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A fixed point theorem for (φ, L)-weak contraction mappings on a partial metric space

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Abstract

In this paper, we explore (φ, L)-weak contractions of Berinde by obtaining Suzuki-type fixed point results. Thus, we obtain generalized fixed point results for Kannan's, Chatterjea's and Zamfirescu's mappings on a 0-complete partial metric space. In this way we obtain very general fixed point theorems that extend and generalize several related results from the literature.

Keywords: (φ, L)-weak contraction, partial metric, 0-complete space
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1. Introduction and preliminaries

Historically, the concept of completeness of metric spaces has interesting and important applications in classical analysis. On the other hand, Banach's fixed point theorem [8] is one of the most useful results in nonlinear analysis. Therefore, many authors considered the equivalence of the existence of fixed points of mappings by proving some equivalence theorems for the completeness of metric spaces. Precisely, Kirk [28] and Subrahmanyam [40] obtained that a metric space (X, d) is complete if and only if each Caristi's mapping defined on X has a unique fixed point, and by [40], (X, d) is complete if and only if each Kannan's mapping

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has a unique fixed point, but this is not the case with Banach's fixed point theorem. Despite of this fact, Suzuki [41] obtained a variant of Banach's fixed point theorem that characterizes metric completeness by using different types of contractions. Subsequently, many authors gave different generalizations of this result (see details in [33]–[42]). Very recently, Paesano and Vetro [38] proved analogous fixed point results for a self-mapping on a partial metric space and on a partially ordered metric space. Moreover, they obtained a characterization of partial metric 0-completeness in terms of fixed point theory. This result extends Suzuki's characterization of metric completeness.

Recently, Berinde introduced some new mappings that he called weak contraction mappings in a metric space [9, 10, 11]. He showed that Banach's, Kannan's, Chatterjea's and Zamfirescu's mappings are weak contractions. Subsequently, a lot of generalizations of these results in different types of spaces appeared in the literature (see, e.g., [1]–[19]; note that Berinde-type weak contractions were later usually called almost contractions).

In [12] Berinde introduced a nonlinear-type weak contraction using a comparison function and proved a fixed point result for such contractions. A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$, is called a comparison function if it satisfies:

(i) φ is increasing,

(ii) $\lim_{n\to+\infty} \varphi^n(t) = 0$ for all $t \in \mathbb{R}^+$ (here, φ^n is the n-th iterate of φ).

If φ satisfies (i) and

(iii) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t \in \mathbb{R}^+$,

then φ is said to be a (c)-comparison function.

One can find some properties and some examples of comparison and (c)-comparison functions in [11]. In particular, it is easy to see that $\varphi(t) < t$ for each comparison function φ and each $t > 0$.

Definition 1.1. Let (X, d) be a metric space and $T : X \to X$ be a self-mapping. T is said to be a weak φ-contraction (or (φ, L)-weak contraction) if there exist a comparison function φ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(y, Tx) \tag{1.1}$$

for all $x, y \in X$.

Then, Berinde proved that if (X, d) is a complete metric space and T is a (φ, L)-weak contraction, then T has a fixed point.

Clearly, any weak contraction (or (δ, L)-weak contraction) is a weak φ-contraction (in Definition 1.1 assume $\varphi(t) = \delta t$, where $\delta \in (0, 1)$), but the converse may not be true. Note that the class of weak φ-contractions includes Matkowski type nonlinear contractions [30].

Similarly to the case of weak contractions, by the symmetry of the metric d, the weak φ-contractiveness of the mapping T means also that

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(x, Ty) \tag{1.2}$$

for all $x, y \in X$. In other words, condition (1.1) can be replaced by

$$d(Tx, Ty) \leq \varphi(d(x, y)) + L \min\{d(y, Tx), d(x, Ty)\}. \tag{1.3}$$

In 2011 Suzuki proved a fixed point theorem [43] for generalized weak contractions with constants in complete metric spaces. Moreover, for each $r \in [0, 1)$, $(1 + r)^{-1}$ is the best constant in the Suzuki's fixed point theorem. Inspired by this paper, we deduce Suzuki-type fixed point results for generalized weak contractions in 0-complete partial metric spaces. Also we obtain that $(1 + r)^{-1}$ is the best constant in these results, for every $r \in [0, 1)$. Hence, in this way we give very general fixed point theorems that extend and generalize several related fixed point results from the literature.

