_x+\phi_y)+h(\rho,\rho_1)\phi(x,t,y,s)\Big]dxdtdyds\\ -\int\int|\rho(1,t)-\rho_1(1,s)|\phi(1,t,1,s)dtds+\int\int|\rho(0,t)-\rho_1(0,s)|\phi(0,t,0,s)dtds\\ -\int\int|\rho(x,T)-\rho_1(y,T)|\phi(x,T,y,T)dxdy+\int\int|\rho^0(x)-\rho_1^0(y)|\phi(x,0,y,0)dxdy\geq 0.\end{aligned}$$

We can choose the test function such that $\phi=0$ at $x=1$. After using the same test function ψ, we end up with

$$\begin{aligned}\int\int\left[|\rho(x,t)-\rho_1(x,t)|\psi_t+q(\rho,\rho_1)\psi_x+h(\rho,\rho_1)\psi(x,t)\right]dxdt+\int|\rho^0(0,t)-\rho_1^0(0,t)|\psi(0,t)dx\\ -\int|\rho(x,T)-\rho_1(x,T)|\psi(x,T)dx+\int|\rho^0(x)-\rho_1^0(x)|\psi(x,0)dx\geq 0.\end{aligned} \tag{5.6}$$

We define $\psi(x,t)$ as following

$$\psi(x,t):=\exp((T-t)\|g\|_{C^0})\left(\chi_{[-M+L_1t+\epsilon,M-L_1t-\epsilon]}*\eta_\epsilon\right)(x),\quad t\in[0,T].$$

Here $*$ denotes the convolution product and $\chi_{[a,b]}$ is the characteristics function of the interval $[a,b]$. We choose M such that $M-L_1t-\epsilon>-M+L_1t+\epsilon$ for $t<T$. To make ψ as an admissible test function, we need to consider it as zero for $t>T$. Now we compute for $t<T$,

$$\begin{aligned}\psi_t&=-\|g\|_{C^0}\psi+\exp((T-t)\|g\|_{C^0})\frac{d}{dt}\int_{-M+L_1t+\epsilon}^{M-L_1t-\epsilon}\eta_\epsilon(x-y)dy\\ &=-L_2\psi-\exp((T-t)\|g\|_{C^0})L_1(\eta_\epsilon(x-M+L_1t+\epsilon)+\eta_\epsilon(x+M-L_1t-\epsilon))\end{aligned}$$

$$\begin{aligned}\psi_x&=\exp((T-t)\|g\|_{C^0})\frac{d}{dx}\int_{-M+L_1t+\epsilon}^{M-L_1t-\epsilon}\eta_\epsilon(x-y)dy\\ &=-\exp((T-t)\|g\|_{C^0})(\eta_\epsilon(x-M+L_1t+\epsilon)-\eta_\epsilon(x+M-L_1t-\epsilon))\end{aligned}$$

Therefore,

$$\begin{aligned}0=\psi_t+L_1|\psi_x|+L_2\psi\geq\psi_t+\frac{q(\rho,\rho_1)}{|\rho-\rho_1|}\psi_x+\frac{h(\rho,\rho_1)}{|\rho-\rho_1|}\psi\\ \Rightarrow|\rho-\rho_1|\psi_t+q(\rho,\rho_1)\psi_x+h(\rho,\rho_1)\psi\leq 0\end{aligned} \tag{5.7}$$

Using (5.7) in (5.6) and letting $\epsilon\to 0$, we get

$$\int_{-M+L_1t}^{M-L_1t}|\rho(x,t)-\rho_1(x,t)|dx\leq\exp(T\|g\|_{C^0})\int_{-M}^{M}|\rho^0-\rho_1^0|dx+\exp(T\|g\|_{C^0})\int_{-M}^{M}|\rho(0,t)-\rho_1(0,t)|dt.$$

Since ρ^0, ρ_1^0, u and u^0 are integrable functions, we will end up with the following

$$\|\rho(.,t)-\rho_1(.,t)\|_1\leq\exp(T\|g\|_{C^0})(\|\rho^0-\rho_1^0\|_1+\|u-u^0\|_1). \tag{5.8}$$

Hence the proof. □

From the above theorem, we can observe the following remark.

