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Positive periodic solution for a nonlinear neutral delay population equation with feedback control

Payam Nasertayoob, S. Mansour Vaezpour*

Department of Mathematics, Amirkabir University of Technology (Polytechnic), Hafez Ave., P. O. Box 15914, Tehran, Iran.

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Abstract

In this paper, sufficient conditions are investigated for the existence of positive periodic solution for a nonlinear neutral delay population system with feedback control. The proof is based on the fixed-point theorem of strict-set-contraction operators. We also present an example of nonlinear neutral delay population system with feedback control to show the validity of conditions and efficiency of our results.

Keywords: Fixed point theory, neutral nonlinear equation, feedback control, strict-set-contraction.
2010 MSC: 34K40, 92D25, 47H10.

1. Introduction and Preliminaries

The preliminary mathematical model of the population growth is given by the following logistic equation:

$$\frac{dx}{dt} = \rho x(t)[1 - ax(t)]. \tag{1.1}$$

In the real problems, conditions for the population of species are more complicated and the simple logistic model (1.1) could be generalized in many ways. For some kinds of population systems, when density of species depends not only on the population at time, but also on the population unit earlier, the equation (1.1) may be recovered as follows [15]:

$$\frac{dx}{dt} = \rho x(t)[1 - ax(t) - bx(t-\tau)]. \tag{1.2}$$

*Corresponding author
Email addresses: `nasertayoob@aut.ac.ir` (Payam Nasertayoob), `vaez@aut.ac.ir` (S. Mansour Vaezpour)

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One can also consider the non-autonomous version of (1.2) [18], i.e.,

$$\frac{dx}{dt} = \rho(t)x(t)[1 - a(t)x(t) - b(t)x(t-\tau)]. \tag{1.3}$$

In the more realistic situation, the biological systems or ecosystems are continuously perturbed via unpredictable forces. These perturbations are generally results of the change in the system's parameters. In the language of the control theory, these perturbation functions may be regarded as control variables and, consequently, one should ask a question that whether or not an ecosystem can withstand those unpredictable perturbations which persist for a finite periodic time. In the mathematical biology, the following population model of systems of differential equations is a famous feedback control model with delays [6],

$$\begin{aligned}\frac{dx}{dt} &= \rho x(t)[1 - ax(t-\tau) - bu(t)],\\ \frac{du}{dt} &= -\eta u(t) + gx(t-\tau),\end{aligned}$$

where, a, b, g, μ, $\rho \in (0,\infty)$ and u represent an indirect feedback control mechanism. In the recent years, study of the feedback control models has been further developed and the literature in mathematical biology has been riched with study of such models [6, 10, 16]. Besides, many scholars have worked on neutral systems. Some results can be found in [1, 9].

In [14], existence of positive periodic solutions for neutral population model was studied. Subsequently, Lu *et. al.*, [13] investigated the positive periodic solutions for such a neutral differential system with feedback control. Besides, several scholars have paid their attention to the nonlinear population dynamics [3, 4, 5, 11], because such kinds of systems could simulate the real world more accurately. Recently, the almost periodic solution of the following nonlinear population dynamics with feedback control has been studied [17],

$$\begin{aligned}\frac{dx}{dt} &= x(t)[\rho(x) - a(x)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - c(t)u(t)],\\ \frac{du}{dt} &= -\eta(t)u(t) + \sum_{i=1}^{n} g(t)x^{\beta_i}(t-\sigma_i).\end{aligned} \tag{1.4}$$

Investigation of the nonlinear system above is based on the properties of almost periodic systems and Lyapunov-Razumikhin technique.

In this paper, we consider the following nonlinear neutral non-autonomous delay population model

$$\begin{aligned}\frac{dx}{dt} &= x(t)[\rho(x) - a(x)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - \gamma(t)x'(t-\tau) - c(t)u(t)],\\ \frac{du}{dt} &= -\eta(t)u(t) + \sum_{i=1}^{n} g(t)x^{\beta_i}(t-\sigma_i),\end{aligned} \tag{1.5}$$

which is a generalization of neutral system dynamics in [13], where, a, b_i, g, μ, ρ, η, γ are positive ω-periodic functions and α, β_i are belong to $(0,\infty)$.

