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Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces

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Abstract

The aim of this paper is to present some common fixed point theorems for g-weakly isotone increasing mappings satisfying a generalized contractive type condition under a continuous function in the framework of ordered b-metric spaces. Our results extend the results of Nashine et al. [H. K. Nashine, B. Samet, C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Math. Comput. Modelling 54 (2011) 712–720] from the context of ordered metric spaces to the setting of ordered b-metric spaces. Moreover, some examples of applications of the main result are given. Finally, we establish an existence theorem for a solution of an integral equation.

Keywords: Common fixed point, b-metric space, partially ordered set, weakly isotone increasing mappings.
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1. Introduction and preliminaries

Recently, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order and obtained many fixed point results in such spaces. For more details on fixed point results, their applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the reader to [1, 2, 4, 6, 7, 8, 11, 19, 24] and the references mentioned therein.

Let $(X, \preceq)$ be a partially ordered set and let f, g be two self-maps on X. We will use the following terminology:

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(a) elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds;

(b) a subset K of X is said to be well ordered if every two elements of K are comparable;

(c) f is called nondecreasing w.r.t. $\preceq$ if $x \preceq y$ implies $fx \preceq fy$;

(d) the pair (f, g) is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$ [6];

(e) f is said to be g-weakly isotone increasing if for all $x \in X$ we have $fx \preceq gfx \preceq fgfx$ [21].

Note that two weakly increasing mappings need not be nondecreasing. There exist some examples to illustrate this fact in [5].

If $f, g : X \to X$ are weakly increasing, then f is g-weakly isotone increasing. Also, in the above definition (e), if $f = g$, we say that f is is weakly isotone increasing. In this case for each $x \in X$, we have $fx \preceq ffx$.

Definition 1.1. Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $x_n \to x$, $n \to \infty$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $y_n \to y$, $n \to \infty$, then $y_n \succeq y$ for all n.

Recently, Nashine et al. [21] have given an improved version of Theorem 2.2 of Ćirić et al. [9], and proved the following theorem.

Theorem 1.2. *Let $(X, \preceq, d)$ be a complete ordered metric space. Assume that there is a continuous function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$ and that $f, g : X \to X$ are two mappings such that*

$$d(fx, gy) \leq \max\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, gy)), \psi(\frac{d(x, gy) + d(y, fx)}{2})\}$$

holds for all comparable $x, y \in X$. Also suppose that f is g-weakly isotone increasing and one of f or g is continuous. Then f and g have a common fixed point.

The aim of this paper is to present some common fixed point theorems for g-weakly isotone increasing mappings satisfying a generalized contractive type condition under a continuous function ψ in the framework of ordered b-metric spaces. Our results extend and generalize the results of Nashine et al. [21] (resp., [20]) from the context of ordered metric spaces (resp., ordered partial metric spaces) to the setting of ordered b-metric spaces.

For this purpose, we need some preliminaries from the literature on b-metric spaces.

Czerwik introduced in [10] the concept of a b-metric space. Since then, several papers dealt with fixed point theory for single-valued and multivalued operators in b-metric spaces (see, e.g., [12, 14, 16, 17, 18, 23, 25, 27]). Pacurar [23] proved results on sequences of almost contractions and fixed points in b-metric spaces. Recently, Hussain and Shah [14] obtained results on KKM mappings in cone b-metric spaces. Khamsi [17], as well as Jovanović et al. [16], showed that each cone metric space over a normal cone induces a b-metric structure.

Consistent with [10], the following definition will be needed in the sequel.

Definition 1.3. ([10]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. We present an easy example to show that in general a b-metric need not be a metric.

Example 1.4. Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

However, (X, ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is the usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$, but it is not a metric on $\mathbb{R}$.

We also need the following definition.

Definition 1.5. Let X be a nonempty set. Then $(X, \preceq, d)$ is called a partially ordered b-metric space if d is a b-metric on a partially ordered set $(X, \preceq)$.

Recently, N. Hussain et al. [12] have presented an example (with slight error) of a b-metric which is not continuous (see corrected version in [13, Example 1]). Thus, while working in b-metric space, the following lemma is useful.

Lemma 1.6. *([3]) Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z).$$

Lemma 1.7. *Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{1.1}$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences

$$d\left(x_{m(k)}, x_{n(k)}\right), \quad d\left(x_{m(k)}, x_{n(k)+1}\right), \quad d\left(x_{m(k)+1}, x_{n(k)}\right) \quad \text{and} \quad d\left(x_{m(k)+1}, x_{n(k)+1}\right),$$

it holds:

$$\begin{aligned}
\varepsilon &\leq \liminf_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon,\\
\frac{\varepsilon}{s} &\leq \liminf_{k\to\infty} \left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon,\\
\frac{\varepsilon}{s} &\leq \liminf_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon,\\
\frac{\varepsilon}{s^2} &\leq \liminf_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.
\end{aligned}$$

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$n(k) > m(k) > k, \; d\left(x_{m(k)}, x_{n(k)-1}\right) < \varepsilon, \; d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon \tag{1.2}$$

for all positive integers k. Now, from (1.2) and using the triangle inequality we have

$$\varepsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq s\left[d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)\right] < s\varepsilon + sd\left(x_{n(k)-1}, x_{n(k)}\right). \tag{1.3}$$

Taking the upper and lower limits as $k \to \infty$ in (1.3), and using (1.1) we obtain that

$$\varepsilon \leq \liminf_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon. \tag{1.4}$$

Using the triangle inequality again we have

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq s\left[d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right)\right] \\ &\leq s^2\left[d\left(x_{m(k)}, x_{n(k)}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right)\right] + sd\left(x_{n(k)+1}, x_{n(k)}\right). \end{aligned} \tag{1.5}$$

Taking the upper and lower limits as $k \to \infty$ in (1.5) and using (1.1) and (1.4), we have

$$\varepsilon \leq s \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^3\varepsilon,$$

or, equivalently,

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon.$$

The remaining two conditions of the lemma can be proved in a similar way. □

Motivated by the work in [20] and [21], we prove some common fixed point theorems for g-weakly isotone increasing mappings satisfying a generalized contractive type condition in partially ordered b-metric spaces. As applications, we present some results on periodic points of self-mappings, and we prove an existence theorem for solutions of an integral equation.

