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Strong convergence of a modified SP-iteration process for generalized asymptotically quasi-nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, we establish strong convergence theorems of the modified SP-iteration generalized asymptotically quasi-nonexpansive mapping in CAT(0) spaces which extend and improve the recent ones announced by Phuengrattana and Suantai [W. Phuengrattana, S. Suantai, J. Comput. Appl. Math., **235** (2011), 3006–3014], Sahin and Basarir [A. Sahin, M. Basarir, J. Inequal. Appl., **2013** (2013), 10 pages], Nanjaras and Panyanak [B. Nanjaras, B. Panyanak, Fixed Point Theory Appl., **2010** (2010), 14 pages] and some others.

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1. Introduction

The initials of the term CAT are in honor of Cartan, Alexandrov, and Toponogov, who have made important contributions to the understanding of curvature via inequalities for the distance function. Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = l$. A geodesic path from x to y is a isometry $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic part is called a geodesic segment. A metric space X is a (uniquely) geodesic space, if every two point of X are joined by only one geodesic segment. We will use $[x, y]$ to denote a geodesic segment joining x and y. A subset C of a geodesic space is said to be convex if $[x, y] \in C$ for any $x, y \in C$.

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A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane.

It is important that the concept of asymptotically nonexpansive, which is closely related to the theory of fixed points in Banach spaces, is introduced by Goebel and Kirk [7]. An early fundamental result due to Goebel and Kirk [7] proved that every asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. Zhou et al. [21] introduced a class of new generalization asymptotically nonexpansive.

Kirk [9, 11] first studied the theory of fixed point in CAT(0) space. Lim [14] introduced the concept of Δ-convergence in a general metric space. In 2008, Kirk and Panyanak [12] specialized Lim's concept to CAT(0) spaces and proved that it is very similar to weak convergence in the Banach space setting. Every nonexpansive (single-valued) mapping defined on closed bounded convex subset of complete CAT(0) space always has a fixed point, since then the fixed point theory in CAT(0) space has been rapidly developed and many paper has appeared [3, 5, 6, 8, 10, 13, 17, 18, 19, 20].

The Man iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n) x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

The Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T x_n + (1-\alpha_n) x_n, \\ y_n = \beta_n T x_n + (1-\beta_n) x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $(0,1)$.

The Noor iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n T x_n + (1-\gamma_n) x_n, \\ y_n = \beta_n T z_n + (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n T y_n + (1-\alpha_n) x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ is a sequence in $(0,1)$.

Recently, Phuengrattana and Suantai [16] introduced the SP-iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in C, \\ z_n = \gamma_n T x_n + (1-\gamma_n) x_n, \\ y_n = \beta_n T z_n + (1-\beta_n) z_n, \\ x_{n+1} = \alpha_n T y_n + (1-\alpha_n) y_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1]$.

The purpose of this paper was to prove strong and Δ-convergence of the modified SP-iteration process for generalized asymptotically quasi-nonexpansive mapping in CAT(0) spaces. Our results extend and improve the corresponding recent results announced by [17]. This paper is organized as follows. In Sections 2 and **??**, we present preliminaries and results of strong and Δ-convergence, respectively.

2. Preliminaries

Complete CAT(0) spaces are often called Hadamard spaces (see [1]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2\left(x, y_1\right) + \frac{1}{2}d^2\left(x, y_2\right) - \frac{1}{4}d^2\left(y_1, y_2\right). \tag{2.1}$$

The inequality (2.1) is the (2.1) inequality of Bruhat and Titz [2].

A geodesic metric spaces is a CAT(0) space if and only if it satisfies the (CN) inequality.

A subset K of a CAT(0) space X is convex if for any $x, y \in K$, we have $[x, y] \subset K$.

Lemma 2.1 ([5]). *Let X be a CAT(0) space.*

(1) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d\left((1-t)\,x \oplus ty, z\right) \leq (1-t)\,d\left(x, z\right) + \alpha d\left(y, z\right). \tag{2.2}$$

(2) For any $x, y, z \in X$ and $t \in [0, 1]$,

$$d^2\left((1-t)\,x \oplus ty, z\right) \leq (1-t)\,d^2\left(x, z\right) + td^2\left(y, z\right) - t\left(1-t\right)d^2\left(x, y\right). \tag{2.3}$$

Let C be nonempty subset of a CAT(0) space. We denote the set of fixed points of T by $F\left(T\right) = \left\{x \in C : Tx = x\right\}$.