2. Partial metric spaces

Recall (see, e.g., [3]–[31]) that a partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(ii) $p(x,x) \leq p(x,y)$,

(iii) $p(x,y) = p(y,x)$,

(iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) of a nonempty set X and a partial metric p on X. It is clear that, if $p(x,y) = 0$ then, from (i) and (ii), $x = y$. However, $p(x,x) = 0$ might not hold for each $x \in X$.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then two equivalent standard metrics, p^s and p^w can be defined on X by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$
$$p^w(x,y) = p(x,y) - \min\{p(x,x), p(y,y)\}.$$

Definition 2.1. A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ (with respect to τ_p) if and only if $\lim_{m,n\to+\infty} p(x_n,x_m) = \lim_{n\to+\infty} p(x_n,x) = p(x,x)$.

Definition 2.2. A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if

$$\lim_{m,n\to+\infty} p(x_m,x_n)$$

exists and is finite.

If $\lim_{m,n\to+\infty} p(x_n,x_m) = 0$, then $\{x_n\}$ is said to be a 0-Cauchy sequence in (X,p).

Definition 2.3. A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $\lim_{m,n\to+\infty} p(x_m,x_n) = p(x,x)$.

(X,p) is 0-complete [39] if every 0-Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ (with respect to τ_p) such that $p(x,x) = 0$. Therefore, (X,p) is 0-complete if and only if every 0-Cauchy sequence converges with respect to τ_{p^s}. It is clear that every 0-Cauchy sequence in (X,p) is a Cauchy sequence in (X,p). Therefore, if (X,p) is complete then it is 0-complete. The opposite is not true (see [39]).

In general, $p(x,y)$ is not a continuous function in two variables, in the sense that $x_n \to x$ and $y_n \to y$ (in τ_p) imply that $p(x_n,y_n) \to p(x,y)$, as $n \to +\infty$. However, the following holds:

Lemma 2.4 ([2, 25])**.** *Let (X,p) be a partial metric space and $\{x_n\} \subset X$. If $x_n \to z$ as $n \to +\infty$ and $p(z,z) = 0$ then, for each $x \in X$,*

$$\lim_{n\to+\infty} p(x_n,x) = p(z,x).$$

Definition 2.5 ([3])**.** Let (X,p) be a partial metric space. A mapping $T : X \to X$ is called a (δ, L)-weak contraction if there exist $\delta \in [0,1)$ and $L \geq 0$ such that

$$p(Tx,Ty) \leq \delta p(x,y) + Lp^w(y,Tx) \tag{2.1}$$

for all $x, y \in X$.

From the symmetry of the partial metric p and the respective standard metric p^w, the (δ, L)-weak contraction condition implicitly includes the following dual one

$$p(Tx, Ty) \leq \delta p(x, y) + Lp^w(x, Ty) \tag{2.2}$$

for all $x, y \in X$. These conditions might be replaced by the following one

$$p(Tx, Ty) \leq \delta p(x, y) + L \min\{p^w(y, Tx), p^w(x, Ty)\}. \tag{2.3}$$

Remark 2.6. In Definition 2.5, if p is an ordinary metric and $\varphi(t) = \delta t$, then the inequalities (2.1)–(2.3) reduce to (1.1)–(1.3), respectively.

Note that any Banach contraction is a (δ, L)-weak contraction. Also, Altun and Acar in [3] obtained that Kannan's and Chatterjea's mappings are (δ, L)-weak contractions on a partial metric space. Hence, they obtained some fixed point results such as Banach's, Kannan's, Chatterjea's, Zamfirescu's contraction-type theorems by using (φ, L)-weak contractions on a partial metric space.