Our investigation in this paper is based on a fixed point theorem of strict-set-contractive operator which goes back to Cac-Gatica [2] and which was used for similar purpose in [13].

We recall some preliminaries that will be used in the further sections.

Definition 1.1. Let X be a Banach space. For the bounded set $\Omega \subset X$, Kuratowski measure of noncompactness defines by:

$$\begin{aligned}\mu_X(\Omega) = \inf\{d > 0 \,:\, &\text{there is a finite number of subsets } \Omega_i \subset \Omega\\ &\text{such that } \Omega = \bigcup_i \Omega_i \text{ and } diam\Omega_i < d\}.\end{aligned}$$

Definition 1.2. ([7, 8]) Let X and Y are Banach spaces. A continuous and bounded map $F : E \subset X \to Y$ is called a k-set contraction if for any bounded set $\Omega \subset E$

$$\mu_Y(F(\Omega)) \leq k\mu_X(\Omega).$$

The operator F is called a strict-set-contraction, where $0 \leq k < 1$.

Theorem 1.3. ([2, 12]) *Let Π be a semi-ordered cone in a Banach space X and*

$$\Pi_{r,R} = \{x \in \Pi : 0 < r \leq \|x\| \leq R\}.$$

Let $F : \Pi_{r,R} \to \Pi$ be a strict-set-contraction, satisfying

$$Fx \ngeq x \qquad \text{for any } x \in \Pi_{r,R} \text{ and } \|x\| = r, \tag{1.6}$$

and

$$Fx \nleq x \qquad \text{for any } x \in \Pi_{r,R} \text{ and } \|x\| = R. \tag{1.7}$$

Then, $F : \Pi_{r,R} \to \Pi$ has at least one fixed point in $\Pi_{r,R}$.

One may easily shows that the following function is a solution of the second equation in (1.5),

$$(\Phi x)(t) = \int_t^{t+\omega} G(t,s)\{\sum_{i=1}^{n} g(s)x^{\beta_i}(s-\sigma_i)\}ds, \tag{1.8}$$

where

$$G(t,s) = \frac{\exp(\int_t^s \eta(\theta)d\theta)}{\exp(\int_0^\omega \eta(\theta)d\theta) - 1}, \qquad s \in [t, t+\omega], \ \ t \in \mathbb{R}.$$

Therefore, existence of the ω-periodic solution of the system (1.5) is equivalent to the existence of the solution of the following equation:

$$\frac{dx}{dt} = x(t)[\rho(t) - a(t)x^{\alpha}(t) - \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) - \gamma(t)x'(t-\tau) - c(t)(\Phi x)(t)]. \tag{1.9}$$

On the other hand, each ω-periodic solution of the following integral equation

$$x(t) = \int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) + \gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds, \tag{1.10}$$

is a solution of equation (1.9), where

$$\widetilde{G}(t,s) = \frac{\exp(-\int_t^s \rho(\theta)d\theta)}{1 - \exp(-\int_0^\omega \rho(\theta)d\theta)}, \qquad s \in [t, t+\omega], \ \ t \in \mathbb{R}.$$

Also, in what follows, we employ the following notations:

$$\overline{f} = \sup_{t\in[t,t+\omega]} f(t), \qquad \underline{f} = \inf_{t\in[t,t+\omega]} f(t),$$

$$\lambda = \exp(-\int_0^\omega \rho(\theta)d\theta) < 1, \qquad \kappa = \exp(\int_0^\omega \eta(\theta)d\theta) > 1,$$

$$\mathcal{C}_\omega = \{x \in \mathcal{C}(\mathbb{R}, (0, +\infty)) \ : \ x(t+\omega) = x(t)\},$$

$$\mathcal{C}^1_\omega = \{x \in \mathcal{C}^1(\mathbb{R},(0,+\infty)) \ : \ x(t+\omega)=x(t)\},$$
$$A=\min\{1,\alpha,\beta_1,...,\beta_n\}, \qquad B=\max\{1,\alpha,\beta_1,...,\beta_n\},$$
$$\Psi(t)=\int_t^{t+\omega} G(t,s)g(s)ds,$$
$$M=\sup_{t\in[t,t+\omega]}[a(t)+\sum_{i=1}^{n} b_i(t)+\gamma(t)+nc(t)\Psi(t)],$$
$$N=\int_0^{\omega}[a(s)+\sum_{i=1}^{n} b_i(s)+\gamma(s)+nc(s)\Psi(s)]ds.$$