Definition 2.2 ([22]). A mapping $T : C \to C$ called:

(1) Nonexpansive if $d\left(Tx, Ty\right) \leq d\left(x, y\right)$ for all $x, y \in C$.
(2) Quasi-nonexpansive if $d\left(Tx, p\right) \leq d\left(x, p\right)$ for all $x \in X$ and for all $p \in F(T)$.
(3) Asymptotically quasi-nonexpansive if there exists $k_n \in [0, 1)$ for all $n \geq 1$ with $\lim\limits_{n\to\infty} k_n = 0$ such that $d\left(T^n x, p\right) \leq \left(1 + k_n\right) d\left(x, p\right)$ for all $x \in C$, for all $p \in F(T)$.
(4) Generalized asymptotically quasi-nonexpansive if $F\left(T\right) \neq \emptyset$ and there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ with $\lim\limits_{n\to\infty} u_n = 0 = \lim\limits_{n\to\infty} c_n$ such that $d\left(T^n x, p\right) \leq d\left(x, p\right) + \left(1 + u_n\right) d\left(x, p\right) + c_n$ for all $x \in C, p \in F(T)$ and $n \geq 1$.
(5) Uniformly L-Lipschitzian if for some $L > 0$, $d\left(T^n x, T^n y\right) \leq Ld\left(x, y\right)$ for all $x, y \in C$ and $n \geq 1$.
(6) Semi-compact if for any bounded sequence $\{x_n\}$ in C with $d\left(x_n, T^n x_n\right) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

Let $\{x_n\}$ be a sequence in a metric space (X, d), and let C be a subset of X. We say that $\{x_n\}$, (1) is of monotone type (A) with respect to C if for each $p \in C$, there exist two sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $\sum\limits_{n=1}^{\infty} r_n < \infty$, $\sum\limits_{n=1}^{\infty} s_n < \infty$ and $d\left(x_{n+1}, p\right) \leq \left(1 + r_n\right) d\left(x_n, p\right) + s_n$, (2) of monotone type (B) with respect to C if there exist sequence $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers such that $d\left(x_{n+1}, C\right) \leq \left(1 + r_n\right) d\left(x_n, C\right) + s_n$.

A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d\left(x, Tx\right) \geq f\left(d\left(x, F(T)\right)\right)$, for all $x \in C$.

Let $\{x_n\}$ be a bounded sequence in CAT(0) space X. For $x \in X$, set

$$r\left(x, \{x_n\}\right) = \limsup_{n\to\infty} d\left(x, x_n\right). \tag{2.4}$$

The asymptotic radius $r\left(\{x_n\}\right)$ of $\{x_n\}$ is given by

$$r\left(\{x_n\}\right) = \inf\left\{r\left(x, \{x_n\}\right) : x \in X\right\}, \tag{2.5}$$

and the asymptotic center $A\left(\{x_n\}\right)$ of $\{x_n\}$ is the set

$$A\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) = r\left(\{x_n\}\right)\right\}. \tag{2.6}$$

Lemma 2.3 ([4]). *If C be a closed convex subset of a complete CAT(0) space X and if $\{x_n\}$ be a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.4 ([4]). *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.5 ([5]). *Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{p\}$ and $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

Lemma 2.6 ([15]). *Let X be a CAT(0) space, $x \in X$ be given point and $\{t_n\}$ be a sequence in $[b, c]$ with $b, c \in (0, 1)$ and $0 < b(1-c) \leq \frac{1}{2}$. Let $\{x_n\}$ and $\{y_n\}$ be any sequence in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{x\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$, for some $r \geq 0$. Then $\lim_{x\to\infty} d(x_n, y_n) = 0$.*

Lemma 2.7 ([20]). *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying*

$$a_{n+1} \leq (1+b_n)a_n + c_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ exists,
(2) If $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we establish some convergence results of SP- iterations to a fixed point for generalized asymptotically quasi-nonexpansive mappings in the general class of CAT(0) spaces.

Theorem 3.1. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} s_n < \infty$ and $\sum\limits_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, the sequence $\{x_n\}$ be the SP-iteration defined as follows:*

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n) x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n) z_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n) y_n, \end{cases} \tag{3.1}$$

where $\{\gamma_n\}, \{\beta_n\}, \{\alpha_n\}$ are real sequence in $[0, 1]$. Then the sequence $\{x_n\}$ is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point q of the mapping T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf_{q\in F(T)} \{d(x, q)\}$.