3. Main results

Before giving the main result, we present the following lemma, which was given in [26, 41] in the case of standard metric. The proof is the same for the partial metric and so we omit it.

Lemma 3.1. *Let (X, p) be a partial metric space and let $T : X \to X$ be a mapping. Let $x \in X$ satisfy $p(Tx, T^2x) \leq rp(x, Tx)$ for some $r \in [0, 1)$. Then for $y \in X$, either*

$$\frac{1}{1+r} p(x, Tx) \leq p(x, y) \quad \text{or} \quad \frac{1}{1+r} p(Tx, T^2x) \leq p(Tx, y)$$

holds.

Now we present the main result.

Theorem 3.2. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be a mapping. Assume there exist a (c)-comparison function φ, $r \in [0, 1)$ and $L \geq 0$ such that*

$$(1 + r)^{-1} p(x, Tx) \leq p(x, y) \implies p(Tx, Ty) \leq \varphi(p(x, y)) + Lp^w(y, Tx) \tag{3.1}$$

for all $x, y \in X$. Then for every $x \in X$, $\{T^n x\}$ converges to a fixed point z of T such that $p(z, z) = 0$.

Proof. Since $(1 + r)^{-1} p(x, Tx) \leq p(x, Tx)$, we have

$$p(Tx, T^2x) \leq \varphi(p(x, Tx)) + Lp^w(Tx, Tx) = \varphi(p(x, Tx)). \tag{3.2}$$

Fix $u \in X$ and define $u_0 = u$, $u_n = Tu_{n-1}$ for $n \in \mathbb{N}$. Then from (3.2), we have

$$p(Tu_{n-1}, Tu_n) = p(T^{n-1}u, T^n u) \leq \varphi(p(u_{n-1}, u_n))$$

and so

$$p(u_n, u_{n+1}) \leq \varphi(p(u_{n-1}, u_n)).$$

We obtain by induction

$$p(u_n, u_{n+1}) \leq \varphi^n(p(u, Tu))$$

for all $n \in \mathbb{N}$. By triangle inequality, for $m > n$, we have

$$\begin{aligned} p(u_n, u_m) &\leq \sum_{i=n}^{m-1} p(u_i, u_{i+1}) - \sum_{i=n}^{m-2} p(u_{i+1}, u_{i+1}) \\ &\leq \sum_{i=n}^{m-1} p(u_i, u_{i+1}) \leq \sum_{i=n}^{+\infty} p(u_i, u_{i+1}) \leq \sum_{i=n}^{+\infty} \varphi^i(p(u, Tu)). \end{aligned}$$

Since φ is a (c)-comparison function, then $\sum_{i=0}^{+\infty}\varphi^i(p(u,Tu))$ is convergent and so $\{u_n\}$ is a 0-Cauchy sequence in X. Since X is 0-complete, it follows that $\{x_n\}$ converges, with respect to τ_p, to a point $z\in X$ such that

$$\lim_{n\to+\infty} p(u_n,z)=p(z,z)=0.$$

Now we claim that $p(z,Tz)=0$. Assume that $p(z,Tz)>0$. Since $u_n\to z$ as $n\to+\infty$, there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, $p(u_n,z)\leq\frac{1}{3}p(z,Tz)$. Also, by Lemma 3.1 and (3.2), we can find a subsequence $\{n_j\}$ of $\{n\}$ such that

$$(1+r)^{-1}p(T^{n_j}u,T^{n_j+1}u)\leq p(T^{n_j}u,z).$$

By (3.1), we have

$$\begin{aligned}
p(z,Tz)&\leq p(z,u_{n_j+1})+p(u_{n_j+1},Tz)\\
&=p(z,u_{n_j+1})+p(Tu_{n_j},Tz)\\
&\leq p(z,u_{n_j+1})+\varphi(p(u_{n_j},z))+Lp^w(z,u_{n_j+1})\\
&\leq p(z,u_{n_j+1})+\varphi\left(\frac{1}{3}p(z,Tz)\right)+Lp^w(z,u_{n_j+1})
\end{aligned}$$

and letting $n\to+\infty$ we obtain

$$0<p(z,Tz)\leq\frac{1}{3}p(z,Tz),$$

which is a contradiction. Therefore, $p(z,Tz)=0$ and $z=Tz$. □

Next we give a simple illustrative example.