Clearly, two spaces $(\mathcal{C}_\omega, \|\|)$ and $(\mathcal{C}^1_\omega, \|\|_1)$ are Banach spaces. Where

$$\|x\|=\max_{t\in[t,t+\omega]}|x(t)|,$$

and

$$\|x\|_1=\max\{\|x(t)\|,\|x'(t)\|\}.$$

We define the following integral operator $F:\Pi\mapsto \mathcal{C}^1_\omega$

$$(Fx)(t)=\int_t^{t+\omega}\widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s)+\sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i)+\gamma(s)x'(s-\tau)+c(s)(\Phi x)(s)]ds, \tag{1.11}$$

where, $\Pi=\{x\in\mathcal{C}^1_\omega \ : \ x(t)\geq \lambda\|x\|_1\}$ is a semi-ordered cone in $\mathcal{C}^1_\omega$.

2. Main results

Lemma 2.1. *Let $R\leq 1$, $\overline{\rho}\leq 1$ and there exists a positive real number Z such that*

$$Z\leq \lambda^B r^{B-A}, \tag{2.1}$$

and

$$M\leq(\underline{\rho}+1)\frac{Z\lambda^2}{(1-\lambda)}N, \tag{2.2}$$

and

$$\gamma(t)\leq Z\{a(t)+\sum_{i=1}^{n} b_i(t)+nc(t)\Psi(t)\}, \tag{2.3}$$

then, the integral operator F maps $\Pi_{r,R}$ into Π.

Proof: For $x\in\Pi_{r,R}$ with $R\leq 1$, we have

$$Z\leq\lambda^B r^{B-A}\leq\lambda^B\|x\|_1^{B-A}\leq\lambda^{\xi}\|x\|_1^{\xi-A}, \qquad for\ any\ \xi\in\{1,\alpha,\beta_1,...,\beta_n\},$$

therefore

$$Z\|x\|_1^A\leq\lambda^{\xi}\|x\|_1^{\xi}\leq x^{\xi}(t), \qquad for\ any\ \xi\in\{1,\alpha,\beta_1,...,\beta_n\}. \tag{2.4}$$

On the other hand, since $\|x\|_1\leq 1$, we obtain

$$\|x\|_1^{\xi}\leq\|x\|_1^A, \qquad for\ any\ \xi\in\{1,\alpha,\beta_1,...,\beta_n\}, \tag{2.5}$$

thus

$$nZ\|x\|_1^A\Psi(t)\leq(\Phi x)(t)\leq n\|x\|_1^A\Psi(t). \tag{2.6}$$

Applying (2.3), (2.4) and (2.6), we have

$$\|x\|_1^A\gamma(t) \leq Z\|x\|_1^A a(t) + Z\|x\|_1^A \sum_{i=1}^{n} b_i(t) + nZ\|x\|_1^A c(t)\Psi(t).$$

Thus

$$\pm x'(t)\gamma(t) \leq \|x\|_1\gamma(t) \leq a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) + c(t)(\Phi x)(t).$$

Consequently,

$$0 \leq a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \pm x'(t)\gamma(t) + c(t)(\Phi x)(t). \tag{2.7}$$

On the other hand, since ρ is a positive and ω-periodic function we obtain

$$\frac{\lambda}{1-\lambda} \leq \widetilde{G}(t,s) \leq \frac{1}{1-\lambda}. \tag{2.8}$$

Step 1. We show $(Fx)(t) \geq \lambda\|Fx\|$.

$$\begin{aligned}
\|Fx\| &= \sup_{t\in[t,t+\omega]} |(Fx)(t)| \\
&= \sup_{t\in[t,t+\omega]} \int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&\leq \frac{1}{1-\lambda}\int_t^{t+\omega} x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&= \frac{1}{\lambda}\{\int_t^{t+\omega} \frac{\lambda}{1-\lambda}x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds\} \\
&= \frac{1}{\lambda}\int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&= \frac{1}{\lambda}(Fx)(t).
\end{aligned} \tag{2.9}$$