Proof. Following (2.2), Definition 2.2(4) and (3.1), we have

$$\begin{aligned} d(z_n, q) &= d(\gamma_n T^n x_n \oplus (1-\gamma_n) x_n, q) \\ &\leq \gamma_n d(T^n x_n, q) + (1-\gamma_n) d(x_n, q) \\ &\leq \gamma_n [(1+s_n) d(x_n, q) + t_n] + (1-\gamma_n) d(x_n, q) \\ &\leq (1+s_n)[\gamma_n + 1 - \gamma_n] d(x_n, q) + \gamma_n t_n \\ &= (1+s_n) d(x_n, q) + \gamma_n t_n \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d\left(y_n, q\right) &= d\left(\beta_n T^n z_n \oplus \left(1-\beta_n\right) z_n, q\right) \\ &\le \beta_n d\left(T^n z_n, q\right) + \left(1-\beta_n\right) d\left(z_n, q\right) \\ &\le \beta_n \left[\left(1+s_n\right) d\left(z_n, q\right) + t_n\right] + \left(1-\beta_n\right) d\left(z_n, q\right) \\ &\le \left(1+s_n\right)\left[\beta_n + 1 - \beta_n\right] d\left(z_n, q\right) + \beta_n t_n \\ &= \left(1+s_n\right) d\left(z_n, q\right) + \beta_n t_n. \end{aligned} \tag{3.3}$$

Substituting (3.2) into (3.3) and combining, we have

$$\begin{aligned} d\left(y_n, q\right) &\le \left(1+s_n\right)\left[\left(1+s_n\right) d\left(x_n, q\right) + \gamma_n t_n\right] + \beta_n t_n \\ &\le \left(1+s_n\right)^2 d\left(x_n, q\right) + \left(1+s_n\right)\gamma_n t_n + \beta_n t_n, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} d\left(x_{n+1}, q\right) &= d\left(\alpha_n T^n y_n \oplus \left(1-\alpha_n\right) y_n, q\right) \\ &\le \alpha_n d\left(T^n y_n, q\right) + \left(1-\alpha_n\right) d\left(y_n, q\right) \\ &\le \alpha_n \left[\left(1+s_n\right) d\left(y_n, q\right) + t_n\right] + \left(1-\alpha_n\right) d\left(y_n, q\right) \\ &\le \left(1+s_n\right)\left[\alpha_n + 1 - \alpha_n\right] d\left(y_n, q\right) + \alpha_n t_n \\ &= \left(1+s_n\right) d\left(y_n, q\right) + \alpha_n t_n. \end{aligned} \tag{3.5}$$

Substituting (3.4) into (3.5) and combining, we have

$$\begin{aligned} d\left(x_{n+1}, q\right) &\le \left(1+s_n\right)\left[\left(1+s_n\right)^2 d\left(x_n, q\right) + \left(1+s_n\right)\gamma_n t_n + \beta_n t_n\right] + \alpha_n t_n \\ &\le \left(1+s_n\right)^3 d\left(x_n, q\right) + \left(1+s_n\right)^2 \gamma_n t_n + \beta_n t_n + \alpha_n t_n \\ &= \left(1+\psi_n\right) d\left(x_n, q\right) + \varphi_n, \end{aligned} \tag{3.6}$$

where $\psi_n = 3s_n + 3s_n^2 + s_n^3$ and $\varphi_n = \left(1+s_n\right)^2 \gamma_n t_n + \beta_n t_n + \alpha_n t_n$. Since $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, it follows that $\sum_{n=1}^{\infty} \psi_n < \infty$ and $\sum_{n=1}^{\infty} \varphi_n < \infty$. Now, from (3.6), we get

$$d\left(x_{n+1}, q\right) \le \left(1+\psi_n\right) d\left(x_n, q\right) + \varphi_n), \tag{3.7}$$

$$d\left(x_{n+1}, F(T)\right) \le \left(1+\psi_n\right) d\left(x_n, F(T)\right) + \varphi_n). \tag{3.8}$$

In these inequalities, respectively, we prove that $\{x_n\}$ is a sequence of monotone type (A) and monotone type (B) with respect to $F(T)$.

Now, we prove that $\{x_n\}$ converges strongly to a fixed point of the mapping T if and only if $\liminf_{n\to\infty} d\left(x_n, F(T)\right) = 0$. If $x_n \to q \in F(T)$, then $\lim_{n\to\infty} d\left(x_n, q\right) = 0$. Since $0 \le \left(x_n, F(T)\right) \le d\left(x_n, q\right)$, we have $\lim_{n\to\infty} \inf d\left(x_n, F(T)\right) = 0$.

