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Global bifurcation analysis of the Lorenz system

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Abstract

We carry out the global bifurcation analysis of the classical Lorenz system. For many years, this system has been the subject of study by numerous authors. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in this system. Using some numerical results and our bifurcational geometric approach, we present a new scenario of chaos transition in the Lorenz system.

Keywords: Lorenz system, bifurcation, singular point, limit cycle, chaos.
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1. Introduction

We consider a three-dimensional dynamical system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(r - z) - y, \quad \dot{z} = xy - bz \tag{1}$$

known as the Lorenz system. Historically, (1) was the first dynamical system for which the existence of an irregular attractor (chaos) was proved for $\sigma = 10$, $b = 8/3$, and $24{,}06 < r < 28$. For many years, the Lorenz system has been the subject of study by numerous authors; see, e. g., [1, 2], [11]–[16]. However, until now the structure of the Lorenz attractor is not clear completely yet, and the most important question at present is to understand the bifurcation scenario of chaos transition in system (1).

In Section 2 of this paper, we recall the classical scenario of chaos transition in the Lorenz system (1). In Section 3, we give for (1) a relatively new chaos transition scenario proposed by N. A. Magnitskii and S. V. Sidorov [12]. In Section 4, we present a different bifurcation scenario for system (1), where $\sigma = 10$, $b = 8/3$, and $r > 0$, using numerical results of [12] and our bifurcational geometric approach to the global qualitative analysis of three-dimensional dynamical systems which we applied earlier in the planar case [3]–[10].

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2. The classical scenario of chaos transition

First, let us briefly recall the contemporary point of view on the structure of the Lorenz attractor and chaos transition [12, 16].

1. The Lorenz system (1) is dissipative and symmetric with respect to the z-axis. The origin $O(0,0,0)$ is a singular point of system (1) for any σ, b, and r. It is a stable node for $r < 1$. For $r = 1$, the origin becomes a triple singular point, and then, for $r > 1$, there are two more singular points in the system: $O_1(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $O_2(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$ which are stable up to the parameter value $r_a = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ ($r_a \approx 24{,}74$ for $\sigma = 10$ and $b = 8/3$). For all $r > 1$, the point O is a saddle-node. It has a two-dimensional stable manifold W^s and a one-dimensional unstable manifold W^u. If $1 < r < r_1 \approx 13{,}9$, then separatrices Γ_1 and Γ_2 issuing from the point O along its one-dimensional unstable manifold W^u are attracted by their nearest stable points O_1 and O_2, respectively.

2. If $r = r_1$, then each of the separatrices Γ_1 and Γ_2 becomes a closed homoclinic loop. In this case, two homoclinic loops are tangent to each other and the z-axis at the point O and form a figure referred to as a homoclinic butterfly. It is assumed that the generation of an unstable homoclinic butterfly is one of the two bifurcations leading to the appearance of the Lorenz attractor.

3. If $r_1 < r < r_2 \approx 24{,}06$, then a saddle periodic trajectory bifurcates from each of the closed homoclinic loops (these trajectories will be denoted by L_1 and L_2, respectively). In this case, separatrices Γ_1 and Γ_2 tend to the stable points O_2 and O_1, respectively. It is usually assumed that stable manifolds of the saddle periodic trajectories L_1 and L_2 are the boundaries of attraction domains of points O_1 and O_2. A curve issuing from the exterior of these domains can make oscillations from the neighborhood of L_1 into a neighborhood of L_2 and conversely until it enters the attraction domain of the attractor O_1 and O_2; the closer is parameter r to the value r_2, the larger is the number of oscillations. This behavior of the system is referred to as metastable chaos. If $r = r_2$, then separatrices Γ_1 and Γ_2 do not tend to the points O_2 and O_1, but wind around the limit saddle cycles L_2 and L_1, respectively. Here the second bifurcation leading to the appearance of the Lorenz attractor takes place. If $r_2 < r < r_3 = r_a$, then points O_1 and O_2 are still stable. In addition, in the phase space, there is an attracting set B referred to as the Lorenz attractor; it is a set of integral curves moving from L_1 to L_2 and vice versa. The saddle point O, together with its separatrices Γ_1 and Γ_2, belongs to the attractor.