2. Main results

Let $(X, \preceq, d)$ be an ordered b-metric space with $s > 1$, and $f, g : X \to X$ be two mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_s(x, y) = \max\left\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, gy)), \psi\left(\frac{d(x, gy) + d(y, fx)}{2s}\right)\right\}, \tag{2.1}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function with $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$.

Theorem 2.1. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$, we have*

$$s^4 d(fx, gy) \leq M_s(x, y). \tag{2.2}$$

Then, the pair (f, g) has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = x_1$ and $x_2 \in X$ such that $gx_1 = x_2$. Continuing in this way, construct a sequence $\{x_n\}$ defined by:

$$x_{2n+1} = fx_{2n}, \quad \text{and} \quad x_{2n+2} = gx_{2n+1},$$

for all $n \geq 0$. As f is g-weakly isotone increasing, we have

$$x_1 = fx_0 \preceq gfx_0 = gx_1 = x_2 \preceq fgfx_0 = fx_2 = x_3.$$

Repeating this process, we obtain $x_n \preceq x_{n+1}$, for all $n \geq 1$.

We will complete the proof in three steps.

Step I. We prove that $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$.

Suppose $d(x_{k_0}, x_{k_0+1}) = 0$, for some k_0. Then, $x_{k_0} = x_{k_0+1}$. In the case $k_0 = 2n$, $x_{2n} = x_{2n+1}$ gives $x_{2n+1} = x_{2n+2}$. Indeed,

$$s^4 d(x_{2n+1}, x_{2n+2}) = s^4 d(fx_{2n}, gx_{2n+1}) \leq M_s(x_{2n}, x_{2n+1}), \tag{2.3}$$

where

$$\begin{aligned} M_s(x_{2n}, x_{2n+1}) &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, fx_{2n})), \psi(d(x_{2n+1}, gx_{2n+1})), \\ &\qquad \psi(\frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2s})\} \\ &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2})), \\ &\qquad \psi(\frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2s})\} \\ &= \max\{0, 0, \psi(d(x_{2n+1}, x_{2n+2})), \psi(\frac{d(x_{2n}, x_{2n+2})}{2s})\} \\ &= \max\{\psi(d(x_{2n+1}, x_{2n+2})), \psi(\frac{d(x_{2n+1}, x_{2n+2})}{2s})\}. \end{aligned}$$

If $M_s(x_{2n}, x_{2n+1}) = \psi(d(x_{2n+1}, x_{2n+2}))$, then, from (2.3), we have

$$s^4 d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

a contradiction.

If $M_s(x_{2n}, x_{2n+1}) = \psi(\frac{d(x_{2n+1}, x_{2n+2})}{2s})$, then, from (2.3), we have

$$s^4 d(x_{2n+1}, x_{2n+2}) \leq \psi(\frac{d(x_{2n+1}, x_{2n+2})}{2s}) < \frac{d(x_{2n+1}, x_{2n+2})}{2s},$$

hence, $[2s^5 - 1]d(x_{2n+1}, x_{2n+2}) < 0$, that is, $x_{2n+1} = x_{2n+2}$.

Similarly, if $k_0 = 2n + 1$, then $x_{2n+1} = x_{2n+2}$ gives $x_{2n+2} = x_{2n+3}$. Consequently, the sequence $\{x_k\}$ becomes constant for $k \geq k_0$ and x_{k_0} is a coincidence point of f and g. For this, let $k_0 = 2n$. Then, we know that $x_{2n} = x_{2n+1} = x_{2n+2}$. Hence,

$$x_{2n} = x_{2n+1} = fx_{2n} = x_{2n+2} = gx_{2n+1}.$$

This means that $fx_{2n} = gx_{2n+1}$. Now, since $x_{2n} = x_{2n+1}$, we have $fx_{2n} = gx_{2n}$.

In the other case, when $k_0 = 2n+1$, similarly, one can show that x_{2n+1} is a coincidence point of the pair (f, g).

Suppose now that $d(x_k, x_{k+1}) > 0$ for each k. We claim that the inequality

$$d(x_{k+1}, x_{k+2}) \leq d(x_k, x_{k+1}) \tag{2.4}$$

holds for each $k = 1, 2, \ldots$.

Let $k = 2n$ and for an $n \geq 0$,

$$d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1}) > 0. \tag{2.5}$$

Then, as $x_{2n} \preceq x_{2n+1}$, using (2.2) we obtain that

$$s^4 d(x_{2n+1}, x_{2n+2}) = s^4 d(fx_{2n}, gx_{2n+1}) \leq M_s(x_{2n}, x_{2n+1}), \tag{2.6}$$

where

$$\begin{aligned} M_s(x_{2n}, x_{2n+1}) &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, fx_{2n})), \psi(d(x_{2n+1}, gx_{2n+1})), \\ &\quad \psi(\frac{d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})}{2s})\} \\ &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2})), \psi(\frac{d(x_{2n}, x_{2n+2})}{2s})\}. \end{aligned} \tag{2.7}$$

If $M_s(x_{2n}, x_{2n+1}) = \psi(d(x_{2n+1}, x_{2n+2}))$, then from (2.6),

$$s^4 d(x_{2n+1}, x_{2n+2}) \le \psi(d(x_{2n+1}, x_{2n+2})) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction.

If $M_s(x_{2n}, x_{2n+1}) = \psi(\frac{d(x_{2n}, x_{2n+2})}{2s})$, then from (2.6),

$$s^4 d(x_{2n+1}, x_{2n+2}) \le \psi(\frac{d(x_{2n}, x_{2n+2})}{2s}) < \frac{d(x_{2n}, x_{2n+2})}{2s} \le \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2} < d(x_{2n+1}, x_{2n+2}),$$

which is also a contradiction.