Example 3.3. Let $X=[0,1]$ endowed with the partial metric $p:X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=\begin{cases}\max\{x,y\}+a, & x\neq y\\ a, & x=y\neq\frac{1}{2}\\ 0, & x=y=\frac{1}{2},\end{cases}$$

where $a\in(0,1)$. Also define $T:X\to X$ by

$$Tx=\begin{cases}\frac{1}{2}, & x\in[0,\frac{1}{2}]\\ \frac{1}{4}, & x\in(\frac{1}{2},1)\\ \frac{1}{3}, & x=1.\end{cases}$$

Let $r\in[0,1)$, $L=2$ and consider a (c)-comparison function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ given by $\varphi(t)=st$ for all $t\in\mathbb{R}^+$, where $0<r<s<1$ and $s>\frac{2a}{2a+1}$.

Since $p(x,x)=a\neq 0$ for all $x\neq\frac{1}{2}$, it follows that p is not a metric. Obviously, (X,p) is 0-complete and

$$p^w(x,y)=\begin{cases}\max\{x,y\}, & x\neq y\neq\frac{1}{2}\\ \max\{x,y\}+a, & x\neq y,\ x=\frac{1}{2}\ (or\ y=\frac{1}{2})\\ 0, & x=y.\end{cases}$$

Now, we discuss the condition (3.1). Indeed, if $x=y$ then the condition (3.1) trivially holds. Let us, therefore, assume that $x\neq y$ and distinguish the following eight cases:

1° $x\in[0,\frac{1}{2}]$, $y\in[0,\frac{1}{2}]$. Then $Tx=Ty=\frac{1}{2}$, $p(Tx,Ty)=0$; hence, the right-hand side of (3.1) trivially holds and so the implication is true.

2° $x \in [0, \frac{1}{2}]$, $y \in (\frac{1}{2}, 1)$. Then $Tx = \frac{1}{2}$, $Ty = \frac{1}{4}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot \left(\frac{1}{2} + a\right) \leq y + a$ and holds true. The right-hand side reduces to

$$\begin{cases} \frac{1}{2} + a \leq s(y+a) + 2y & \text{if } x \neq \frac{1}{2} \\ \frac{1}{2} + a \leq s(y+a) + 2(y+a) & \text{if } x = \frac{1}{2} \end{cases}$$

and, in both cases, holds true since $y > \frac{1}{2}$.

3° $x \in [0, \frac{1}{2}]$, $y = 1$. Then $Tx = \frac{1}{2}$, $Ty = \frac{1}{3}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot \left(\frac{1}{2} + a\right) \leq 1 + a$ and holds true. The right-hand side reduces to

$$\begin{cases} \frac{1}{2} + a \leq s(1+a) + 2 & \text{if } x \neq \frac{1}{2} \\ \frac{1}{2} + a \leq s(1+a) + 2(1+a) & \text{if } x = \frac{1}{2} \end{cases}$$

and, in both cases, holds true.

4° $x \in (\frac{1}{2}, 1)$, $y \in [0, \frac{1}{2}]$. Then $Tx = \frac{1}{4}$, $Ty = \frac{1}{2}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot (x + a) \leq x + a$ and holds true. The right-hand side reduces to

$$\begin{cases} \frac{1}{2} + a \leq s(x+a) + 2\max\{y, \frac{1}{4}\} & \text{if } y \neq \frac{1}{2} \\ \frac{1}{2} + a \leq s(x+a) + 2\max\{y, \frac{1}{4}\} + 2a & \text{if } y = \frac{1}{2} \end{cases}$$

and, in both cases, holds true for all x, y.

5° $x \in (\frac{1}{2}, 1)$, $y \in (\frac{1}{2}, 1)$. Then $Tx = Ty = \frac{1}{4}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot (x + a) \leq \max\{x, y\} + a$ and holds true. The right-hand side reduces to $a \leq s(\max\{x, y\} + a) + 2y$.