4. If $r \to r_3 = r_a$, then the saddle limit cycles L_1 and L_2 shrink to the points O_1 and O_2; for $r = r_3$, they vanish and coincide with these points as a result of the Andronov–Hopf subcritical bifurcation.

5. If $r_3 < r < r_4 \approx 30{,}1$, then the Lorenz attractor is the unique stable limit set of system (1). It is usually assumed that this set is a branching surface S lying near the plane $x - y = 0$ and consisting of infinitely many sheets tied together and infinitely close to each other. A phase trajectory issuing on the left from the z-axis comes untwisted along a spiral around the point O_1 until the transition to the right of the z-axis, after which it becomes untwisted along a spiral around the point O_2 in the opposite direction. The number of rotations around the points O_1 and O_2 varies irregularly; thus the motion looks chaotic. It is assumed that the attractor is not a two-dimensional manifold and has a fractal structure [12]. If $r_4 < r \leq 313$, then the structure of solutions of the system of Lorenz equations becomes extremely complicated with alternation of chaotic and periodic modes. It is usually assumed that there may be infinitely many periodicity windows in the system, and each of such windows is a direct subharmonic cascade of bifurcations, which terminates with a basic stable limit cycle. For further growth of r, each of such cycles is destroyed by an intermittency, and the appearance of periodicity windows is preceded by the inverse cascade of bifurcations [12].

6. If $r > 313$, then the unique stable limit cycle is an attractor in the Lorenz system.

3. The Magnitskii–Sidorov scenario

It was shown in [12], that actually in the Lorenz system absolutely another scenario of chaos transition would be realized. It turns out that all cycles from infinite family of unstable cycles, generating Lorenz attractor, have crossing with an one-dimensional unstable not invariant manifold V^u of the point O (do not confuse with the invariant unstable manifold W^u). This result follows from the theory of dynamical chaos stated in [12]. After the derivation of analytic formulas for the manifold V^u, it becomes possible to reduce the problem of establishing and proving the existence of unstable cycles in the Lorenz system to the one-dimensional case, namely, to finding stable points of the one-dimensional first return mapping defined on the unstable manifold [12]. By this method, it is shown in [12] that items 2 and 3 of the above-represented classical scenario of transition to chaos in the Lorenz system (1) are invalid. Some assertions of items 4–6 fail, while other require a more detailed investigation.

1. This item remains the same as item 1 of the classical scenario.

2. If $r = r_1 \approx 13{,}9$, then the separatrices Γ_1 and Γ_2 do not form two separate homoclinic loops. Here we have a bifurcation with the generation of a single closed contour surrounding both stationary points O_1 and O_2; the end of the separatrix Γ_1 enters the beginning of the separatrix Γ_2, and vice versa, the end of Γ_2 enters the beginning of Γ_1. As r grows, from this contour, a closed cycle C_0 appears there first. It is an eight-shaped figure surrounding both points O_1 and O_2.

3. If $r_1 < r < r_2 \approx 24{,}06$, then cycles L_1 and L_2 surrounding the points O_1 and O_2, respectively, do not appear; but with further growth of r, pairs of cycles C_n^+, C_n^-, $n = 0, 1, \ldots$, are successively generated. They determine the generation of the Lorenz attractor. The cycle C_n^+ makes n complete rotations in the half-space containing the point O_1 and one incomplete rotation around the point O_2. Conversely, the cycle C_n^- makes n complete rotations around the point O_2 and one incomplete rotation around the point O_1.