If $M_s(x_{2n}, x_{2n+1}) = \psi(d(x_{2n}, x_{2n+1}))$, then from (2.6),

$$s^4 d(x_{2n+1}, x_{2n+2}) \le \psi(d(x_{2n}, x_{2n+1})) < d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction.

Hence, (2.5) is false, that is, $d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1})$ holds for all n. Therefore, (2.4) is proved for $k = 2n$.

Similarly, it can be shown that

$$d(x_{2n+2}, x_{2n+3}) \le d(x_{2n+1}, x_{2n+2}).$$

Hence, $\{d(x_k, x_{k+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an $r \ge 0$ such that

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = r.$$

Assume that $r > 0$. From (2.7), we have

$$\begin{aligned} M_s(x_{2n}, x_{2n+1}) &< \max\{\psi\left(d(x_{2n}, x_{2n+1})\right), \psi\left(d(x_{2n+1}, x_{2n+2})\right), \psi(\frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2})\} \\ &< \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\}. \end{aligned} \tag{2.8}$$

Now, taking the upper limit as $n \to \infty$ in (2.8), we obtain

$$\limsup_{n\to\infty} M_s(x_{2n}, x_{2n+1}) \le r. \tag{2.9}$$

Taking the upper limit as $n \to \infty$ in (2.6), and using (2.9), we have $s^4 r \le r$. Therefore $(s^4 - 1)r \le 0$, a contradiction. Hence,

$$r = \lim_{k\to\infty} d(x_k, x_{k+1}) = 0.$$

Step II. We now show that $\{x_n\}$ is a b-Cauchy sequence in X. That is, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n \ge k$, $d(x_m, x_n) < \varepsilon$.

We assume to the contrary, that $\{x_n\}$ is not a b-Cauchy sequence. Then from Lemma 1.7, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \ge k$ and:

(a) $m(k) = 2t$ and $n(k) = 2t' + 1$, where t and t' are nonnegative integers,

(b) $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, and
(c) $n(k)$ is the smallest number such that the condition (b) holds; i.e., $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$.
Then we have

$$\varepsilon \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon, \tag{2.10}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon, \tag{2.11}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon, \tag{2.12}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon. \tag{2.13}$$

Since $n(k) > m(k)$, we have $x_{m(k)} \preceq x_{n(k)}$, so from (2.2), we have

$$s^4 d(x_{m(k)+1}, x_{n(k)+1}) = s^4 d(fx_{m(k)}, gx_{n(k)}) \leq M_s(x_{m(k)}, x_{n(k)}), \tag{2.14}$$

where

$$\begin{aligned} M_s(x_{m(k)}, x_{n(k)}) &= \max\{\psi(d(x_{m(k)}, x_{n(k)})), \psi(d(x_{m(k)}, fx_{m(k)})), \psi(d(x_{n(k)}, gx_{n(k)})), \\ &\quad \psi(\frac{d(x_{m(k)}, gx_{n(k)}) + d(x_{n(k)}, fx_{m(k)})}{2s})\} \\ &= \max\{\psi(d(x_{m(k)}, x_{n(k)})), \psi(d(x_{m(k)}, x_{m(k)+1})), \psi(d(x_{n(k)}, x_{n(k)+1})), \\ &\quad \psi(\frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s})\} \\ &< \max\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\quad \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\}. \end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.2), (2.10), (2.11), and (2.12), we have

$$\begin{aligned} \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) &\leq \max\{\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}), \limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}), \limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1}), \\ &\quad \limsup_{k\to\infty} \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\} \\ &\leq \max\{s\varepsilon, 0, 0, \frac{s^2\varepsilon + s^2\varepsilon}{2s}\} = s\varepsilon. \end{aligned}$$

Hence, by taking the upper limit as $k \to \infty$ in (2.14), and using (2.13) we have

$$s^4 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq s\varepsilon,$$

which implies that $\limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \frac{\varepsilon}{s^3} < \frac{\varepsilon}{s^2}$, a contradiction to (2.13). Hence $\{x_n\}$ is a b-Cauchy sequence.

Step III. We will show that f and g have a common fixed point.

Since $\{x_n\}$ is a b-Cauchy sequence in the complete b-metric space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_{2n}, z) = \lim_{n\to\infty} d(x_{2n+1}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = 0. \tag{2.15}$$

By the triangle inequality, we have

$$d(fz, z) \leq s[d(fz, fx_{2n}) + d(fx_{2n}, z)] = s[d(fz, fx_{2n}) + d(x_{2n+1}, z)]. \tag{2.16}$$

Suppose that f is continuous. Letting $n \to \infty$ in (2.16) and applying (2.15) we have

$$d(fz, z) \leq s[\lim_{n\to\infty} d(fz, fx_{2n}) + \lim_{n\to\infty} d(fx_{2n}, z)] = 0,$$

which implies that $fz = z$.

Let $d(z, gz) > 0$. As z and z are comparable, by (2.2) we have

$$s^4 d(z, gz) = s^4 d(fz, gz) \leq M_s(z, z), \tag{2.17}$$

where

$$M_s(z, z) = \max\{\psi(d(z, z)), \psi(d(z, fz)), \psi(d(z, gz)), \psi(\frac{d(z, gz) + d(z, fz)}{2s})\} < d(z, gz).$$

Hence, (2.17) gives $s^4 d(z, gz) < d(z, gz)$, which is a contradiction. Thus, $d(z, gz) = 0$.

Similarly, if g is continuous, the desired result is obtained. □

Remark 2.2. In the case when ψ is a nondecreasing function, the contractive condition (2.2) is equivalent to the condition

$$s^4 d(fx, gy) \leq \psi(M'_s(x, y)),$$

where

$$M'_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\right\}.$$

However, if ψ is not monotone, the condition (2.2) is weaker (see [20, Example 3.8]).