Step 2. We show $(Fx)'(t) \leq Fx$. Based on the Leibniz integral rule, relations (2.2) and (2.8), we obtain

$$\begin{aligned}
(Fx)'(t) &= \widetilde{G}(t+\omega,t)x(t+\omega) \\
&\quad \times\ [a(t+\omega)x^{\alpha}(t+\omega) + \sum_{i=1}^{n} b_i(t+\omega)x^{\beta_i}(t+\omega-\sigma_i) \\
&\quad + \gamma(t+\omega)x'(t+\omega-\tau) + c(t+\omega)(\Phi x)(t+\omega)] \\
&\quad -\ \widetilde{G}(t)x(t)[a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \\
&\quad + \gamma(t)x'(t-\tau) + c(t)(\Phi x)(t)] \\
&= (\frac{\lambda}{1-\lambda} - \frac{1}{1-\lambda})x(t)[a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \\
&\quad + \gamma(t)x'(t-\tau) + c(t)(\Phi x)(t)] - \rho(t)(Fx)(t) \\
&\leq \rho(t)(Fx)(t) \leq (Fx)(t).
\end{aligned} \tag{2.10}$$

Step 3. We show $-(Fx)'(t) \le (Fx)(t)$. Applying (2.2), (2.10), (2.5), (2.8) and (2.4), we obtain

$$\begin{aligned}
-(Fx)'(t) &= x(t)[a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \\
&\quad + \gamma(t)x'(t-\tau) + c(t)(\Phi x)(t)] - \rho(t)(Fx)(t) \\
&\le \|x\|_1^{A+1} \sup_{t\in[t,t+\omega]} [a(t) + \sum_{i=1}^{n} b_i(t) + \gamma(t) + nc(t)\Psi(t)] - \rho(t)(Fx)(t) \\
&\le \|x\|_1^{A+1} M - \underline{\rho}(Fx)(t) \\
&\le \|x\|_1^{A+1}\{\underline{\rho}+1\}\frac{Z\lambda^2}{(1-\lambda)}N - \underline{\rho}(Fx)(t) \\
&= \{\underline{\rho}+1\}\int_0^{\omega} \frac{\lambda}{1-\lambda}(\lambda\|x\|_1)[a(s)(Z\|x\|_1^A) + \sum_{i=1}^{n} b_i(s)(Z\|x\|_1^A) \\
&+ \gamma(s)(Z\|x\|_1^A) + c(s)n(Z\|x\|_1^A)\Psi(t)]ds - \underline{\rho}(Fx)(t) \\
&\le \{\underline{\rho}+1\}\int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&+ \gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds - \underline{\rho}(Fx)(t) \\
&= \{\underline{\rho}+1\}(Fx)(t) - \underline{\rho}(Fx)(t) = (Fx)(t).
\end{aligned}$$

Steps 1, 2 and 3 result that $(Fx)(t) \ge \lambda\|Fx\|_1$. Thus, $Fx \in \Pi$ and the proof is completed. □

Lemma 2.2. *Let the relation* (2.2) *satisfies and* $R\overline{\gamma} \le 1$*, then* $F : \Pi_{r,R} \mapsto \Pi$ *is a strict-set-contraction.*

Proof: Clearly, one may indicate that F is continuous and bounded operator. Let $\Omega \subset \Pi_{r,R}$ be any bounded set and $\mu_{\mathcal{C}_\omega^1}(\Omega) = d$, then, for any positive real number $\varepsilon \le R\overline{\gamma}d$ there exists a finite family $\{\Omega_i\}$ such that $\Omega = \bigcup_i \Omega_i$ and $diam\Omega_i \le d + \varepsilon$. Thus,

$$\|x-y\|_1 \le d+\varepsilon, \qquad for\ any\ x,y \in \Omega_i. \tag{2.11}$$

On the other hand, Ω_i is precompact in $\mathcal{C}_\omega$ thus there is a finite family of subsets Ω_{ij} such that $\Omega_i = \bigcup_j \Omega_{ij}$ and

