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Best proximity point iteration for nonexpensive mapping in Banach spaces

Mohammad Reza Haddadi[a,*]

[a] *Faculty of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran.*

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Abstract

In this paper we prove existence theorems of best proximity points in Banach spaces. Also an iterative approximation of the best proximity point of a nonexpensive mapping in Banach space is developed.

Keywords: Best proximity pair, best proximity point, cyclic contractive map, P-property.
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1. Introduction and Preliminaries

Let X be a metric space and A and B be nonempty subsets of X. Put

$$\begin{aligned} A_0 &= \{x \in A : d(x, y) = dist(A, B) \ for \ some \ y \in B\}, \\ B_0 &= \{x \in B : d(x, y) = dist(A, B) \ for \ some \ y \in A\}. \end{aligned}$$

If there is a pair $(x_0, y_0) \in A \times B$ for which $d(x_0, y_0) = dist(A, B)$,that $dist(A, B)$ is distance of A and B, then the pair (x_0, y_0) is called a best proximity pair for A and B. Best proximity pair evolves as a generalization of the concept of best approximation.

We can find the best proximity points of the set A, by considering a map $T : A \to B$. We say that the point $x \in A$ is a best proximity point of the pair (A, B), if $d(x, Tx) = dist(A, B)$ and we denote the set of all best proximity points of A by $P_T(A)$, that is

$$P_T(A) := \{x \in A : d(x, Tx) = dist(A, B)\}.$$

*Corresponding author
Email address: haddadi@abru.ac.ir (Mohammad Reza Haddadi)

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A huge number of generalizations of this principle appear in the literature. Fixed point theory is one of the most fruitful and effctive tools in mathematics which has many applications with in as well as out side mathematics [1], [8]. The best proximity point theorem for contractive mappings has been detailed in Sadiq Basha [9]. Anthony Eldred et al. [3] have elicited a best proximity point theorem for relatively nonexpansive mappings, an alternative treatment to which has been focused in Sankar Raj and Veeramani [11]. A best proximity point theorem for contraction has been obtained in Sadiq Basha [10]. Raj V. Sankar in [12] have discussed best proximity point theorems for contractive non-self-mappings and Abkar and Gabeleh continued it in [2], [5]. Best proximity point theorems for various variants of contractions have been explored Eldred and Veeramani [4], Haddadi and Moshtaghioun [6]. P-property introduced for studied the existence of best proximity point for weakly contractive mapping, for instance [2],[5],[12]. In the following we give some definitions and preliminaries that is need for main results.

Definition 1.1. ([12]) Let A and B be nonempty subsets of a metric space X with $A_0 \neq \emptyset$. The pair (A, B) is said to have P-property if

$$\begin{cases} d(x_1, y_1) = dist(A, B) \\ \\ d(x_2, y_2) = dist(A, B) \end{cases} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

Definition 1.2. ([4]) Let X be a complete metric space and A and B subsets of X. A map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies:

(i) $T(A) \subset B, T(B) \subset A$
(ii) $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$ for all $x \in A$ and $y \in B$.

Lemma 1.3. *([13]) Let $\{a_n\} \subset [0, \infty)$, $\{b_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, 1)$ be sequences of real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n \quad \forall n \in \mathbb{N},$$
$$\sum_{n=1}^{\infty} c_n = \infty, \sum_{n=1}^{\infty} b_n < \infty.$$

Then, $\lim_{n\to\infty} a_n = 0$.

Definition 1.4. ([12]). Let (A, B) be a pair of two nonempty subsets of a complete metric space X. A map $T : A \to B$ is said to be a weakly contractive mapping if

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad \forall x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $lim_{t\to\infty}\psi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

Theorem 1.5. *([12]) Let (A, B) be a pair of two nonempty closed subsets of a complete metric space X such that A_0 is nonempty. Let $T : A \to B$ be a weakly contractive mapping such that $T(A_0) \subseteq B_0$. Assume that the pair (A, B) has the P-property. Then there exists a unique $x_0 \in A$ such that $d(x_0, Tx_0) = dist(A, B)$.*

2. Main results

In the following, we give some existence best proximity point theorems of nonexpensive mapping.