In the following theorem, we omit the assumption of continuity of f or g.

Theorem 2.3. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$, we have*

$$s^4 d(fx, gy) \leq M_s(x, y).$$

Then the pair (f, g) has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Following the proof of Theorem 2.1, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Now we prove that z is a common fixed point of f and g.

Since $x_{2n+1} \to z$, as $n \to \infty$, from regularity of X, $x_{2n+1} \preceq z$. Therefore, from (2.2), we have

$$s^4 d(fz, gx_{2n+1}) \leq M_s(z, x_{2n+1}), \tag{2.18}$$

where

$$M_s(z, x_{2n+1}) = \max\{\psi(d(z, x_{2n+1})), \psi(d(z, fz)), \psi(d(x_{2n+1}, gx_{2n+1})), \psi(\frac{d(z, gx_{2n+1}) + d(x_{2n+1}, fz)}{2s})\}.$$

Taking the limit as $n \to \infty$ in (2.18) and using Lemma 1.6, we obtain that

$$\begin{aligned} s^3 d(fz, z) = s^4 \frac{1}{s} d(fz, z) &\leq s^4 \limsup_{n\to\infty} d(fz, gx_{2n+1}) \leq \limsup_{n\to\infty} M_s(z, x_{2n+1}) \\ &= \max\{\limsup_{n\to\infty} d(z, x_{2n+1}), \limsup_{n\to\infty} d(z, fz), \limsup_{n\to\infty} d(x_{2n+1}, gx_{2n+1}), \\ &\quad \limsup_{n\to\infty} \frac{d(z, gx_{2n+1}) + d(x_{2n+1}, fz)}{2s}\} \\ &\leq \max\{0, d(z, fz), 0, \frac{d(z, fz)}{2}\} = d(z, fz), \end{aligned}$$

which implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. □

Taking $f = g$ in Theorems 2.1 and 2.3, we obtain the following fixed point result:

Corollary 2.4. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space. Let $f : X \to X$ be a mapping such that f is weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$ we have*

$$s^4 d(fx, fy) \leq M_s(x, y), \tag{2.19}$$

where

$$M_s(x, y) = \max\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, fy)), \psi(\frac{d(x, fy) + d(y, fx)}{2s})\}.$$

Then f has a fixed point z in X if either:
(a) f is continuous, or
(b) X is regular.
Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

We present now results for so-called quasicontractions.

Theorem 2.5. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space with $s > 1$. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$, we have*

$$s^4 d(fx, gy) \leq N(x, y), \tag{2.20}$$

where

$$N(x, y) = \max\{\psi(d(x, y)), \psi(d(x, fx)), \psi(d(y, gy)), \psi(d(x, gy)), \psi(d(y, fx))\},$$

and $\psi : [0, \infty) \to [0, \infty)$ is a continuous function with $\psi(t) < \frac{t}{2s}$ for each $t > 0$ and $\psi(0) = 0$. Then, the pair (f, g) has a common fixed point z in X if one of f or g is continuous. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Define the sequence $\{x_n\}$ as given in the proof of Theorem 2.1. We will complete the proof in three steps.

Step I. We prove that $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$.

Suppose $d(x_{k_0}, x_{k_0+1}) = 0$, for some k_0. Then, $x_{k_0} = x_{k_0+1}$. In the case $k_0 = 2n$, $x_{2n} = x_{2n+1}$ gives $x_{2n+1} = x_{2n+2}$. If $d(x_{2n+1}, x_{2n+2}) > 0$, then from (2.20) we have

$$s^4 d(x_{2n+1}, x_{2n+2}) = s^4 d(fx_{2n}, gx_{2n+1}) \leq N(x_{2n}, x_{2n+1}), \tag{2.21}$$

where

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, fx_{2n})), \psi(d(x_{2n+1}, gx_{2n+1})), \\ &\quad \psi(d(x_{2n}, gx_{2n+1})), \psi(d(x_{2n+1}, fx_{2n}))\} \\ &= \max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2})), \\ &\quad \psi(d(x_{2n}, x_{2n+2})), \psi(d(x_{2n+1}, x_{2n+1}))\} \\ &= \max\{0, 0, \psi(d(x_{2n+1}, x_{2n+2})), \psi(d(x_{2n}, x_{2n+2}))\} \\ &= \max\{\psi(d(x_{2n+1}, x_{2n+2})), \psi(d(x_{2n+1}, x_{2n+2}))\} \\ &= \psi(d(x_{2n+1}, x_{2n+2})). \end{aligned}$$

Then, from (2.21), we have

$$s^4 d(x_{2n+1}, x_{2n+2}) \leq \psi(d(x_{2n+1}, x_{2n+2})) < \frac{1}{2s} d(x_{2n+1}, x_{2n+2}).$$

Hence, $[2s^5 - 1]d(x_{2n+1}, x_{2n+2}) < 0$, which is a contradiction, so $x_{2n+1} = x_{2n+2}$.

Similarly, if $k_0 = 2n+1$, then $x_{2n+1} = x_{2n+2}$ gives $x_{2n+2} = x_{2n+3}$. Consequently, the sequence $\{x_k\}$ becomes constant for $k \ge k_0$ and x_{k_0} is a coincidence point of f and g. For this, let $k_0 = 2n$. Then, we know that $x_{2n} = x_{2n+1} = x_{2n+2}$. Hence,

$$x_{2n} = x_{2n+1} = fx_{2n} = x_{2n+2} = gx_{2n+1}.$$

This means that $fx_{2n} = gx_{2n+1}$. Now, since $x_{2n} = x_{2n+1}$, we have $fx_{2n} = gx_{2n}$.

In the other case, when $k_0 = 2n+1$, similarly, one can show that x_{2n+1} is a coincidence point of the pair (f, g).

Suppose that $d(x_k, x_{k+1}) > 0$, for each k. We now claim that the inequality

$$d(x_{k+1}, x_{k+2}) \le d(x_k, x_{k+1}) \tag{2.22}$$

holds for each $k = 1, 2, \dots$.