$$\max\{\|x-y\|, \|x^{\alpha}-y^{\alpha}\|, \|x^{\beta_1}-y^{\beta_1}\|, ..., \|x^{\beta_n}-y^{\beta_n}\|\} \le \varepsilon, \tag{2.12}$$

for any $x, y \in \Omega_{ij}$. Also, $F(\Omega)$ is precompact in $\mathcal{C}_\omega$. To see this, note that

$$\begin{aligned}
|(Fx)(t)| &\le \frac{1}{1-\lambda}\|x\|_1^{A+1}\int_0^{\omega}\{a(t) + \sum_{i=1}^{n} b_i(t) + \gamma(t) + nc(t)\Psi(t)\}dt \\
&\le \frac{N}{\overline{\gamma}^{A+1}(1-\lambda)}.
\end{aligned}$$

This inequality together with (2.6) give

$$\begin{aligned}
|(Fx)'(t)| &= |x(t)[a(t)x^{\alpha}(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \\
&\quad + \gamma(t)x'(t-\tau) + c(t+\omega)(\Phi x)(t)] - \rho(t)(Fx)(t)| \\
&\le \|x\|_1^{A+1}|a(t) + \sum_{i=1}^{n} b_i(t) + \gamma(t) + nc(t)\Psi(t)| + \frac{1}{\lambda}|(Fx)(t)| \\
&\le \frac{1}{\overline{\gamma}^{A+1}}\{M + \frac{N}{\lambda(1-\lambda)}\} = \varrho.
\end{aligned} \tag{2.13}$$

Suppose $\{\xi_m\}$ is an arbitrary sequences on Ω. Clearly, $\{\xi_m\}$ is bounded. Based on the definition of integral operator F in (1.10) the function $(F\xi_m)(t)$ is differentiable for all $m \in \mathbb{N}$ and $t \in [0,\omega]$. For given $\varepsilon > 0$, if we consider $\delta = \frac{\varepsilon}{\varrho}$, then for all $m \in \mathbb{N}$ and $t, t' \in [0,\omega]$ with $|t - t'| < \delta$ we obtain

$$|(F\xi_m)(t) - (F\xi_m)(t')| \leq \varrho |t - t'| \leq \varepsilon.$$

Thus, $\{(F\xi_m)(t)\}$ as a sequence of functions on $[0,\omega]$ is equicontinuous. Therefore, based on Arzela-Ascoli theorem there exists a subsequent of $\{(F\xi_m)(t)\}$, say $\{(F\xi_{m_i})(t)\}$, which is uniformly convergence on $[0,\omega]$. Consequently, F is a compact bounded operator and $F(\Omega)$ is precompact in $\mathcal{C}_\omega$. As a result, there exists a family of subsets Ω_{ijk} such that $\Omega_{ij} = \bigcup_k \Omega_{ijk}$ and

$$\|Fx - Fy\| \leq \varepsilon, \qquad for\ any\ x, y \in \Omega_{ijk}. \tag{2.14}$$

On the other hand, applying (2.10), (2.11), (2.12) and (2.14), for any $x, y \in \Omega_{ijk}$, we obtain