For each r, $r_1 < r < r_2$, there exists the number $n(r)$ ($n(r) \to \infty$ as $r \to r_2$) such that in the phase-space of (1), there are unstable cycles C_0, C_k^+, C_k^-, $k = 0, \ldots, n$, and cycles C_{km}^+, C_{km}^-, $k, m < n$, which make k rotations around the point O_1 and m rotations around the point O_2 and are various combinations of the cycles C_n^+ and C_n^-, and many other cycles generated by bifurcations of the cycles C_n^+ and C_n^- [12]. Points of intersection of all these cycles with the manifold V_u have the following arrangement on the curve V_u for $0 \le z_{min} \le z \le z_{max} < r - 1$. The point z_{min} corresponds to the right large single loop of the cycle C_n^-. This loop is the larger face of the right truncated cone of the set S. Further, the trajectory of the cycle passes into the left half-plane and makes n clockwise rotations around the point O_2. The smallest first loop around the point O_2 is the smaller face of the truncated cone of the set S. The point z_{max} corresponds to the smallest loop of the cycle C_n^+ around the point O_1. This loop is the smaller face of the right truncated cone. Further, the trajectory of this cycle makes n rotations around the point O_1 clockwise, passes into the left half-plane, and makes one large rotation around the point O_2. This rotation is the larger face of the left truncated cone. Between the points z_{min} and z_{max} there is a point z_0 corresponding to the main cycle C_0.

Boundaries of the attraction domains of the stable points O_1 and O_2 are given by the smallest loops of the cycles C_n^+ and C_n^-, whose size decay as r grows. Therefore, for some $r = r_m$, the attraction domain of the set B no longer intersects the attraction domains of points O_1 and O_2, and the set B becomes an attractor. Therefore, in the Lorenz system ($a = 10$, $b = 8/3$), metastable chaos exists only in the interval $r_1 < r < r_m$, and in the interval $r_m < r < r_2$, the system has three stable limit sets, namely, O_1 and O_2 and the Lorenz attractor.

If $r \to r_2$, then the eye size decreases as the number of rotations of the cycles C_n^+ and C_n^- around the points O_1 and O_2, respectively, grows. The value z_{max} grows, and z_{min} decays; moreover, $z_{min} \to 0$ as $r \to r_2$. The lengths of generatrices of truncated cones grow, since additional rotations are added to the cone vertex and diminish the size of the smaller face. Conversely, the larger face grows. If $r = r_2$, then $z_{min} = 0$, but $z_{max} < r - 1$; thus, the larger face of each cone achieves its maximal size, while the smaller face is not contracted into a point, the cone vertex. The following bifurcation takes place. In the limit as $n \to \infty$, each set of cycles C_n^+ (respectively, C_n^-) forms a point-cycle heteroclinic structure consisting of two separatrix contours of the point O. The first contour consists of a separatrix issuing from the point O along its unstable manifold and spinning on the appearing (only for $r = r_2$) saddle cycle L_1 (respectively, L_2) of the point O_1 (respectively, O_1). The second contour consists of the separatrix spinning out from the saddle cycle L_1 (respectively, L_2) and entering the point O along its stable manifold.

As mentioned above, the described bifurcation does not lead to generation of the Lorenz attractor for $r = r_2$. It is more correct to say that it is only a prerequisite of destruction of the attractor as r decays. The attractor itself, existing in the system for $r = r_2$, is formed from finitely many stable cycles $C_k^\pm$, $k = 0, \ldots, l$, for $r < 313$. It contains neither separatrices Γ_1 and Γ_2 of the point O nor infinitely many unstable cycles $C_n^{\pm}$ existing in the neighborhood of the point-cycle heteroclinic structure.