Remark 5.3. For any $\epsilon > 0$, if there exists η (depending on ϵ) such that

$$\|\rho^0 - \rho_1^0\|_1 + \|u - u^0\|_1 < \eta,$$

then we have

$$\|\rho(.,t) - \rho_1(.,t)\|_1 < \epsilon, \quad \forall t \in [0,T].$$

We have shown the stability of the solution with respect to the initial condition and boundary data. Now we focus on the solution procedure of the presented nonlinear hyperbolic equation. The analysis can be carried out in similar way for serial suppliers case.

6. Solution Procedure

A rich amount of research have been done in the direction of solution procedure of homogeneous scalar equation with smooth flux function. Since in our model we have nonhomogeneous hyperbolic equation with non-convex continuous flux function, one can approach through semigroup theory [18] in this aspect. Here we make use of front tracking method. The idea to construct solution $\rho(x,t)$ for all times t by front tracking algorithm is given in [2, 10]. We basically start with a step function $\rho_0(x)$ and solve at each point of a jump discontinuity a Riemann problem. The evaluated solution $\rho(x,t),\ t > 0$ is again a step function with discontinuities traveling at constant speed. Since the approximate the flux function is continuous and piecewise linear, all discontinuities of the solution will be referred as fronts.

Due to the nonhomogeneous term, the solution will be constructed by operator splitting method. There will be PDE part due to convection and ODE part due to the yield loss. We observe that if the velocity function in a production network is constant, then the flux function is continuous and piecewise linear. So, we can directly apply the front tracking method. For non-constant velocity function, we need to approximate the flux function as piecewise linear function. Similarly, for non-constant input values we need to approximate by piecewise constant functions.

With piecewise constant input data and piecewise linear flux function we can construct the solution of the PDE part. At each point of jump discontinuity we need to solve the PDE repeatedly with modified input data. It is already shown in [16] that the number of interactions between the discontinuities are finite. More precisely, let us fix a large positive number N such that $\rho_i = i\delta$, for $0 \le i\delta \le N$, where ρ_i's are the break points of the flux function. For each positive δ and each piecewise constant input data taking values from ρ_i, there are only a finite number of intersections between discontinuities of the weak solution for $t > 0$. The positivity of the characteristic velocity and boundedness from above ensure the restriction on the interaction of shock waves. Therefore, the procedure is well-defined and generates a solution for the homogeneous part.

For the ODE part, continuity of the yield loss term ensures the existence of solution. ODE part will be considered after incorporating the nature of the solution coming from the PDE. An approximate solution will be obtained by combining both parts in operator splitting. Now we shall show that the constructed approximate solution actually converges to the weak solution (4.1).

6.1. Convergence

We show that for general initial data $\rho_0 \in L^1[0,1]$ and boundary data $u \in L^1[0,T]$, the front tracking method will converge i.e., if a sequence of step functions $\{\rho_0^n\}$ converges to ρ_0 and another sequence of step functions $\{u^n\}$ converges to u in L^1, then the corresponding front-tracking solution $\{\rho^n(x,t)\}$ converge to $\rho(x,t)$ in L^1. Clearly, (5.8) demonstrates this. Now we show that $\rho(x,t)$ is indeed a weak solution of the equation (2.1).

Let $\phi(x,t)$ be a fixed test function. We choose a constant M such that

$$M \ge \max\{\|\phi\|_\infty, \|\phi_t\|_\infty, \|\phi_x\|_\infty\}$$

and $\phi(x,t) = 0$ for $t \geq T$ and also for $x \geq 1$. As all $\rho^n(x,t)$ are weak solutions, we have

$$\int_0^T \int_0^1 [\rho^n(x,t)\phi_t + f(\rho^n)\phi_x + g(\rho^n)\phi]dxdt + \int_0^T u^n(t)\phi(0,t)dt + \int_0^1 \phi(x,0)\rho_0^n(x)dx = 0. \tag{6.1}$$