$$\begin{aligned}
\|(Fx)' - (Fy)'\| &= \sup_{t\in[t,t+\omega]} |(Fx)'(t) - (Fy)'(t)| \\
&\leq \sup_{t\in[t,t+\omega]} |\rho(t)(Fx)'(t) - \rho(t)(Fy)'(t)| \\
&+ \sup_{t\in[t,t+\omega]} |x(t)[a(t)x^\alpha(t) + \sum_{i=1}^{n} b_i(t)x^{\beta_i}(t-\sigma_i) \\
&+ \gamma(t)x'(t-\tau) + c(t)(\Phi x)(t)] - y(t)[a(t)y^\alpha(t) \\
&+ \sum_{i=1}^{n} b_i(t)y^{\beta_i}(t-\sigma_i) + \gamma(t)y'(t-\tau) + c(t)(\Phi y)(t)]| \\
&\leq \|\rho\| \|(Fx)' - (Fy)'\| \\
&+ \sup_{t\in[t,t+\omega]} |(x(t) - y(t))[a(t)y^\alpha(t) + \sum_{i=1}^{n} b_i(t)y^{\beta_i}(t-\sigma_i) \\
&+ \gamma(t)y'(t-\tau) + c(t)(\Phi y)(t)]| \\
&\leq \frac{1}{\lambda}\varepsilon + \{\overline{a}R^\alpha + \sum_{i=1}^{n} \overline{b_i}R^{\beta_i} + \overline{\gamma}R + \overline{cg}\frac{\kappa}{\kappa-1}\sum_{i=1}^{n} R^{\beta_i}\omega\}\varepsilon \\
&+ R\{\overline{a}\varepsilon + \sum_{i=1}^{n} \overline{b_i}\varepsilon + \overline{\gamma}(d+\varepsilon) + \overline{cg}\frac{\kappa}{\kappa-1}n\varepsilon\omega\} \\
&\leq R\overline{\gamma}d + J\varepsilon,
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
J &= \frac{1}{\lambda} + \overline{a}\{R^\alpha + R\} + \sum_{i=1}^{n} \overline{b_i}\{R^{\beta_i} + R\} \\
&+ 2\overline{\gamma}R + \overline{cg}\frac{\kappa}{\kappa-1}\omega\{\sum_{i=1}^{n} R^{\beta_i} + R\}.
\end{aligned}$$

Therefore, from (2.14) and (2.15) and the condition $\varepsilon \leq R\overline{\gamma}d$, we obtain

$$\|Fx - Fy\|_1 \leq R\overline{\gamma}d + J\varepsilon, \qquad for\ any\ x, y \in \Omega_{ijk}.$$

Since ε is arbitrary small, we have

$$\mu_{\mathcal{C}^1_\omega}(F(\Omega)) \leq R\overline{\gamma}\mu_{\mathcal{C}^1_\omega}(\Omega),$$

and the proof of lemma is completed. □

Theorem 2.3. *Let conditions of the Lemma 2.1 hold, also*

$$r < \{\frac{1-\lambda}{N}\}^{\frac{1}{A}}, \tag{2.16}$$

and

$$\{\frac{1-\lambda}{Z\lambda N}\}^{\frac{1}{A}} < R, \tag{2.17}$$

then, the integral operator (1.11) *has at least one periodic solution in* $\Pi_{r,R}$.

Proof: **step 1**. Let $x \in \Pi_{r,R}$ and $\|x\| = r$. If $Fx = x$, then the operator F has fixed point. Let $Fx > x$. This means that $Fx - x \in \Pi - \{0\}$, which implies that $(Fx)(t) - x(t) \geq \lambda \|Fx - x\|_1$, consequently,

$$\|x\| \leq \|Fx\|. \tag{2.18}$$

Applying (2.8), (2.5) and inequality (2.16), we have

$$\begin{aligned}
(Fx)(t) &= \int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&\leq \frac{1}{1-\lambda}\int_t^{t+\omega} x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x^{\beta_i}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&\leq \frac{1}{1-\lambda}\int_t^{t+\omega} \|x\|[a(s)\|x\|_1^{\alpha} + \sum_{i=1}^{n} b_i(s)\|x\|_1^{\beta_i} \\
&\quad +\gamma(s)\|x\|_1 + c(s)\sum_{i=1}^{n}\|x\|_1^{\beta_i}\Psi(s)]ds \\
&\leq \frac{\|x\|\|x\|_1^{A}}{1-\lambda}\int_0^{\omega}[a(s) + \sum_{i=1}^{n} b_i(s) + \gamma(s) + nc(s)\Psi(s)]ds \\
&= \frac{\|x\|r^{A}N}{1-\lambda} < \|x\|,
\end{aligned} \tag{2.19}$$

therefore,

$$\|x\| \leq \|Fx\| < \|x\|.$$

which is a contradiction.