Theorem 2.1. *Let a pair (A, B) be nonempty closed convex subsets of a Banach space X such that $A_0 \neq \emptyset$ and weakly compact and also the pair (A, B) has the P-property. $T : A \to B$ a nonexpansive mapping such that $T(A_0) \subseteq B_0$. If $I - T$ is demiclosed, then $P_T(A, B) \neq \emptyset$.*

Proof. Suppose $T_n : A \to B$ define $T_n x = (1 - \frac{1}{n})u + \frac{1}{n}Tx$ where $u \in B_0$. It is clear that T_n is a non-self contractive mapping $T_n(A_0) \subseteq B_0$. By Theorem 1.5 there is $x_n \in A_0$ such that $\|x_n - Tx_n\| = dist(A, B)$. Hence $x_n - Tx_n \to u$. Since A_0 is compact, there is subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup v$. Hence u lies in the closure of $(I - T)(A)$. Because $I - T$ is closed, there exists a point $v \in A$ such that $(I - T)v = u$. Hence $\|v - Tv\| = dist(A, B)$. $\square$

Theorem 2.2. *Let a pair (A, B) be nonempty closed convex subsets of a Banach space X with the Opial condition such that $A_0 \neq \emptyset$ and weakly compact and also the pair (A, B) has the P-property. $T : A \to B$ a nonexpansive mapping such that $T(A_0) \subseteq B_0$. Then $P_T(A, B) \neq \emptyset$.*

Proof. By Theorem 2.1 there exists a sequence $\{x_n\}$ in A such that $lim_{n\to\infty}\|x_n - Tx_n\| = dist(A, B)$. By the weakly compactness of A_0, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. Now show that $I - T$ is demiclosed. Let $lim_{k\to\infty}\|(I - T)x_{n_k} - y\| = 0$ for some $y \in X$. Observe that

$$\|x_{n_k} - Tx - y\| \leq \|x_{n_k} - Tx_{n_k} - y\| + \|Tx_{n_k} - Tx\|,$$

which implies that

$$\liminf_{k\to\infty} \|x_{n_k} - Tx - y\| \leq \liminf_{k\to\infty} \|x_{n_k} - x\|.$$

By the Opial condition, we have

$$\liminf_{k\to\infty} \|x_{n_k} - x\| < \liminf_{k\to\infty} \|x_{n_k} - Tx - y\|,$$

a contradiction. Therefore, $(I - T)x = y$ and so by Theorem 2.1 $\|x - Tx\| = dist(A, B)$. $\square$

In the following, we give a best proximity point of nonexpensive mapping that is obtained as the limit of an iteratively sequence.

Theorem 2.3. *Let (A, B) be a pair of two nonempty closed convex subsets of a normed space X with the P-property such that $A_0 \neq \emptyset$ and compact. Suppose $T : A \to B$ be a nonexpensive map such that $TA_0 \subseteq B_0$ and $f : A \to A$ contractive map and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $u_1 \in A_0$, $d(x_n, u_n) = dist(A, B)$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subseteq [0, 1]$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then there are subsequences of $(\{u_n\}, \{x_n\})$ that converge to (x, Tx) such that $d(x, Tx) = dist(A, B)$.

Proof. Since A is compact, we have $\{u_n\}$ bounded, and since $\|u_n - x_n\| = dist(A, B)$, therefore $\{x_n\}$ is bounded. If $M = max\{sup\|f(x_n)\|, sup\|Tu_n\|\}$ we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)Tu_n - (\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1})Tu_{n-1})\| \\ &= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) + (1 - \alpha_n)Tu_n \\ &- [\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_n)Tu_{n-1} - (1 - \alpha_n)Tu_{n-1} + (1 - \alpha_{n-1})Tu_{n-1}]\| \\ &\leq \alpha_n a\|x_n - x_{n-1}\| + 2M|\alpha_n - \alpha_{n-1}| + (1 - \alpha_n)\|u_n - u_{n-1}\|, \end{aligned}$$

where $a \in [0, 1)$. We have $\|x_n - u_n\| = \|x_{n-1} - u_{n-1}\| = dist(A, B)$, and so by P-property $\|x_n - x_{n-1}\| = \|u_n - u_{n-1}\|$. Hence

$$\|x_{n+1} - x_n\| \leq (1 - \alpha_n(1 - a))\|x_n - x_{n-1}\| + 2M|\alpha_n - \alpha_{n-1}|.$$