Now using the constants L_1 and L_2 as defined in (5.6)-(5.7), choice of M and using equation (6.1), we have the following

$$\begin{aligned}
&\Big|\int_0^T \int_0^1 (\rho(x,t)\phi_t + f(\rho(x,t))\phi_x + g(\rho)\rho(x,t))dxdt + \int_0^T u(t)\phi(0,t)dt + \int_0^1 \phi(x,0)\rho_0(x)dx\Big| \\
&= \Big|\int_0^T \int_0^1 [(\rho(x,t) - \rho^n(x,t))\phi_t + (f(\rho) - f(\rho^n))\phi_x + (g(\rho) - g(\rho^n))\phi(x,t)]dxdt \\
&\qquad + \int_0^T (u - u^n)\phi(0,t)dt + \int_0^1 \phi(x,0)(\rho_0(x) - \rho_0^n(x))dx\Big| \\
&\leq \int_0^T \Big(M\|\rho(.,t) - \rho^n(.,t)\|_1 + ML_1\|\rho(.,t) - \rho^n(.,t)\|_1 + ML_2\|\rho(.,t) - \rho^n(.,t)\|_1\Big)dt \\
&\qquad + M\|u - u^n\|_1 + M\|\rho_0 - \rho_0^n\|_1 \\
&\leq M((1 + L_1 + L_2)T + 1)(\|\rho_0 - \rho_0^n\|_1 + \|u - u^n\|_1).
\end{aligned}$$

Since the right side of the above equation can be made arbitrarily small, $\rho(x,t)$ is a weak solution of the nonhomogeneous differential equation (2.1) with prescribed initial and boundary data.

6.2. *Estimates*

To get an overview of whole production network, the following aspects are quite important.

- Demand forecasting
- Production decisions
- Aggregate planning
- Production scheduling

The estimation of the solution should be beneficial in the continuous model to incorporate the above aspects. It will provide an ample insight on the demand forecasting and outflux in the production system. It is not difficult to verify the following estimates:

$$\|\rho(t,.)\|_{L^\infty} \leq \exp(T\|g\|_{C^0}) \max\{\|\rho_0\|_{L^\infty}, \|u\|_{L^\infty}\}$$

provided, ρ_0 and u (where $u(t) = \rho(0,t)$) are essentially bounded.

If the velocity in production system is non-constant i.e., function of density, then due to front tracking method we need to approximate the flux function f by piecewise linear function f^δ. Let ρ and ρ^δ be the weak solutions of

$$\partial_t \rho + \partial_x f(\rho) + y_l(\rho) = 0, \qquad \partial_t \rho^\delta + \partial_x f^\delta(\rho^\delta) + y_l(\rho^\delta) = 0,$$

with initial condition

$$\rho(x,0) = \rho^\delta(x,0) = \rho_0(x),$$

and boundary condition

$$\rho(0,t) = \rho^\delta(0,t) = u(t).$$

Here we consider $y_l(\rho) = \alpha\rho$, where α is nonnegative constant. With the help of [10], we can deduce the following and the inequality holds until the first collision.

$$\frac{d}{dt}\big(\exp(-\alpha t)\|\rho(.,t) - \rho^{\delta}(.,t)\|_1\big) \le \|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)).$$

This implies

$$\begin{aligned}\exp(-\alpha t)\|\rho(.,t) - \rho^{\delta}(.,t)\|_1 &\le t\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u))\\ \Rightarrow \|\rho(.,t) - \rho^{\delta}(.,t)\|_1 &\le t\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)),\end{aligned}$$

where T.V. represents the total variation of the function.
The above estimates will valid until the interaction of fronts for either ρ or ρ^{δ}. After the interaction (say t_1 collision time), ρ^{δ_1} be the front tracking solution of

$$\rho_t^{\delta_1} + f(\rho^{\delta_1})_x + y_l(\rho^{\delta_1}) = 0$$

with the condition

$$\rho^{\delta_1}(x,t_1) = \rho^{\delta}(x,t_1).$$

Until the next interaction, we have

$$\begin{aligned}\|\rho(t) - \rho^{\delta}(t)\|_1 &\le \|\rho(t) - \rho^{\delta_1}(t)\|_1 + \|\rho^{\delta_1}(t) - \rho^{\delta}(t)\|_1\\ &\le \|\rho(t_1) - \rho^{\delta_1}(t_1)\|_1 + (t-t_1)\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho^{\delta}(t_1)) + \text{T.V.}(u(t_1))\\ &\le \|\rho(t_1) - \rho^{\delta}(t_1)\|_1 + (t-t_1)\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho^{\delta}(t_1)) + \text{T.V.}(u(t_1))\\ &\le t_1\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)) + (t-t_1)\\ &\qquad \exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho^{\delta}(t_1)) + \text{T.V.}(u(t_1))\\ &\le t_1\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)) + t\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}\\ &\quad (\text{T.V.}(\rho_0) + \text{T.V.}(u)) - t_1\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)) \quad (\text{as } t \ge t_1)\\ &\le t\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)).\end{aligned}$$