Let $k = 2n$ and, for an $n \ge 0$,

$$d(x_{2n+1}, x_{2n+2}) > d(x_{2n}, x_{2n+1}) > 0. \tag{2.23}$$

Then, as $x_{2n} \preceq x_{2n+1}$, using (2.20) we obtain that

$$s^4 d(x_{2n+1}, x_{2n+2}) = s^4 d(fx_{2n}, gx_{2n+1}) \le N(x_{2n}, x_{2n+1}), \tag{2.24}$$

where

$$\begin{aligned} N(x_{2n}, x_{2n+1}) = \max\{&\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n}, fx_{2n})), \psi(d(x_{2n+1}, gx_{2n+1})), \\ &\psi(d(x_{2n}, gx_{2n+1})), \psi(d(x_{2n+1}, fx_{2n}))\} \\ = \max\{&\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2})), \psi(d(x_{2n}, x_{2n+2}))\}. \end{aligned} \tag{2.25}$$

If $N(x_{2n}, x_{2n+1}) = \psi(d(x_{2n+1}, x_{2n+2}))$, then from (2.24),

$$s^4 d(x_{2n+1}, x_{2n+2}) \le \psi(d(x_{2n+1}, x_{2n+2})) < \frac{1}{2s} d(x_{2n+1}, x_{2n+2}),$$

which is a contradiction.

If $N(x_{2n}, x_{2n+1}) = \psi(d(x_{2n}, x_{2n+2}))$, then from (2.24),

$$\begin{aligned} s^4 d(x_{2n+1}, x_{2n+2}) &\le \psi(d(x_{2n}, x_{2n+2})) < \frac{1}{2s} d(x_{2n}, x_{2n+2}) \\ &\le \frac{1}{2s} s[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] < d(x_{2n+1}, x_{2n+2}), \end{aligned}$$

which is a contradiction.

If $N(x_{2n}, x_{2n+1}) = \psi(d(x_{2n}, x_{2n+1}))$, then from (2.24),

$$s^4 d(x_{2n+1}, x_{2n+2}) \le \psi(d(x_{2n}, x_{2n+1})) < \frac{1}{2s} d(x_{2n}, x_{2n+1}) < \frac{1}{2s} d(x_{2n+1}, x_{2n+2}),$$

a contradiction.

Hence, (2.23) is false, that is, $d(x_{2n+1}, x_{2n+2}) \le d(x_{2n}, x_{2n+1})$, for all n. Therefore, (2.22) is proved for $k = 2n$.

Similarly, we have

$$d(x_{2n+2}, x_{2n+3}) \le d(x_{2n+1}, x_{2n+2}),$$

for all n. Hence $\{d(x_k, x_{k+1})\}$ is a nondecreasing sequence of nonnegative real numbers. Therefore, there is an $r \ge 0$ such that

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = r.$$

Assume that $r > 0$. From (2.25), we have

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &< \frac{1}{2s} \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2})\} \\ &\leq \frac{1}{2s} \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), sd(x_{2n}, x_{2n+1}) + sd(x_{2n+1}, x_{2n+2})\}. \end{aligned} \tag{2.26}$$

Now, taking the upper limit as $n \to \infty$ in (2.26), we obtain

$$\limsup_{n\to\infty} N(x_{2n}, x_{2n+1}) \leq \frac{1}{2s} \max\{r, 2sr\} = r. \tag{2.27}$$

Taking the upper limit as $n \to \infty$ in (2.24), and using(2.27), we have $s^4 r \leq r$. Therefore $(s^4 - 1)r \leq 0$, a contradiction with $s > 1$. Hence,

$$r = \lim_{k\to\infty} d(x_k, x_{k+1}) = 0. \tag{2.28}$$

Step II. We now show that $\{x_n\}$ is a b-Cauchy sequence in X. That is, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $m, n \geq k$, $d(x_m, x_n) < \varepsilon$.

We assume to the contrary, that $\{x_n\}$ is not a b-Cauchy sequence. Then from Lemma 1.7 there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) \geq k$ and:

(a) $m(k) = 2t$ and $n(k) = 2t' + 1$, where t and t' are nonnegative integers,

(b) $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, and

(c) $n(k)$ is the smallest number such that the condition (b) holds; i.e., $d(x_{m(k)}, x_{n(k)-1}) < \varepsilon$.

We have the following relations:

$$\varepsilon \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq s\varepsilon, \tag{2.29}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\varepsilon, \tag{2.30}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\varepsilon, \tag{2.31}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq s^3\varepsilon. \tag{2.32}$$

Since $n(k) > m(k)$, we have $x_{m(k)} \preceq x_{n(k)}$, so from (2.20), we have

$$s^4 d(x_{m(k)+1}, x_{n(k)+1}) = s^4 d(fx_{m(k)}, gx_{n(k)})) \leq N(x_{m(k)}, x_{n(k)}), \tag{2.33}$$

where

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}) &= \max\{\psi(d(x_{m(k)}, x_{n(k)})), \psi(d(x_{m(k)}, fx_{m(k)})), \psi(d(x_{n(k)}, gx_{n(k)})), \\ &\quad \psi(d(x_{m(k)}, gx_{n(k)})), \psi(d(x_{n(k)}, fx_{m(k)}))\} \\ &= \max\{\psi(d(x_{m(k)}, x_{n(k)})), \psi(d(x_{m(k)}, x_{m(k)+1})), \psi(d(x_{n(k)}, x_{n(k)+1})), \\ &\quad \psi(d(x_{m(k)}, x_{n(k)+1})), \psi(d(x_{n(k)}, x_{m(k)+1}))\} \\ &< \frac{1}{2s} \max\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\quad d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})\}. \end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (2.28), (2.29), (2.30), and (2.31), we have

$$\begin{aligned} \limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)}) &\leq \frac{1}{2s} \max\{\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}), \limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}), \limsup_{k\to\infty} d(x_{n(k)}, x_{n(k)+1}), \\ &\quad \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}), \limsup_{k\to\infty} d(x_{n(k)}, x_{m(k)+1})\} \\ &\leq \frac{1}{2s} \max\{s\varepsilon, 0, 0, s^2\varepsilon, s^2\varepsilon\} = \frac{s\varepsilon}{2}. \end{aligned}$$

Taking the upper limit as $k \to \infty$ in (2.33) and using (2.32), we have

$$s^2\varepsilon = s^4\frac{\varepsilon}{s^2} \le s^4 \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \le \limsup_{k\to\infty} N(x_{m(k)}, x_{n(k)}) \le \frac{s\varepsilon}{2},$$

which implies that $s \le \frac{1}{2}$, a contradiction to $s > 1$. Hence $\{x_n\}$ is a b-Cauchy sequence.