6° $x \in (\frac{1}{2}, 1)$, $y = 1$. Then $Tx = \frac{1}{4}$, $Ty = \frac{1}{3}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot (x + a) \leq 1 + a$ and holds true. The right-hand side reduces to $\frac{1}{3} + a \leq s(1 + a) + 2$ and holds true.

7° $x = 1$, $y \in [0, \frac{1}{2}]$. Then $Tx = \frac{1}{3}$, $Ty = \frac{1}{2}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r} \cdot (1 + a) < 1 + a$ and holds true. The right-hand side reduces to

$$\begin{cases} \frac{1}{2} + a \leq s(1+a) + 2\max\{y, \frac{1}{3}\} & \text{if } y \neq \frac{1}{2} \\ \frac{1}{2} + a \leq s(1+a) + 2\max\{y, \frac{1}{3}\} + 2a & \text{if } y = \frac{1}{2} \end{cases}$$

and, in both cases, holds true.

8° $x = 1$, $y \in (\frac{1}{2}, 1)$. Then $Tx = \frac{1}{3}$, $Ty = \frac{1}{4}$. The left-hand side of (3.1) reduces to $\frac{1}{1+r}(1 + a) < 1 + a$ and holds true. The right-hand side reduces to $\frac{1}{3} + a \leq s(1 + a) + 2y$ and holds true since $y > \frac{1}{2}$.

Thus, in all possible cases the condition (3.1) is satisfied. Thus we can apply Theorem 3.2 to this example and $z = \frac{1}{2}$ is a fixed point of T.

As a direct consequence of Theorem 3.2, we obtain Theorem 3 of Altun and Acar in [3]:

Corollary 3.4 ([3])**.** *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be a (φ, L)-weak contraction, where φ is a (c)-comparison function and $L \geq 0$. Then T has a fixed point.*

Also, taking $\varphi(t) = rt$ for all $t \in \mathbb{R}^+$ and some $r \in [0, 1)$, we obtain the following result.

Corollary 3.5. *Let (X, p) be a 0-complete partial metric space and $T : X \to X$ be a mapping. Assume there exist $r \in [0, 1)$ and $L \geq 0$ such that*

$$(1 + r)^{-1} p(x, Tx) \leq p(x, y) \implies p(Tx, Ty) \leq r\, p(x, y) + Lp^w(y, Tx)$$

for all $x, y \in X$. Then for every $x \in X$, $\{T^n x\}$ converges to a fixed point of T.

Now, we show that $(1+r)^{-1}$ is the best constant in the previous corollary, i.e., for (r, L)-weak contraction mappings.

Theorem 3.6. *For each $r \in [0,1)$, there exist a 0-complete partial metric space (X,p), a mapping $T : X \to X$ and $L \geq 0$ such that T has no fixed points and*

$$(1+r)^{-1}p(x,Tx) < p(x,y) \implies p(Tx,Ty) \leq r\,p(x,y) + Lp^w(y,Tx) \tag{3.3}$$

for all $x,y \in X$.

Proof. Let $X = [0,1]$ and define a 0-complete (partial) metric by

$$p(x,y) = p^w(x,y) = \begin{cases} \max\{x,y\}, & x \neq y \\ 0, & x = y. \end{cases}$$

We define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}) \\ \frac{1}{4}, & x \in [\frac{1}{2}, 1) \\ \frac{1}{3}, & x = 1. \end{cases}$$

Then we take $L = 2$ and show that T has no fixed points.

If $x = y$ then $p(x,y) = 0$; it follows that the left-hand side of (3.3) cannot hold and the thesis of the theorem is true. Let us, therefore, assume that $x \neq y$ and distinguish the following eight cases:

1° $x \in [0, \frac{1}{2})$, $y \in [0, \frac{1}{2})$. Then $Tx = Ty = \frac{1}{2}$, $p(Tx,Ty) = 0$; hence, the right-hand side of (3.3) trivially holds and so the implication is true.