Using Lemma 1.3, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

$$\begin{aligned} \|x_n - Tu_n\| &= \|\alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})Tu_{n-1} - Tu_n\| \\ &\leq \alpha_{n-1}\|x_{n-1} - Tu_n\| + (1-\alpha_{n-1})\|Tu_n - Tu_{n-1}\| \end{aligned}$$

and so

$$\lim_{n\to\infty} \|x_n - Tu_n\| = 0. \tag{2.1}$$

$$\|u_n - Tu_n\| \leq \|u_n - x_n\| + \|x_n - Tu_n\|$$

$$\lim_{n\to\infty} \|u_n - Tu_n\| = dist(A,B). \tag{2.2}$$

Since A is compact, there is subsequence $\{u_{n_k}\} \to x \in A_0$ and so by (2.2) we have $\|x - Tx\| = dist(A,B)$. Also by (2.1) we have $\|x_{n_k} - Tu_{n_k}\| \to 0$. Therefore $\|x_{n_k} - Tx\| \to 0$ and so $x_{n_k} \to Tx$. □

Theorem 2.4. *Let (A,B) be a pair of two nonempty closed convex subsets of a normed space X such that $A_0 \neq \emptyset$ and compact. Suppose $T : A\cup B \to A\cup B$ be a cyclic contractive map and let a sequence $\{t_n\}$ of real numbers and a sequence $\{x_n\}$ and be sequences generated by $x_1 \in A_0$ and*

$$x_{n+1} = (1-t_n)T^3x_n + t_nTx_n \quad \forall n \in \mathbb{N}, \tag{2.3}$$

where $\{t_n\} \subseteq [0,1]$ satisfy $\lim_{n\to\infty} t_n = 1$. Then there are subsequences of $\{x_n\}$ that converge to $x \in A$ such that $d(x,Tx) = dist(A,B)$.

Proof. Fix $x_1 \in A$ and define a sequence $\{x_n\}$ in $A \cup B$ by (2.3). First we show that $\lim_{n\to\infty} \|x_n - x_{n-1}\| = dist(A,B)$. Note

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1-t_n)T^3x_n + t_nTx_n - ((1-t_{n-1})T^3x_{n-1} + t_{n-1}Tx_{n-1})\| \\ &\leq (1-t_n)k^3\|x_n - x_{n-1}\| + t_nk\|x_n - x_{n-1}\| + (1-t_n)(1-k^3)dist(A,B) \\ &\leq (1-t_n(1-k))\|x_n - x_{n-1}\| + (1-k^3)dist(A,B) \\ &\;\;\vdots \\ &\leq \prod_{i=1}^{n}(1-t_i(1-k))\|x_2 - x_1\| + (1-k^{3n})dist(A,B) \\ &\leq e^{-\sum_{i=1}^{n} t_i(1-k)}\|x_2 - x_1\| + (1-k^{3n})dist(A,B). \end{aligned}$$

Hence $\|x_n - x_{n-1}\| \to dist(A,B)$.

Because A_0 is compact $\{x_{2n}\}$ has convergent subsequence $x_{2n_k} \to x \in A$. Now

$$dist(A,B) \leq \|x - x_{2n_k-1}\| \leq \|x - x_{2n_k}\| + \|x_{2n_k} - x_{2n_k-1}\|.$$

Thus we have $\|x - x_{2n_k-1}\|$ converges to $dist(A,B)$. Since

$$\begin{aligned} dist(A,B) &\leq \|x_{2n_k} - Tx\| \\ &\leq (1-t_{2n_k-1})\|T^3x_{2n_k-1} - Tx\| + t_{2n_k-1}\|Tx_{2n_k-1} - Tx\| \\ &\leq (1-t_{2n_k-1})\|T^2x_{2n_k-1} - x\| + t_{2n_k-1}\|x_{2n_k-1} - x\|. \end{aligned}$$

Since $t_{2n_k-1} \to 1$, hence we have

$$dist(A,B) \leq \lim_{k\to\infty} \|x_{2n_k} - Tx\| \leq dist(A,B).$$

Therefore $d(x,Tx) = dist(A,B)$. □

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