Step III. We will show that f and g have a common fixed point.

Since $\{x_n\}$ is a b-Cauchy sequence in the complete b-metric space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_{2n}, z) = \lim_{n\to\infty} d(x_{2n+1}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = 0, \tag{2.34}$$

By the triangle inequality, we have

$$d(fz, z) \le s[d(fz, fx_{2n}) + d(fx_{2n}, z)] = s[d(fz, fx_{2n}) + d(x_{2n+1}, z)]. \tag{2.35}$$

Suppose that f is continuous. Letting $n \to \infty$ in (2.35) and applying (2.34) we have

$$d(fz, z) \le s[\lim_{n\to\infty} d(fz, fx_{2n}) + \lim_{n\to\infty} d(fx_{2n}, z)] = 0,$$

which implies that $fz = z$.

Let $d(z, gz) > 0$. As z and z are comparable, by (2.20) we have

$$s^4 d(z, gz) = s^4 d(fz, gz) \le N(z, z), \tag{2.36}$$

where

$$N(z, z) = \max\{\psi(d(z, z)), \psi(d(z, fz)), \psi(d(z, gz)), \psi(d(z, gz)), \psi(d(z, fz))\} < d(z, gz).$$

Hence, (2.36) gives $s^4 d(z, gz) < d(z, gz)$, which is a contradiction. Hence, $d(z, gz) = 0$.

Similarly, if g is continuous, the desired result is obtained. □

In the following theorem, we omit the assumption of continuity of f or g.

Theorem 2.6. *Let $(X, \preceq, d)$ be a complete partially ordered b-metric space. Let $f, g : X \to X$ be two mappings such that f is g-weakly isotone increasing. Suppose that for every two comparable elements $x, y \in X$, we have*

$$s^4 d(fx, gy) \le N(x, y).$$

Then, the pair (f, g) has a common fixed point z in X if X is regular. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Proof. Following the proof of Theorem 2.5, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Now we prove that z is a common fixed point of f and g.

Since $x_{2n+1} \to z$, as $n \to \infty$, from regularity of X, $x_{2n+1} \preceq z$. Therefore, from (2.20), we have

$$s^4 d(fz, gx_{2n+1}) \le N(z, x_{2n+1}), \tag{2.37}$$

where

$$N(z, x_{2n+1}) = \max\{\psi(d(z, x_{2n+1})), \psi(d(z, fz)), \psi(d(x_{2n+1}, gx_{2n+1})), \psi(d(z, gx_{2n+1})), \psi(d(x_{2n+1}, fz))\}.$$

Taking the limit as $n \to \infty$ in (2.37) and using Lemma 1.6, we obtain that

$$\begin{aligned} s^3 d(fz,z) = s^4 \frac{1}{s} d(fz,z) &\le s^4 \limsup_{n\to\infty} d(fz, gx_{2n+1}) \le \limsup_{n\to\infty} N(z, x_{2n+1}) \\ &< \frac{1}{2s} \max\{\limsup_{n\to\infty} d(z, x_{2n+1}), \limsup_{n\to\infty} d(z, fz), \limsup_{n\to\infty} d(x_{2n+1}, gx_{2n+1}), \\ &\quad \limsup_{n\to\infty} d(z, gx_{2n+1}), \limsup_{n\to\infty} d(x_{2n+1}, fz)\} \\ &\le \frac{1}{2s} \max\{0, d(z,fz), 0, 0, d(z,fz)\} = \frac{1}{2s} d(z,fz), \end{aligned}$$

which implies that $fz = z$.

Similarly, it can be shown that z is a fixed point of g. □

We illustrate our results by the following examples.

Example 2.7. Let $X = \{0,1,2\}$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,x) = 0$ for $x \in X$, $d(0,1) = d(1,2) = 1$, $d(0,2) = \frac{9}{4}$, $d(x,y) = d(y,x)$ for $x, y \in X$. Then, by [26], (X,d) is a b-metric space (with $s = \frac{9}{8} > 1$) which is not a metric space. Define an order on X by $\preceq = \{(0,0),(1,1),(2,2),(1,0),(2,0)\}$ and obtain a complete ordered b-metric space.

Consider the mapping $f : X \to X$ given by

$$f : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $f^2 : \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and it follows that $fx \preceq f^2x$, i.e., f is weakly isotone increasing and, obviously, continuous. Finally, take $\psi(t) = kt$ for any k satisfying $\frac{729}{1024} < k < 1$. The contractive condition (2.19) have to be checked only for $x = 0$, $y = 1$ and for $x = 0$, $y = 2$. In the first case, $fx = fy = 0$ and (2.19) is trivially satisfied. In the second case, we get that

$$\begin{aligned} s^4 d(f0, f2) = \left(\frac{9}{8}\right)^4 d(0,1) = \frac{9^4}{4^6} = \frac{9^3}{4^5} \cdot \frac{9}{4} &< k \cdot d(0,2) \\ &= k \cdot \max\{d(0,2), d(0,f0), d(2,f2), \frac{4}{9}(d(0,f2) + d(2,f0))\} = M_s(0,2), \end{aligned}$$

and all the conditions of Corollary 2.4 are satisfied. The mapping f has a unique fixed point $z = 0$.