2° $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1)$. Then $Tx = \frac{1}{2}$, $Ty = \frac{1}{4}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} \cdot \frac{1}{2} < y$ and holds true (except if $r = 0$ and $y = \frac{1}{2}$). The right-hand side reduces to $\frac{1}{2} \leq ry + 2y$ and holds true since $y \geq \frac{1}{2}$.

3° $x \in [0, \frac{1}{2})$, $y = 1$. Then $Tx = \frac{1}{2}$, $Ty = \frac{1}{3}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} \cdot \frac{1}{2} < 1$ and holds true. The right-hand side reduces to $\frac{1}{2} \leq r + 2$ and holds true.

4° $x \in [\frac{1}{2}, 1)$, $y \in [0, \frac{1}{2})$. Then $Tx = \frac{1}{4}$, $Ty = \frac{1}{2}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} \cdot x < x$ and holds true (except if $r = 0$). The right-hand side reduces to $\frac{1}{2} \leq rx + 2\max\{y, \frac{1}{4}\}$ and holds true for all x, y.

5° $x \in [\frac{1}{2}, 1)$, $y \in [\frac{1}{2}, 1)$. Then $Tx = Ty = \frac{1}{4}$, $p(Tx,Ty) = 0$; again, the right-hand side of (3.3) trivially holds and so the implication is true.

6° $x \in [\frac{1}{2}, 1)$, $y = 1$. Then $Tx = \frac{1}{4}$, $Ty = \frac{1}{3}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} \cdot x < 1$ and holds true. The right-hand side reduces to $\frac{1}{3} \leq r + 2$ and holds true, too.

7° $x = 1$, $y \in [0, \frac{1}{2})$. Then $Tx = \frac{1}{3}$, $Ty = \frac{1}{2}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} < 1$ and holds true (except if $r = 0$). The right-hand side reduces to $\frac{1}{2} \leq r + 2\max\{y, \frac{1}{3}\}$ and holds true.

8° $x = 1$, $y \in [\frac{1}{2}, 1)$. Then $Tx = \frac{1}{3}$, $Ty = \frac{1}{4}$. The left-hand side of (3.3) reduces to $\frac{1}{1+r} < 1$ and holds true (except if $r = 0$). The right-hand side reduces to $\frac{1}{3} \leq r + 2y$ and holds true since $y \geq \frac{1}{2}$.

Thus, in all possible cases the condition (3.3) is satisfied. This completes the proof of the theorem. □

As in the metric case, the hypotheses of Theorem 3.2 do not guarantee the uniqueness of the fixed point. However, we conclude this paper giving a sufficient condition for uniqueness, which is similar to the ones in the metric case (see [7]) and in the partial metric case (see [3]) for "non-Suzuki"-type mappings.

Theorem 3.7. *Let (X,p) be a 0-complete partial metric space and $T : X \to X$ be a (φ, L)-weak contraction. Also suppose there exist a comparison function φ_1 and $L_1 \geq 0$ such that*

$$\frac{1}{1+r}p(x,Tx) \leq p(x,y) \implies p(Tx,Ty) \leq \varphi_1(p(x,y)) + L_1 p^w(x,Tx) \tag{3.4}$$

for all $x,y \in X$. Then T has a unique fixed point z such that $p(z,z) = 0$.

Proof. By Theorem 3.2, there is a point $z \in X$ such that $Tz = z$ and $p(z, z) = 0$. Suppose that there is another point $w \in X$ such that $Tw = w$ and $p(w, w) = 0$ and assume that $p(z, w) > 0$. Then

$$\frac{1}{1+r}p(z, Tz) = 0 \leq p(z, w),$$

so we can put $z = x$ and $y = w$ in (3.4) to obtain

$$0 < p(z, w) = p(Tz, Tw) \leq \varphi_1(p(z, w)) + L_1 p^w(z, Tz) = \varphi_1(p(z, w)) < p(z, w),$$

a contradiction. Hence, $p(z, w) = 0$ and $z = w$. □

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