Conversely, suppose that $\lim_{n\to\infty} \inf d\left(x_n, F(T)\right) = 0$. From (3.8) using Lemma 2.7, we have that $\lim_{n\to\infty} d\left(x_n, F(T)\right)$ exists. Further, by hypothesis $\liminf_{n\to\infty} d\left(x_n F(T)\right) = 0$, we conclude that $\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence. Since $1 + a \le e^a$ for $a \ge 0$, hence from (3.7), we have

$$\begin{aligned} d\left(x_{n+m}, q\right) &\le \left(1+\psi_{n+m-1}\right) d\left(x_{n+m-1}, q\right) + \varphi_{n+m-1} \\ &\le e^{\psi_{n+m-1}} d\left(x_{n+m-1} q\right) + \varphi_{n+m-1} \\ &\le e^{\psi_{n+m-1}}\left[e^{\psi_{n+m-1}} d\left(x_{n+m-2}, q\right) + \varphi_{n+m-2}\right] + \varphi_{n+m-1} \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
&\leq e^{\sum\limits_{k=n}^{n+m-1}\psi_k} d\left(x_n,q\right) + e^{\sum\limits_{k=n}^{n+m-1}\psi_k}\left(\sum_{k=n}^{n+m-1}\varphi_k\right)\\
&\leq Md\left(x_n,q\right) + M\left(\sum_{k=n}^{n+m-1}\varphi_k\right),
\end{aligned}
$$

where $M = e^{\sum\limits_{k=n}^{n+m-1}\psi_k}$, for $M>0$ and for the natural numbers m,n and $q \in F(T)$. Since $\lim\limits_{n\to\infty} d\left(x_n, F(T)\right) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number N_0 such that $d\left(x_n, F(T)\right) < \frac{\varepsilon}{8M}$ and $\sum\limits_{k=n}^{n+m-1}\varphi_k < \frac{\varepsilon}{4M}$ for all $n > n_0$. And so, we can find $q^* \in F(T)$ such that $d\left(x_{n_0}, q^*\right) < \frac{\varepsilon}{4M}$. thus, for all $n > n_0$ and $m \geq 1$, we have

$$
\begin{aligned}
d\left(x_{n+m},x_n\right) &\leq d\left(x_{n+m},q^*\right) + d\left(x_n,q^*\right)\\
&\leq Md\left(x_{n_0},q^*\right) + M\sum_{k=n_0}^{\infty}\varphi_k + Md\left(x_{n_0},q^*\right) + M\sum_{k=n_0}^{\infty}\varphi_k\\
&= 2M\left(d\left(x_{n_0},q^*\right) + \sum_{k=n_0}^{\infty}\varphi_k\right) \leq 2M\left(\frac{\varepsilon}{4M}+\frac{\varepsilon}{4M}\right) = \varepsilon.
\end{aligned}
\tag{3.9}
$$

This proves that $\{x_n\}$ is a Cauchy sequence. Hence, By the completeness of X. we assume that $\lim\limits_{n\to\infty} x_n = a$. Since C is closed, therefore $a \in C$. Next, we show that $a \in F(T)$. Following two inequalities:

$$
\begin{aligned}
d\left(a,q\right) &\leq d\left(a,x_n\right) + d\left(x_n,q\right) \quad \forall q \in F(T),\ n \geq 1,\\
d\left(a,x_n\right) &\leq d\left(a,q\right) + d\left(x_n,q\right) \quad \forall q \in F(T),\ n \geq 1,
\end{aligned}
\tag{3.10}
$$

give that

$$
-d\left(a,x_n\right) \leq d\left(a,F(T)\right) - d\left(x_n,F(T)\right) \leq d\left(a,x_n\right), \quad n \geq 1. \tag{3.11}
$$

That is

$$
\left|d\left(a,F(T)\right) - d\left(x_n,F(T)\right)\right| \leq d\left(a,x_n\right), \quad n \geq 1. \tag{3.12}
$$

Since $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} d\left(x_n,F(T)\right) = 0$, we conclude that $a \in F(T)$. The proof is completed. □

Corollary 3.2. *Let (X,d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a generalized asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0,\infty)$ such that $\sum\limits_{n=1}^{\infty} s_n < \infty$ and $\sum\limits_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by* (3.1)*.Then the sequence $\{x_n\}$ converges strongly to a common fixed point q of the mapping T if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $q \in F(T)$.*