Note that if we considered the same example without order, then we would have also to consider the case $x = 1$, $y = 2$. However, then the contractive condition would not hold, since it would reduce to $9^4/4^6 < k$ which is not true since $k < 1$.

Motivated by Example 2 in [25], we present the following example.

Example 2.8. Let $X = [0,\infty)$ be equipped with the b-metric $d(x,y) = |x-y|^2$, $x, y \in X$ where $s = 2^{2-1} = 2$ according to Example 1.4, and define a relation $\preceq$ on X by $x \preceq y$ iff $y \le x$, where $\le$ is the usual ordering on $\mathbb{R}$. Define functions $f, g : X \to X$ by

$$fx = \ln\left(1 + \frac{x}{9}\right) \quad \text{and} \quad gx = \ln\left(1 + \frac{x}{7}\right).$$

Define $\psi : [0,+\infty) \to [0,+\infty)$ by $\psi(t) = kt$ with $\frac{4}{9} \le k < 1$. Then we have the following:

(1) $(X, \preceq, d)$ is a complete partially ordered b-metric space.

(2) f is g-weakly isotone increasing with respect to $\preceq$.

(3) f and g are continuous.

(4) for every two comparable elements $x, y \in X$ the inequality (2.2) holds, where $M_s(x,y)$ is given by (2.1).

Proof. The proof of (1) is clear. To prove (2), for each $x \in X$, we have $1+\frac{x}{9} \leq e^{\frac{x}{9}}$ and $1+\frac{x}{7} \leq e^{\frac{x}{7}}$. Hence, $fx = \ln(1+\frac{x}{9}) \leq x$, $gx = \ln(1+\frac{x}{7}) \leq x$. Thus for each $x \in X$ we have $gfx = \ln(1+\frac{fx}{7}) \leq fx$ and $fgfx = \ln(1+\frac{gfx}{9}) \leq gfx$, i.e., $fx \preceq gfx \preceq fgfx$. Thus, f is g-weakly isotone increasing with respect to $\preceq$. It is easy to see that f and g are continuous. To prove (4), consider $x, y \in X$ with $x \preceq y$, i.e., $y \leq x$. We have the following cases:

Case 1: If $\frac{y}{7} \leq \frac{x}{9}$ then we have

$$1 \leq \frac{1+\frac{x}{9}}{1+\frac{y}{7}} \leq \frac{1+\frac{x}{7}}{1+\frac{y}{7}} \implies 0 \leq \ln\left(\frac{1+\frac{x}{9}}{1+\frac{y}{7}}\right) \leq \ln\left(\frac{1+\frac{x}{7}}{1+\frac{y}{7}}\right).$$

Now, by using the mean value theorem for the function $\ln(1+t)$ on $t \in \left[\frac{y}{7}, \frac{x}{7}\right]$ we have

$$\begin{aligned} s^4 d(fx, gy) &= 16\left(\ln(1+\frac{x}{9}) - \ln(1+\frac{y}{7})\right)^2 = 16\left(\ln\left(\frac{1+\frac{x}{9}}{1+\frac{y}{7}}\right)\right)^2 \leq 16\left(\ln\left(\frac{1+\frac{x}{7}}{1+\frac{y}{7}}\right)\right)^2 \\ &= 16\left(\ln(1+\frac{x}{7}) - \ln(1+\frac{y}{7})\right)^2 \leq 16\left(\frac{x}{7}-\frac{y}{7}\right)^2 \leq kd(x,y) = \psi\left(d(x,y)\right) \leq M_2(x,y). \end{aligned}$$

So we have $s^4 d(fx, fy) \leq M_s(x,y))$.

Case 2: If $\frac{x}{9} < \frac{y}{7}$ then we have

$$0 < \frac{y}{7} - \frac{x}{9} \leq \frac{y}{7} \implies \left(\frac{y}{7}-\frac{x}{9}\right)^2 \leq \frac{y^2}{49}.$$

By using the mean value theorem for the function $\ln(1+t)$ on $t \in [\frac{x}{9}, \frac{y}{7}]$ we have

$$\begin{aligned} s^4 d(fx, gy)) &= 16\left(\ln(1+\frac{x}{9}) - \ln(1+\frac{y}{7})\right)^2 \leq 16\left(\frac{y}{7}-\frac{x}{9}\right)^2 \leq \frac{16}{49}y^2 \\ &= \frac{4}{9}\left(\frac{6y}{7}\right)^2 \leq k\left(\frac{6y}{7}\right)^2 \leq k\left(y - \ln(1+\frac{y}{7})\right)^2 = \psi\left(d(y, gy)\right) \leq M_2(x,y). \end{aligned}$$

Hence, again $s^4 d(fx, fy) \leq M_s(x,y))$ holds.

By combining all cases together, we conclude that f, g and ψ satisfy all the hypotheses of Theorem 2.1 and hence f and g have a common fixed point. Indeed, 0 is the unique common fixed point of f and g. □

3. Periodic point results

Let X be a nonempty set and denote by $F(f) = \{x \in X : fx = x\}$ the fixed point set of a mapping $f : X \to X$. Clearly, $F(f) \subset F(f^n)$ for every $n \in \mathbb{N}$, but the converse does not hold. If $F(f) = F(f^n)$ for every $n \in \mathbb{N}$, then f is said to have property P. For more details, we refer the reader to [1, 15] and the references mentioned therein.

Theorem 3.1. *Let X and f be as in Corollary 2.4. Then f has the property P.*

Proof. From Corollary 2.4, $F(f) \neq \emptyset$. Let $u \in F(f^n)$ for some $n > 1$. We will show that $u = fu$.