step 2. Let $x \in \Pi_{r,R}$ and $\|x\| = R$. If $Fx = x$, then the operator F has fixed point. Let $Fx > x$. This means that $x - Fx \in \Pi - \{0\}$, which implies that $x(t) - (Fx)(t) \geq \lambda\|Fx - x\|_1$, consequently, for any $t \in [0, \omega]$ we have

$$(Fx)(t) \leq x(t). \tag{2.20}$$

Applying (2.20), (2.8), (2.4) and (2.17), we have

$$\begin{aligned}
x(t) &\geq (Fx)(t) \\
&= \int_t^{t+\omega} \widetilde{G}(t,s)x(s)[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x_i^{\beta}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds \\
&\geq \frac{\lambda\|x\|_1}{1-\lambda}\int_t^{t+\omega}[a(s)x^{\alpha}(s) + \sum_{i=1}^{n} b_i(s)x_i^{\beta}(s-\sigma_i) \\
&\quad +\gamma(s)x'(s-\tau) + c(s)(\Phi x)(s)]ds
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{\lambda\|x\|_1}{1-\lambda}\int_t^{t+\omega}[a(s)(\lambda\|x\|_1)^{\alpha}+\sum_{i=1}^{n}b_i(s)(\lambda\|x\|_1)^{\beta_i}\\
&\quad +\gamma(s)(\lambda\|x\|_1)+c(s)\sum_{i=1}^{n}(\lambda\|x\|_1)^{\beta_i}\Psi(s)]ds\\
&\leq \frac{Z\lambda\|x\|_1\|x\|_1^{A}}{1-\lambda}\int_0^{\omega}[a(s)+\sum_{i=1}^{n}b_i(s)+\gamma(s)+nc(s)\Psi(s)]ds\\
&= \frac{Z\lambda\|x\|_1R^{A}}{1-\lambda}N>\|x\|_1.
\end{aligned}$$

That is a contradiction. Therefore, (1.6) and (1.7) hold. By Theorem 1.3 we see that the integral operator F has at least one fixed point in $\Pi_{r,R}$ under appropriate conditions. □

Remark 2.4. Note that

$$\begin{aligned}
N &= \int_0^{\omega}[a(s)+\sum_{i=1}^{n}b_i(s)+\gamma(s)+nc(s)\Psi(s)]ds\\
&\leq \int_0^{\omega}\sup_{t\in[t,t+\omega]}[a(t)+\sum_{i=1}^{n}b_i(t)+\gamma(t)+nc(t)\Psi(t)]ds\\
&\leq \omega M.
\end{aligned}$$

Thus, for any arbitrary positive ω-periodic functions a, b_i, γ, c and g the real number ω is bounded below by $\frac{N}{M}$.
On the other hand, inequality (2.2) yields

$$\frac{1-\lambda}{\lambda^2 Z\{\underline{\rho}+1\}}\leq\frac{N}{M}\leq\omega. \tag{2.21}$$

This shows that ω is also bounded below by $(1-\lambda)(\lambda^2Z\{\underline{\rho}+1\})^{-1}$. However, λ depends on the both select of the function ρ and period number ω. This means that ρ is a ω-periodic function with the following property:

$$\exp(2\int_0^{\omega}\rho(\theta)d\theta)-\exp(\int_0^{\omega}\rho(\theta)d\theta)\leq\omega Z(\underline{\rho}+1),$$

that is valid for $\rho(t)=\frac{1+\sin(2\pi t)}{32}$, $\omega=1$ and $Z=0.9$.

Remark 2.5. According to the Lemma 2.2, $F:\Pi_{r,R}\mapsto\Pi$ is a strict-set-contraction operator so long as $R\overline{\gamma}\leq 1$. Or

$$\gamma(t)\leq\frac{1}{R}.$$

Thus, with due attention to inequality (2.17), we obtain

$$\gamma(t)\leq\{\frac{Z\lambda N}{1-\lambda}\}^{\frac{1}{A}}.$$

On the other hand, inequality (2.16) yields $N^{\frac{1}{A}}<\{1-\lambda\}^{\frac{1}{A}}r^{-1}$, consequently,

$$\gamma(t)<\frac{\{Z\lambda\}^{\frac{1}{A}}}{r}. \tag{2.22}$$

Thus, γ is bounded above by $\{Z\lambda\}^{\frac{1}{A}}r^{-1}$.