Corollary 3.3. *Let (X,d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0,\infty)$ such that $\sum\limits_{n=1}^{\infty} s_n < \infty$ and $\sum\limits_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by* (3.1)*. Then the sequence x_n is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point q of the mapping T if and only if*

$$
\liminf_{n\to\infty} d\left(x_n,F(T)\right) = 0.
$$

Proof. Follows from Theorem 3.1 with $t_n = 0$ for all $n \geq 1$. □

Corollary 3.4. *Let X be Banach space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T)$ is closed. For arbitrarily chosen $x_1 \in C$, let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Then the sequence x_n is of monotone type (A) and monotone type (B) with respect to $F(T)$. Moreover, $\{x_n\}$ converges strongly to a fixed point q of the mapping T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. Take $\lambda x \oplus (1-\lambda) y = \lambda x + (1-\lambda) y$ in Corollary 3.2.

Lemma 3.5. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a uniformly continuous generalized asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T) \neq \emptyset$. Let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then*

(1) $\lim_{n\to\infty} d(T^n y_n, y_n) = 0,$

(2) $\lim_{n\to\infty} d(T^n z_n, z_n) = 0,$

(3) $\lim_{n\to\infty} d(T^n x_n, x_n) = 0.$

Proof. Let $q \in F(T)$. By Theorem 3.1, we have $\lim_{n\to\infty} d(x_n, q)$ exists and $\{x_n\}$ is bounded. Without loss of generality. Let $\lim_{n\to\infty} d(x_n, q) = b \geq 0$. Taking lim sup on both sides in inequality (3.4), we have

$$\limsup_{n\to\infty} d(y_n, q) \leq b. \tag{3.13}$$

Since

$$d(T^n y_n, q) \leq (1 + s_n) d(y_n, q) + t_n, \tag{3.14}$$

we have

$$\limsup_{n\to\infty} d(T^n y_n, q) \leq b. \tag{3.15}$$

On the other hand, since

$$\lim_{n\to\infty} d(x_{n+1}, q) = \lim_{n\to\infty} d(\alpha_n T^n y_n \oplus (1-\alpha_n) y_n, q) = b, \tag{3.16}$$

by Lemma 2.6, we have

$$\lim_{n\to\infty} d(T^n y_n, y_n) = 0. \tag{3.17}$$

Hence assertion (1) of the lemma is proved.
In addition, since

$$\begin{aligned} d(x_{n+1}, q) &\leq d(x_{n+1}, T^n y_n) + d(T^n y_n, q) \\ &\leq (1-\alpha_n) d(y_n, T^n y_n) + (1+s_n) d(y_n, q) + t_n, \end{aligned} \tag{3.18}$$

we have $\liminf_{n\to\infty} d(y_n, q) \geq b$. By combined (3.16) and it yields that $\lim_{n\to\infty} d(y_n, q) = b$. This implies

$$\lim_{n\to\infty} d(\beta_n T^n z_n \oplus (1-\beta_n) z_n, q) = b. \tag{3.19}$$

Taking lim sup on both sides in inequality (3.3), we have

$$\limsup_{n\to\infty} d(z_n, q) \leq b. \tag{3.20}$$

Since

$$d(T^n z_n, q) \leq (1 + s_n) d(z_n, q) + t_n, \tag{3.21}$$

we have

$$\limsup_{n\to\infty} d\left(T^n z_n, q\right) \le b. \tag{3.22}$$

By Lemma 2.6, we have

$$\lim_{n\to\infty} d\left(T^n z_n, z_n\right) = 0. \tag{3.23}$$

Hence assertion (2) of the lemma is proved.
Thus, by the same method, we can prove that

$$\lim_{n\to\infty} d\left(T^n x_n, x_n\right) = 0. \tag{3.24}$$

Hence assertion (3) of the lemma is proved. The proof is completed. □

Lemma 3.6. *Let (X, d) be a complete CAT(0) space and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mapping with $\{s_n\}, \{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Suppose that $F(T) \ne \emptyset$. Let $\{x_n\}$ be the SP-iteration sequence defined by (3.1). Let $\{\alpha_n\} \subset [\delta, 1-\delta]$ and $\{\beta_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0, 1)$. Then $\lim_{n\to\infty} d\left(T x_n, x_n\right) = 0$.*

Proof. From Lemma 3.5, we have

$$\lim_{n\to\infty} d\left(T^n z_n, z_n\right) = 0,\ \lim_{n\to\infty} d\left(T^n y_n, y_n\right) = 0 \text{ and } \lim_{n\to\infty} d\left(T^n x_n, x_n\right) = 0. \tag{3.25}$$