Assume to the contrary, that $u \neq fu$, i.e., $d(u, fu) > 0$. We have $f^{n-1}u \preceq f^n u$, as f is weakly isotone increasing. Using (2.19), we obtain that

$$d(fu, u) = d(f^{n+1}u, f^n u) = d(ff^n u, ff^{n-1}u) \leq \frac{1}{s^4} M_s(f^n u, f^{n-1}u),$$

where

$$\begin{aligned} M_s(f^n u, f^{n-1}u) &= \max\{d(f^n u, f^{n-1}u), d(f^n u, f^{n+1}u), d(f^{n-1}u, f^n u), \frac{d(f^n u, f^n u) + d(f^{n-1}u, f^{n+1}u)}{2s}\} \\ &\leq \max\{d(f^n u, f^{n-1}u), d(f^n u, f^{n+1}u), \frac{d(f^{n-1}u, f^n u) + d(f^n u, f^{n+1}u)}{2}\} \\ &= \max\{d(f^n u, f^{n-1}u), d(f^n u, f^{n+1}u)\}. \end{aligned}$$

If $M_s(f^n u, f^{n-1}u) = d(f^n u, f^{n+1}u)$, then from (3.1), we have

$$d(u, fu) = d(f^n u, f^{n+1}u) \leq \frac{1}{s^4} d(f^n u, f^{n+1}u),$$

which is a contradiction since $s > 1$. Hence, we have

$$d(fu, u) = d(f^{n+1}u, f^n u) \leq \frac{1}{s^4} d(f^n u, f^{n-1}u).$$

Starting from $d(f^{n-1}u, f^n u)$, and repeating the above process, we get

$$\begin{aligned} d(u, fu) = d(f^{n+1}u, f^n u) &\leq \frac{1}{s^4} d(f^n u, f^{n-1}u) \leq [\frac{1}{s^4}]^2 [d(f^{n-2}u, f^{n-1}u) \\ &\dots \\ &\leq [\frac{1}{s^4}]^n d(u, fu) < d(u, fu), \end{aligned}$$

which is again a contradiction. Thus, $u = fu$. □

4. Existence theorem for a solution of an integral equation

Consider the integral equation

$$x(t) = \int_0^T K(t, r, x(r))\, dr + g(t), \qquad t \in [0, T], \tag{4.1}$$

where $T > 0$. The purpose of this section is to give an existence theorem for a solution of (4.1) that belongs to $X = C(I, \mathbb{R})$ (the set of continuous real functions defined on $I = [0, T]$), by using the obtained result in Corollary 2.4. Obviously, this space with the b-metric given by

$$d(x, y) = \max_{t \in I} |x(t) - y(t)|^p,$$

for all $x, y \in X$ is a complete b-metric space with $s = 2^{p-1}$ and $p \geq 1$ (see Example 1.4).

We endow X with the partial order $\preceq$ given by

$$x \preceq y \iff x(t) \leq y(t), \qquad \text{for all } t \in [0, T].$$

It was proved in [22] that $(X, \preceq, d)$ is regular (the proof is valid also in the b-metric case).

We suppose that $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous. Now, we define $F : X \to X$

$$Fx(t) = \int_0^T K(t, r, x(r))\, dr + g(t), \qquad t \in I$$

for all $x \in X$. Then, a solution of (4.1) is a fixed point of T. We will prove the following result.

Theorem 4.1. *Suppose that the following hypotheses hold:*

(i) for all $t, r \in I$ *and* $u \in X$*, we have*

$$K(t,r,u(t)) \leq K(t,r,\int_0^T K(r,\tau,u(\tau))\,d\tau + g(r));$$

(ii) there exist a continuous function $\xi : I \times I \to [0,\infty)$ *and a non-decreasing continuous function* $\psi : [0,\infty) \to [0,\infty)$ *with* $\psi(l) < l$ *for each* $l > 0$ *and* $\psi(0) = 0$ *such that*

$$|K(t,r,x(r)) - K(t,r,y(r))|^p \leq \xi(t,r)\psi\left(|x(r)-y(r)|^p\right),$$

for all $t, r \in I$ *and* $x, y \in X$ *with* $x \preceq y$*;*

(iii) $\max_{t\in I}(\int_0^T \xi(t,r)\,dr) \leq \frac{1}{2^{4p-4}T^{p-1}}$.

Then, the integral equation (4.1)has a solution $u^* \in X$.

Proof. From (i), for all $t \in I$, we have

$$\begin{aligned} Fx(t) &= \int_0^T K(t,r,x(r))\,dr + g(t) \leq \int_0^T K(t,r,\int_0^T K(r,\tau,x(\tau))\,d\tau + g(r))\,dr + g(t) \\ &= \int_0^T K(t,r,Fx(r))dr + g(t) = F(Fx)(t). \end{aligned}$$

Hence, we have $Fx \preceq F(Fx)$ for all $x \in X$.

Now let $1 \leq q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $x, y \in X$ such that $y \preceq x$, by (ii) and (iii), we have

$$\begin{aligned} 2^{4p-4}|Fx(t) - Fy(t)|^p &\leq 2^{4p-4}(\int_0^T |K(t,r,x(r)) - K(t,r,y(r))|\,dr)^p \\ &\leq 2^{4p-4}[(\int_0^T 1^q\,dr)^{\frac{1}{q}}(\int_0^T |K(t,r,x(r)) - K(t,r,y(r))|^p\,dr)^{\frac{1}{p}}]^p \\ &= 2^{4p-4}T^{\frac{p}{q}}(\int_0^T |K(t,r,x(r)) - K(t,r,y(r))|^p\,dr) \\ &\leq 2^{4p-4}T^{\frac{p}{q}}(\int_0^T \xi(t,r)\psi(|x(r)-y(r)|^p)\,dr) \\ &\leq 2^{4p-4}T^{\frac{p}{q}}(\int_0^T \xi(t,r)\psi(d(x,y))\,dr) \\ &\leq 2^{4p-4}T^{p-1}(\int_0^T \xi(t,r)\,dr)\psi(M_s(x,y)) \\ &\leq \psi(M_s(x,y)). \end{aligned}$$

Thus, from Corollary 2.4, the integral equation (4.1) has a solution $u^* \in X$. □

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