Remark 2.6. Combining inequalities (2.16) and (2.17), we obtain

$$\frac{1-\lambda}{Z\lambda R^A} < N < \frac{1-\lambda}{r^A}, \tag{2.23}$$

that indicate that N is bounded above and below.

3. Illustrative Example

Consider the following system of neutral population dynamics with delay and feedback control

$$\begin{aligned}
\frac{dN}{dt} &= N(t)[1+\sin(2\pi t) - w(1-\cos(2\pi t))N^{\frac{31}{32}}(t) \\
&- \quad w(1+\frac{1}{2}\cos(2\pi t))(N^{\frac{32}{33}}(t-\sigma_1)+N^{\frac{33}{34}}(t-\sigma_2)) \\
&- \quad w(2+\cos(2\pi t))N'(t-\tau) - w(1+\frac{10}{18}\cos(2\pi t))u(t)] \\
\frac{du}{dt} &= (-1-\cos(2\pi t))u(t) + \frac{1-\sin(2\pi t)}{0.582}(N^{\frac{32}{33}}(t-\sigma_1)+N^{\frac{33}{34}}(t-\sigma_2)),
\end{aligned}$$

which is an example of nonlinear population system (1.5), with

$$n=2, \quad \omega = 1,$$

$$a(t) = w(1-\cos(2\pi t)), \qquad c(t) = w(1+\frac{10}{18}\cos(2\pi t)),$$

$$g(t) = \frac{1-\sin(2\pi t)}{0.582}, \qquad \rho(t) = \frac{1+\sin(2\pi t)}{32},$$

$$\gamma(t) = w(2+\cos(2\pi t)), \qquad \eta(t) = 1+\cos(2\pi t),$$

$$b_1(t) = b_2(t) = w(1+\frac{1}{2}\cos(2\pi t)),$$

$$\alpha = \frac{31}{32}\ ,\beta_1 = \frac{32}{33},\ \beta_2 = \frac{33}{34},$$

$$\lambda = \exp(-\int_0^1 \rho(\theta)d\theta) = \exp(-\int_0^1 (\frac{1+\sin(2\pi\theta)}{32})d\theta) = \frac{1}{\sqrt[32]{e}} = 0.9692 < 1,$$

$$\kappa = \exp(\int_0^1 \eta(\theta)d\theta) = \exp(\int_0^1 1+\cos(2\pi\theta)d\theta)) = e > 1,$$

$$R = 0.99, \ \ Z = 0.9, \ \ r = 0.5.$$

Taking into consideration aforesaid data, we have

$$Z = 0.9 < 0.9692 \times (\frac{1}{2})^{\frac{1}{32}} = 0.948 = \lambda^B r^{B-A}, \tag{3.1}$$

and

$$\frac{M}{N} = \frac{10.31w}{5.27w} = 1.97 < (\underline{\rho}+1)\frac{Z\lambda^2}{(1-\lambda)} = 27.44. \tag{3.2}$$

Besides, according to the inequality

$$\frac{1}{e-1} \le G(s,t),$$

we obtain,

$$\begin{aligned}
2+\cos(2\pi t) &\leq 4.5+\cos(2\pi t)\\
&= 0.9\{1-\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)+2(1+\frac{10}{18}\cos(2\pi t))\}\\
&= 0.9\{1-\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)\\
&+ 2(1+\frac{10}{18}\cos(2\pi t))\frac{1}{e-1}\int_0^1 \frac{1-\sin(2\pi t)}{0.582}\}\\
&\leq 0.9\{1-\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)+1+\frac{1}{2}\cos(2\pi t)\\
&+ 2(1+\frac{10}{18}\cos(2\pi t))\int_0^1 G(s,t)\frac{1-\sin(2\pi t)}{0.582}\}.
\end{aligned} \tag{3.3}$$

These three expressions, (3.1), (3.2) and (3.3) show that conditions (2.1), (2.2) and (2.3) in Lemma 2.1 are valid for our example. At the end, for the inequality (2.23) we obtain $0.0439 < N < 0.0745$, that is valid by proper choosing of w.

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