If $r_2 < r < r_3$, then points O_1 and O_2 are still stable, and their attraction domains are bound by the appearing limit cycles L_1 and L_2 contracting to points as $r \to r_3$. But the Lorenz attractor B is not a set of integral curves going from L_1 to L_2 and back, and separatrices Γ_1 and Γ_2 of the saddle point O do not belong to the attractor. Cycles L_1 and L_2 have already made their job at $r = r_2$ and no longer have anything to do with the attractor. If $r_2 < r < r_3$, then, just as in the case of $r_1 < r < r_2$, the cycles C_n^+ and C_n^- appear again from separatrix contours. The attractor is determined by finitely many such cycles [12].

4. For $r = r_3$, the saddle cycles L_1 and L_2 disappear. In the system, there is a unique limit set, namely, the Lorenz attractor.

5. There exist one more important value of the parameter r which affects the formation of the Lorenz attractor. This is a point $r_4 \approx 30{,}485$. If r grows from r_3 to r_4, then the number of rotations of the cycles C_n^+ and C_n^- first rapidly decays, then grows again. In this case, eyes by separatrices of the point O are much smaller than attractor eyes and begin to grow as r increases. Therefore, almost heteroclinic and almost homoclinic contours exist in system (1) at the point r_4.

The process of generation of the Lorenz attractor in system (1) as r decays from the value 313 up to r_4 is referred to as the incomplete double homoclinic cascade [12]. The complete cascade occurs if the r-axis passes exactly through the point of existence of two homoclinic contours. Note that in systems with a single homoclinic contour, there can be a simple complete or incomplete homoclinic cascade of bifurcations of transition to chaos, and in [12], a detailed description of transition to chaos through the double homoclinic (complete or incomplete) cascade of bifurcations is given. Just as in item 6 of the classical scenario, if $r > 313$, then in the system, there exists a unique stable limit cycle C_0 surrounding both points. If $r \approx 313$, then the cycle C_0 becomes unstable and generates two stable cycles C_0^+ and C_0^- which also surround the points O_1 and O_2 but have deflections in the direction of corresponding halves of the unstable manifold V^u of the point O. This is the point where the double homoclinic cascade of bifurcations really begins. In case of an incomplete cascade, it consists of finitely many stages of appearance of stable cycles $C_k^{\pm}$, $k = 0, \ldots, l$, and their infinitely many further bifurcations. But in case of a complete cascade, the number of stages is infinite, and at the limit of $l \to \infty$, cycles tend to homoclinic contours of the points O_1 and O_2, respectively. At the k-th stage of the cascade, originally stable cycles $C_k^{\pm}$ undergo a subharmonic cascade of bifurcations and form two band-form attractors that consist of infinitely many unstable limit cycles intersecting the respective domains of the unstable manifold V^u of the point O. Then these two bands merge and form a single attractor surrounding both the points O_1 and O_2, after which there is a cascade of bifurcations of cycles generated as a result of the merger and making rotations separately around the points O_1 and O_2 and simultaneously around both the points. The last cascade of bifurcations has the property of self-organization, since it is characterized by simplification of the structure of cycles and the generation of new stable cycles with a smaller number of rotations around the points O_1 and O_2 as r decays. Each cycle of the cascade of self-organization bifurcations undergoes its own subharmonic cascade of bifurcations, after which all cycles formed during infinitely many bifurcations of all subharmonic cascades and cascades of self-organization bifurcations of cycles become unstable and form some set B_k. After an incomplete homoclinic cascade of bifurcations, we obtain a set $B = \bigcup B_k$ consisting of infinitely many possible unstable cycles appearing at all stages of the cascade. These cycles generate an incomplete double homoclinic attractor, that is the classical Lorenz attractor.

6. This item remains the same as item 6 of the classical scenario.

4. The bifurcational geometric scenario

Revising the above scenarios, we carry out the global bifurcation analysis of the Lorenz system (1) and present a new scenario of chaos transition in this system for $\sigma = 10$, $b = 8/3$, and $r > 0$.