If we repeat the same argument for every collision time, then for all positive time we obtain the following

$$\|\rho(t) - \rho^{\delta}(t)\|_1 \le t\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0) + \text{T.V.}(u)). \tag{6.2}$$

We want to combine the equations (5.1) and (6.2) to get a comparison result. Let ρ, ρ^{δ} and ρ^{δ_1} be the solutions of the following differential equations

$$\begin{aligned}\rho_t + f(\rho)_x + y_l(\rho) &= 0, & \rho(x,0) &= \rho_0(x), & \rho(0,t) &= u(t)\\ \rho_t^{\delta} + f^{\delta}(\rho^{\delta})_x + y_l(\rho^{\delta}) &= 0, & \rho^{\delta}(x,0) &= \rho_0^{\delta}(x), & \rho^{\delta}(0,t) &= u^0(t)\\ \rho_t^{\delta_1} + f(\rho^{\delta_1})_x + y_l(\rho^{\delta_1}) &= 0, & \rho^{\delta_1}(x,0) &= \rho_0^{\delta}(x), & \rho^{\delta_1}(0,t) &= u^0(t)\end{aligned}$$

respectively. We have

$$\begin{aligned}\|\rho(.,t) - \rho^{\delta}(.,t)\|_1 &\le \|\rho(.,t) - \rho^{\delta_1}(.,t)\|_1 + \|\rho^{\delta_1}(.,t) - \rho^{\delta}(.,t)\|_1\\ &\le \exp(T\|g\|_{C^0})(\|\rho_0(x) - \rho^{\delta_1}(x,0)\|_1 + \|u - \rho^{\delta_1}(0,t)\|_1)\\ &\quad + t\exp(T\|g\|_{C^0})\|f - f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0^{\delta}) + \text{T.V.}(u^0)).\end{aligned}$$

If we had defined ρ^{δ_1} as the solution of the following differential equation

$$\partial_t\rho^{\delta_1} + \partial_x f(\rho^{\delta_1}) + y_l(\rho^{\delta_1}) = 0, \quad \rho^{\delta_1}(x,0) = \rho_0(x), \quad \rho^{\delta_1}(0,t) = u(t),$$

then we would obtain the following:

$$\begin{aligned}\|\rho(.,t)-\rho^{\delta}(.,t)\|_1 \leq \exp(T\|g\|_{C^0})(\|\rho_0^{\delta}(x)-\rho^{\delta_1}(x,0)\|_1+\|u^0-\rho^{\delta_1}(0,t)\|_1)\\ +t\exp(T\|g\|_{C^0})\|f-f^{\delta}\|_{Lip}(\text{T.V.}(\rho_0)+\text{T.V.}(u)).\end{aligned}$$

Finally, we have the estimates as following

$$\begin{aligned}\|\rho(.,t)-\rho^{\delta}(.,t)\|_1 \leq \exp(T\|g\|_{C^0})\Big[\|\rho_0-\rho_0^{\delta}\|_1+\|u-u^0\|_1+t\|f-f^{\delta}\|_{Lip}\\ \big(\min\{\text{T.V.}(\rho_0),\text{T.V.}(\rho_0^{\delta})\}+\min\{\text{T.V.}(u),\text{T.V.}(u^0)\}\big)\Big].\end{aligned}$$

7. Conclusion

Dynamics of production system incorporating yield loss can be represented through nonlinear conservation laws. Results on existence and stability analysis motivate us to study the model numerically. Front tracking method provides an approximate solution which converges to weak solution of the PDE model. The density of products in a production system can be estimated by the total variation of initial and boundary data. Extensions of the presented model to complex network and numerical study are under process. Moreover, optimization perspective like maximum outflux and minimum costs are also be desired for the proposed model.

Acknowledgements:

First author would like to thank Council of Scientific and Industrial Research (CSIR), New Delhi for the financial support (Ref. no. 09/084(0505)/ 2009-EMR-I).

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