Hence, we get

$$\begin{aligned} d\left(x_{n+1}, y_n\right) &= d\left(\alpha_n T^n y_n \oplus \left(1-\alpha_n\right) y_n, y_n\right) \\ &\le \alpha_n d\left(T^n y_n, y_n\right) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.26}$$

Similarly, we have

$$d\left(y_n, z_n\right) \le \beta_n d\left(T^n z_n, z_n\right) \to 0 \quad as \quad n \to \infty. \tag{3.27}$$

and

$$d\left(z_n, x_n\right) \le \alpha_n d\left(T^n x_n, x_n\right) \to 0 \quad as \quad n \to \infty. \tag{3.28}$$

It follows that

$$d\left(x_{n+1}, x_n\right) \le d\left(x_{n+1}, y_n\right) + d\left(y_n, z_n\right) + d\left(z_n, x_n\right) \to 0 \quad as \quad n \to \infty. \tag{3.29}$$

Since T is uniformly L-Lipschitzian, we have

$$\begin{aligned} d\left(T x_n, x_n\right) &\le d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, T^{n+1} x_{n+1}\right) \\ &\quad + d\left(T^{n+1} x_{n+1}, T^{n+1} x_n\right) + d\left(T^{n+1} x_n, T x_n\right) \\ &\le (1+L)\, d\left(x_{n+1}, x_n\right) + d\left(x_{n+1}, T^{n+1} x_{n+1}\right) + L d\left(T^n x_n, x_n\right) \\ &\quad \to 0 \quad as \quad n \to \infty, \end{aligned} \tag{3.30}$$

which implies

$$\lim_{n\to\infty} d\left(T x_n, x_n\right) = 0. \tag{3.31}$$

The proof is completed. □

Theorem 3.7. *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of* Theorem 3.1. *Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. By Lemma 3.6, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, In fact, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F(T)$. This implies that sequence $\{x_n\}$ is bounded. Let $W_\Delta(x_n) = \cup A(\{u_n\}) \subseteq F(T)$ and $W_\Delta(x_n)$ consists exactly of one point. In fact, let $u \in W_\Delta(x_n)$, then there exists subsequence $\{u_n\}$ of $\{x_n\}$ such that $\cup A(\{u_n\}) = \{u\}$. By Lemma 2.4 and Lemma 2.3, there exists a subsequence $\{r_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} r_n = r \in C$. By Lemma 2.6, $r \in F(T)$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, r)$ exists. Assume that $u \neq r$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(r_n, r) &< \limsup_{n\to\infty} d(r_n, u)\\ &\le \limsup_{n\to\infty} d(u_n, u)\\ &\le \limsup_{n\to\infty} d(u_n, r)\\ &= \limsup_{n\to\infty} d(x_n, r)\\ &\le \limsup_{n\to\infty} d(r_n, r).\end{aligned} \tag{3.32}$$

This is a contradiction. Hence $u = r \in F(T)$. Finally, we prove $\{x_n\}$ Δ-converges a fixed point of T. We claim that $x = r$. If not, then the existence of $\lim_{n\to\infty} d(x_n, r)$ and uniqueness of asymptotic centers imply that there exists a contradiction as (3.32) and therefore $x = r \in F(T)$. Thus, $W_\Delta(x_n) = \{x_n\}$. This shows that $\{x_n\}$ Δ-converges a fixed point of T. The proof is completed. □

Theorem 3.8. *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of* Theorem 3.1. *Assume, in addition that T is semi-compact. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 3.1, sequence $\{x_n\}$ is bounded. By Lemma 3.6, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and by the semi-compactness of T, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $q \in C$. By uniformly continuity of T, we have

$$d(Tq, q) = \lim_{n\to\infty} d(Tx_{n_k}, x_{n_k}) = 0. \tag{3.33}$$

This implies that $q \in F(T)$. By Theorem 3.1, $\lim_{n\to\infty} d(x_n, q)$ exists. Thus, q is the strong limit of sequence $\{x_n\}$. The sequence $\{x_n\}$ converges strongly to a fixed point q of T. The proof is completed. □

Theorem 3.9. *Let $X, C, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of* Theorem 3.1. *Assume, in addition that T satisfies condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 3.1, $\lim_{n\to\infty} d(x_n, F(T))$ exists. By condition (I) and Lemma 3.6, we have

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.34}$$

This is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$, for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. By Theorem 3.1 implies that sequence $\{x_n\}$ converges strongly to a fixed point q of T. The proof is completed. □

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