1. If $r < 1$, the unique singular point O of system (1) is a stable node. For $r = 1$, it becomes a triple singular point, and then, for $r > 1$, there are two more singular points in the system: O_1 and O_2 which

are stable up to the parameter value $r_a \approx 24{,}74$. For all $r > 1$, the point O is a saddle-node. It has a two-dimensional stable manifold W^s and an one-dimensional unstable manifold W^u. If $1 < r < r_l = r_1 \approx 13{,}9$, then the separatrices Γ_1 and Γ_2 issuing from the point O along its one-dimensional unstable manifold W^u are attracted by their nearest stable points O_1 and O_2, respectively.

2. If $r = r_l$, then each of the separatrices Γ_1 and Γ_2 becomes a closed homoclinic loop. In this case, two unstable homoclinic loops, C_0^+ and C_0^-, are formed around the points O_1 and O_2, respectively. They are tangent to each other and the z-axis at the point O and form together a homoclinic butterfly.

3. If $r_l < r < r_a \approx 24{,}74$, then, unfortunately, neither the classical scenario (see, e. g., [16]) nor the scenario of [12] can be realized. The reason is that, in both cases, trajectories of system (1) should intersect the two-dimensional stable manifold W^s of the point O. Since this is impossible, the only way to overcome the contradiction is to suppose that a cascade of period-doubling bifurcations [12] will begin immediately in each of the half-spaces with respect to the manifold W^s, when $r > r_l$. In this case, each of the homoclinic loops C_0^+ and C_0^- generates an unstable limit cycle of period 2 and a stable limit cycle of period 1 lying between the coils of the cycle of period 2 in the corresponding half-spaces containing the points O_1 and O_2, respectively. With further growth of r, each of the cycles of period 2 generates an unstable limit cycle of period 4 with a stable limit cycle of period 3 inside of it and each of the cycles of period 1 generates a stable limit cycle of period 2 with an unstable limit cycle of period 1 inside of it. Then, after next doubling, we will have in each of the half-spaces an unstable limit cycle of period 8 with an inserted stable limit cycle of period 7 and a stable limit cycle of period 6 with an inserted unstable limit cycle of period 5, and a stable limit cycle of period 4 with an inserted unstable limit cycle of period 3, and an unstable limit cycle of period 2 with an inserted stable limit cycle of period 1. Continuing this process further, we will obtain limit cycles of all periods from one to infinity, and the space between these cycles will be filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. These cycles are inserted into each other, they make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points and form geometric constructions (limit periodic sets) which look globally like very flat truncated cones described in the chaos transition scenario of [12].

4. For $r = r_a$, the biggest unstable limit cycles of infinite period disappear through the Andronov–Shilnikov bifurcation [11, 14] in each of the half-spaces containing the points O_1 and O_2 (the cone vertices are at these points), and these points become unstable saddle-foci generating two small stable limit cycles lying on two-dimensional focus manifolds of O_1 and O_2.

5. If $r_a < r < +\infty$, then a cascade of period-halving bifurcations [12] occurs in each of the half-spaces with respect to the manifold W^s. We have got again two symmetric with respect to the z-axis limit periodic sets consisting of limit cycles of all periods which are inserted into each other and make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points, and the space between the cycles is filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. With further growth of r, the period-halving process makes the limit periodic sets more and more flat. The obtained geometric constructions are the only stable limit sets of system (1). The spirals of the unstable saddle-foci O_1 and O_2 and the trajectories issuing from infinity tend to these limit periodic sets (more precisely, to their stable limit cycles) as $t \to +\infty$. Just these stable limit periodic sets form two symmetric parts of the so-called Lorenz attractor, and this really looks very chaotic.

6. If $r \to +\infty$ (numerically, when $r > 313$), then the period-halving process will be finishing and system (1) will have two stable limit cycles lying on the two-dimensional focus manifolds of the unstable saddle-foci O_1 and O_2 in two phase half-spaces of (1) containing these points. This completes our scenario of chaos transition in the Lorenz system (1).

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